Chapter 2
Topic-Specific Design Research: An Introduction

Koeno Gravemeijer and Susanne Prediger

Abstract Design research has become a powerful research methodology of increasing relevance in mathematics education research. This chapter provides an overview and selected insights for novice researchers who want to find out if this research methodology is suitable for their own projects, and what possible research outcomes can look like. As topic-specificity is the feature that distinguishes didactical design research from generic educational design research, different models for topic-specific design research are presented.

Keywords Design research · Design experiment · Learning processes · Realistic mathematics education · Structuring the learning content

2.1 Introduction

Design research is a research methodology that has grown during the last 30 years, starting with early work in the 1980 and 1990s (Cobb and Steffe 1983; Gravemeijer and Koster 1988; Wittmann 1995; Artigue 1992; see Prediger et al. 2015, for a historical overview). In this chapter, we present its main ideas and common features, but also different versions of design research. We focus on topic-specific design research aiming at local instruction theories for different mathematical topics.

In this chapter, we present design research with its aims, common characteristics and usual procedures (Sect. 2.2) and offer insights into two example projects (Sect. 2.3). Section 2.4 provides categories for reflection on design research.

K. Gravemeijer (✉)
Eindhoven University of Technology, Eindhoven, The Netherlands
e-mail: koeno@gravemeijer.nl

S. Prediger
Technical University Dortmund, Dortmund, Germany
e-mail: prediger@math.uni-dortmund.de

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2.2 What Is Design Research?

2.2.1 Dual Aims and Common Characteristics

Design research combines *instructional design* (aiming at developing teaching-learning arrangements for classrooms) and *educational research* (aiming at investigating and understanding the initiated teaching learning processes, and what brings this process about). Instead of executing those activities in sequence, design-researchers perform both simultaneously and intertwine them in several cycles in order to reach the dual aims (Cobb et al. 2003; Kelly et al. 2008; Van den Akker et al. 2006).

Even if design research approaches can differ in their concrete realization, they usually share five common characteristics (Cobb et al. 2003; Prediger et al. 2015). They are

(1) *interventionist*, i.e., the intent of design research is to create and study new forms of instruction, in this sense, it must be intended to intervene in the classroom practices (interventionist) rather than just to involve observation of regular classroom practices (naturalistic);
(2) *theory generative*, i.e., the goal of design research is to generate theories about the process of learning and the means of supporting that learning (see above); generating theories here means both developing and refining theories in terms of inventing categories and generating hypotheses (but rarely 'testing hypotheses' in the narrow sense of experimental psychology);
(3) *prospective and reflective,* i.e., design experiments create conditions for developing theory (prospective), however, these theories are in turn the subject of critical examination (reflective);
(4) *iterative*, i.e., theory is developed in an iteration of cycles of conjecturing, testing, and revising;
(5) *pragmatic roots and humble theories*, i.e., design experiments accept the complexity of the classroom as a research setting, and theories are domain- or even topic-specific and are meant to have practical implications.

2.2.2 General Structure of a Design Experiment

These characteristics are realized by design experiments (Cobb et al. 2003). Very roughly speaking, what design researchers do in design experiments is not very different from what teachers do as reflective practitioners, but researchers combine this practice with theory development.

We may observe that what teachers do when teaching a lesson involves three kinds of activity:

- *preparing,* the teacher designs or selects instructional activities with an eye on the learning goals;
- *enacting,* the instructional activities are enacted and the teacher observes the students' actions and utterances with an eye on the intended learning process;
- *reflecting,* the teacher analyzes what has transpired in the classroom, contrasts this with what was anticipated, and revises or adapts the instructional activities.

Typically, reflective practitioners search for the best solution to a concrete practical problem (possibly in an action research mode, e.g., Breen 2003). In retrospect, they might ask themselves, what have I learned? However, teachers rarely start out with the aim of learning from a specific lesson. Moreover, if that would be a teacher's goal, he or she would have to consider how to facilitate reaching that goal, for instance, by explicating the goals and expectations about the learning process in advance, and considering how to keep track of the learning of the students and the factors that might influence that learning.

Likewise, these are important considerations for design researchers. In design research the overriding goal is to contribute to theory development that transcends an individual classroom or lesson. Design researchers may also aim at solving concrete problems, but the aims always include gaining insights into the learning processes, the means of support and typical obstacles and conditions of success (Cobb et al. 2003; Bakker and van Eerde 2015). As design researchers want to make a contribution to the scientific community, an additional feature comes to the fore, that of ensuring a sound empirical and theoretical basis as support for theoretical claims, which may emerge from the design experiment.

Summarizing, we may argue that at its core, design research resembles what reflective practitioners do when designing, enacting and reflecting on individual lessons. However, the goal of generating empirically grounded theory brings a host of demands that are not part of everyday teaching. We elaborate this point in the following, by showing how the three phases, preparing, enacting and reflecting, are worked out in design research.

Preparing for the Design Experiment

In preparation for the design experiment, the researchers need to clarify the learning goals and the instructional starting points, and to develop a conjectured, or provisional, local instruction theory. Such a local instruction theory includes theories about a possible learning process, and theories about possible means of supporting that learning process. Further decisions will have to be made about the theoretical intent of the design experiment and about data gathering and data analysis.

As a rule, the research team cannot simply adopt the educational goals that are current in a given domain—as in general these goals may be determined largely by history and tradition. The researchers will have to problematize the topic under consideration from a disciplinary perspective, search for the core ideas in the given domain, and establish what the most relevant or useful goals are (Gravemeijer and Cobb 2006).

In order to be able to develop a conjectured local instruction theory, one also has to consider the instructional starting points. The focus here is to understand the consequences of earlier instruction, upon which one can build in the further design experiment cycles.

Once the potential end points and the instructional starting points are established, the design research team can start to formulate the conjectured local instruction theory. The term *conjectured* is used as the expectation is that this theory will be revised under the influence of how the students' thinking and understanding evolves when the planned (and later revised) instructional activities are enacted in the classroom. Simon's (1995) conception of a hypothetical learning trajectory may serve as a paradigm here. The enactment of the instructional activities is always tightly interwoven with the envisioned classroom culture and the proactive role of the teacher, so this must be part of the planning as well.

We may further note, that even though one of the primary aims of a design experiment is to support the constitution of an empirically grounded local instruction theory, another aim might be to study classroom events as instances of more encompassing issues. Such issues are, for instance, the role of symbolizing and modeling or the proactive role of the teacher. In practice, this type of aim may be identified prior to the design experiment, during the experiment, or even afterwards.

As part of the preparation, decisions have to be made about the types of data that need to be generated in the course of the experiment. A general guideline here is that the data have to make it possible to address the issues that were identified as research goals at the start of the design experiment.

Next to data gathering one also has to consider how the data are to be interpreted. Here the theoretical frameworks may play a dual role. We may take the emergent perspective on the classroom culture (Yackel and Cobb 1996) as an example. On the one hand, the concepts of social norms and socio-mathematical norms reveal what norms to aim for in order to make the design experiment successful. On the other hand, the same framework offers an interpretative framework for analyzing classroom discourse and communication.

Enacting the Design Experiment

The second phase consists of actually conducting the design experiment. At the heart of the design experiment lies a cyclic process of (re)designing, and testing instructional activities and other aspects of the design. The scope of such a cycle may vary over research projects, from individual activities or lessons, to a complete course. In each cycle, the research team conducts an anticipatory thought experiment by envisioning how the proposed instructional activities might be realized in interaction in the classroom, and what students might learn as they participate in them. During the enactment of the instructional activities in the classroom, and afterwards, the research team tries to analyze the actual process of the students' participation and learning. On the basis of this analysis, the research team later makes decisions about the validity of the conjectures that underlay the instructional activities, and about the consequences for the next activity. This often also implies an adaptation of the local instruction theory.

Reflecting on the Design Experiment, the Retrospective Analysis

One of the primary aims of a design experiment is typically to contribute to the development of a local instruction theory. Other goals may concern more encompassing issues. The manner in which the retrospective analysis is conducted will vary, as differences in theoretical frameworks and objectives will result in differences in the retrospective analyses. Instead of trying to offer a general description, we return to this issue in the discussion of the cases presented as examples.

2.2.3 Differences Between Various Design Research Approaches

Most design research approaches can be subsumed under the three main steps of preparing, enacting, and reflecting, as sketched in Sect. 2.3, and exhibiting the five characteristics presented in Sect. 2.2. However, design research approaches take a large variability of forms, depending on their origin, their actual context, and the specific needs they are supposed to fulfill. Hence, literature of the last decades pays tribute to this variety (e.g., Kelly et al. 2008; Plomp and Nieveen 2013; Van den Akker et al. 2006, for educational design research in different domains). Surveying the field in mathematics education, Prediger et al. (2015) have classified the differences with respect to the following:

- *age groups*: These may vary from Kindergarten to university mathematics.
- *the reasons for doing design research:* Design research approaches vary in their prioritization of the dual aims, focusing more towards solving practical problems or more towards generating theory and understanding the teaching learning processes.
- *the type of results:* Depending on the prioritization of aims, the former purpose may aim at producing artifacts that can be used directly in classrooms. In contrast, the latter may aim at local instruction theories or more general insights, and is often embedded in a larger research program.
- *the scale of the design project:* This may vary from the nano level (of individuals and single tasks), through the micro level (classrooms and teaching units), the meso level (e.g. school-specific curriculum), the macro level (e.g. national syllabi or core objectives) up to the supra level (international or internationally comparative aspects), as specified by Van den Akker (2013, p. 55).
- *the background theory:* Finally, implicit or explicit *background theories* on teaching and learning will strongly influence both the conception and the results of research, e.g., socio-constructivism will lead to other decisions in design and analytic focus than a purely individualistic background theory.

Here, we add an additional source of variation, the degree to which the design research takes into account the *topic-specificity* of the instructional design and the research.

2.2.4 Striving for Topic-Specific Design Research Rather Than Only Generic Educational Design Research

Design research approaches are successfully applied in generic educational research as well as in different subject matter didactics, such as mathematics education research. The collection of 51 case studies involving design research (Plomp and Nieveen 2013) shows that both versions are insightful. Whereas generic design research projects mostly focus on specific design principles or design elements (e.g., How can the use of tutorial computer systems enhance students' motivation for independent work?), many subject matter education research projects pose more didactical questions, concerning the specific mathematical topics to be learned.

In topic-specific design research, which is the theme of this chapter, didactical issues are put at the center. These concern both the question of *how* to teach a given topic and the question of *what* should be taught and *in which structure* (i.e. sequencing order and sense making relations). These questions are treated in all three phases, starting with the preparation phase. It is important to emphasize that the choice of the topic itself is not an empirical question. But the empirical part of topic-specific design research can provide new insights into the structure of the topic to be learned (Hußmann and Prediger 2016). In the following sections, we explain what we mean by this kind of topic-specificity, because we consider it an important quality within the areas of subject matter research.

Apart from promoting topic-specific design research, we take the position that, in general, topic-independent principles must be enriched by very concrete, topic-specific design research striving for local instruction theory on the concrete topic. The design research aims at finding concrete ways of realization as well as specialized knowledge regarding typical, topic-specific learning and teaching processes, organized in hypothetical learning trajectories, as explained in Sect. 2.2.

Although some elements of the local theories are of course transferable to the next topic (e.g., from algebraic expressions to fractions), this transfer is usually investigated in a subsequent topic-specific design research project.

2.3 Learning from Examples of Topic-Specific Design Research

In this section, two projects are presented briefly, in order to provide insights into the processes and typical outcomes. Although sharing the topic-specificity (see Sect. 1.4) and the strong focus on learning processes (Prediger et al. 2015), these projects are different in terms of interesting aspects: Firstly, both examples offer guidelines for instructional design as a significant part of design research, although in different forms. Secondly, both examples have an open eye for what is happening in the design experiment, however, in significantly different ways. The second example on fruitful starting points and obstacles for students' learning pathways, focuses on what the underlying difficulties are and how those can be addressed, following a highly structured approach. The first example aims at developing a local instruction theory from scratch; there is no history of teaching this topic to the given age group. In this sense the research project is exploratory. It aims at finding out which opportunities arise and what possibilities emerge, what ingenuity the students bring to the table and how this can be utilized in the design of highly innovative instruction.

2.3.1 Exploratory Design Research—An Example Project for Instantaneous Speed in Grade 5

In this section, a design research project on instantaneous speed in 5th Grade, carried out as a Ph.D. study by de Beer (2016), is presented as an example of exploratory design research. The section starts with a brief introduction to the design research tradition in which this research project was embedded, namely, Realistic Mathematics Education (RME).

Realistic Mathematics Education as a Research Tradition

Design research in the RME tradition has its roots in Freudenthal's (1973) proposal to organize mathematics education as a process of guided reinvention. Analyzing instructional sequences that tried to do justice to this principle, Treffers (1987) formulated the domain-specific theory for realistic mathematics education implicit in those sequences. This theory was later cast in terms of three instructional design heuristics (Gravemeijer 2004), guided reinvention, didactical phenomenology, and emergent modeling—which are discussed later in this chapter.

The aim of RME is that students be enabled to construct their own mathematics under their own steam. However, the goal is not for the students to construct idiosyncratic mathematics; the mathematics the students construct has to be compatible with the conventional mathematics of the wider society. Thus the teacher has to support students in building on their own knowledge and ideas, while at the same time keeping an eye on the endpoints for which he or she is aiming. This goal points to an interactive process in which the teacher adapts to the students' thinking. To support such a process, RME design research aims at developing local instruction theories, which can function as frameworks of reference for teachers. On the basis of these frameworks, teachers may develop hypothetical learning trajectories (Simon 1995), tailored to their preferences, their goals and their classrooms. In line with this conception, RME design research aims at developing theory about student learning, together with theories about the means of support—such as instructional activities, tasks and tools, and a fitting classroom culture. Thus the goal in RME design research is not just the development of instruction that fits the idea of guided reinvention, for a given topic. Key is also to come to understand how that instruction works. In relation to this point we may observe that there are often a number of key insights that emerge during one or more research projects. Such insights may emerge in all three phases of a research project.

Being open to new insights is critical here. In relation to this openness, we may refer to Smaling's (1992) methodological conception of objectivity. Smaling (1992) argues that there are two components of objectivity, (1) avoiding distortion, and (2), *"letting the object speak"*. Design research aiming at new insights, relies heavily on the latter—of course without neglecting the need to avoid distortion.

As indicated above, a design research project on instantaneous speed in Grade 5 is presented here as an example of exploratory design research within this research tradition. This research project consisted of a series of design experiments, which were extensively reported on by de Beer (2016). Here, we do not describe the individual design experiments, but try to give a more general overview. We use the three phases of a design experiment to structure our elucidation, even though there were actually three design experiments, and thus for each experiment, three of cycles of preparation, enactment, and retrospective analysis, were completed.

Preparing for the Design Experiment on Instantaneous Speed

In the preparation phase, we established the starting points, the potential end points and the preliminary local instruction theory. In conventional education, instantaneous speed is approached by taking the limit of average speed for a time interval approaching zero. This is of course beyond the reach of primary school students. We therefore aimed at an informal conception of instantaneous speed. Following Kaput and Schorr (2007), we further inferred that interactive dynamic computer representations might offer support.

While preparing for the experiment, we drew on the RME instructional design heuristics concerning guided reinvention, didactical phenomenology and emergent modeling (Gravemeijer 1999).

Guided reinvention. Though there is a tradition in RME of describing goals as procedures in relation to the reinvention of algorithms, our interest has shifted towards mathematical relations and conceptual understanding (Gravemeijer in preparation). In case of the local instruction theory on speed, we may characterize the goal for the students as developing a framework of mathematical relations, which involve co-variance, tangent lines, rise-over-run, and eventually, speed as a variable. But at the start of the design experiment on instantaneous speed, it was not clear what would be within the reach of 5th grade students. The belief that understanding speed is closely linked to graphing, however, provided a clear direction for the design. When following the guided reinvention design recommendation to look at the history, we also found strong links between trying to come to grips with speed and the use of graphs. We further made a connection with a historical definition of speed, which preceded the notion of average speed.

Around 1335, William Heytesbury reasoned that one could define instantaneous speed on the basis of the distance that would be traveled if the speed would stay constant for a given period of time (Clagett 1959).

Didactical phenomenology. The didactical-phenomenology design heuristic advises the researcher-designer to look for the phenomena that are organized by the tool, concept, or procedure one wants the students to reinvent (Freudenthal 1991). In our case, the phenomenon of moving objects presents itself as an obvious candidate. As researchers, however, we were concerned that the students' use of the language learned at school in connection with motion might make it very difficult to establish the students' actual understanding of speed. Looking for an alternative we found various descriptions in the literature, of students reasoning about the speed with which the water level in glassware rises (e.g., Swan 1985). On the basis of this characterization, we inferred that they have a basic understanding of the relation between the width of the glass and the rising speed, and of the relation of the latter with the shape of the corresponding graph. In terms of the didactical phenomenology design heuristic, the changing water height became the phenomenon that could be organized by the tool (a graph), and the concept (speed) we wanted the students to reinvent.

Emergent modeling. The emergent modeling design heuristic was not elaborated at the start of the design experiment. Nevertheless, the idea that the visual representations of changing water heights had to play a central role, was already indicted by the other design heuristics. Modeling was given a more prominent place in the 2nd design experiment cycle, when we decided to integrate the idea of modeling-based learning. Consequently modeling changing water heights received a more prominent place. Gradually we started to realize that the initial model and the final model could respectively be described as *model of changing water heights*, and, *model for reasoning about the rising speed* (Gravemeijer 1999), while 'the' model could be loosely defined as 'visual representations of the filling process'.

Enacting the Design Experiment on Instantaneous Speed

As we felt we did not know enough of the students' starting points, we started with a number of one-on-one teaching experiments to get a sense of potential starting points. Those one-on-one teaching experiments showed us that the students were quite able to reason about cylindrical glasses. They realized that the water would rise with a constant speed, and they effortlessly drew linear graphs to illustrate this. With cocktail glasses of conic shape, however, they ran into problems. They initially believed that the water height in the cocktail glass would develop similarly to the cylindrical glass. When they were shown a computer simulation of how the glass filled up, they quickly realized that the rising speed slowed down when the water level went up. They also realized the logic behind it; as the glass got wider the rising speed would go down. As a rule, however, the students in the sample were unable to draw a seemingly correct graph. Our explanation for this was that the students had virtually no experience with drawing graphs. All they had was some experience with interpreting segmented line graphs.

Reflecting on these findings, we connected Heytesbury's idea of "speed staying the same" to the notion of constant speed. In other words, the (instantaneous) rising speed at a given point (at given width) in an arbitrary glass could be equated with the constant speed in a cylindrical glass of precisely that width. In line with this idea, we decided to ask the students, when the rising speeds in a cylindrical and a cocktail glass would be the same. Once this link was established, we would elaborate on it. The linear graph of the fitting cylindrical glass would not only show the speed at a point, but might also be developed as the tangent line at that point, with help of dynamic computer software (Fig. 2.1).

The end point we would be aiming for, therefore concerned the idea of the tangent line at a point of a height versus time graph, signifying the constant speed that would correspond with the instantaneous speed at that point.

Conceptually, the conjectured learning process starts from the students' informal understanding of instantaneous speed, builds on their insight that the rising speed at a given height is defined by the width of the glass at that height, deepens that insight by equating the instantaneous speed with a constant speed that corresponds with that width, and expressing this speed with a linear graph corresponding with a tangent line to the graph of water height versus time.

In deviation from the ideal of short micro-design cycles in which adaptations occur during each design experiment, adaptations were mainly made in between subsequent design experiments. The research conditions did not allow for changes on the spot. Moreover the design experiments had to be carried out in a limited series of lessons, as the topic of instantaneous speed was not part of the regular curriculum.

Because we were not clear about how the students were thinking in the first teaching experiment, we borrowed the idea of modeling-based learning (MbL) from science education (Louca and Zacharia 2012), which aims at engaging students in the socially mediated development and use of an explanatory model. In doing so, we tried to foster that the students would express their thinking with their models. This attention to modeling resulted in the second teaching experiment being better aligned with the RME design heuristic of emergent modeling.

The students were asked to make drawings that would show how the rising speed in a cocktail glass would develop, and next to improve on them. As expected, the students initially came up with realistic drawings that had the character of snapshots (Fig. 2.2).

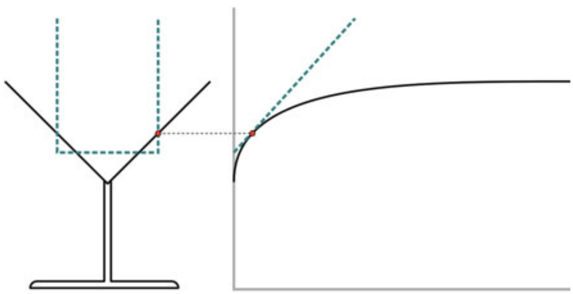

Fig. 2.1 Computer software linking constant speed in an imaginary cylindrical glass with a tangent line

Fig. 2.2 One student's snapshots showing how the water level changed over time: "It goes up increasing more and more slowly"

In subsequent lessons the representations were discussed and the students were asked to make minimalist models with only the necessary elements to describe the situation. This resulted in a pivotal episode—on which we reported earlier (de Beer et al. 2015).

In a whole class discussion in one of the two classrooms in the 3rd design experiment, some students drew a segmented line graph on the white board to model how the water level changed over time (see Fig. 2.3).

When the teacher asked the class if they could find the speed in this graph, a student remarked that a straight line did not fit his understanding of how the cocktail glass fills up. He argued, that "it should go a bit bent", and he drew a curve (Fig. 2.4).

The student explained that at a certain moment, the graph would almost not rise any more. Reactions by other students in the classroom suggested that they agreed with this line of reasoning. The students in the parallel classroom did not come up with the idea of a curved graph by themselves. Here the teacher introduced the idea of shrinking the intervals in discrete graphs. On the basis of this suggestion, these students too came to see the curved continuous graph as an adequate model for

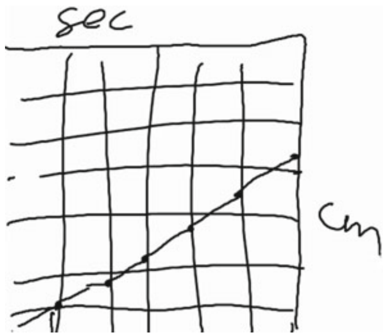

Fig. 2.3 Segmented line graph as a model of changing water heights

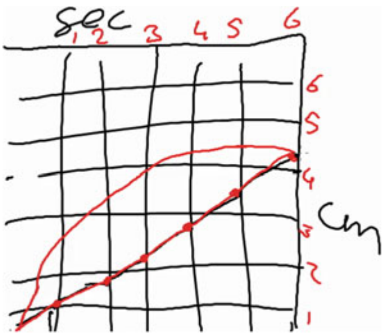

Fig. 2.4 Curve as improvement on the straight line

describing changing speed. In retrospect, we believe that the discrepancy between the segmented line graphs and the constantly diminishing speed may be exploited to support other students in reinventing the curved graph.

Retrospective Analysis

As explained above, abduction played an important role in the retrospective analysis (de Beer et al. 2015). The retrospective analysis was based on the comparative method of Glaser and Strauss (1967), and more specifically on the elaboration of this method by Cobb and Whitenack (1996). The analysis consisted of two steps: First formulating *conjectures of what happened* and testing those conjectures against the available data, and second, formulating *conjectures of why this happened*, which were also tested against the data. Although all data were taken into account, especially the transcriptions of whole-class discussions and the students' products proved valuable in formulating and testing conjectures.

This two-step analysis was carried out after each design experiment, each time the findings of the earlier experiment informed the following one, building on the conjectures that were corroborated, and revising conjectures that were rejected. The latter were used to improve the design, and to generate new explanatory conjectures. We do not have enough room to work out the potential local instruction theory that emerged for this design experiment. We do believe, however, that the gist of it can be deducted from the above account. A more detailed description can be found in the thesis of de Beer (2016). Instead, here we highlight the key insights that emerged from this project:

- fifth graders understand the relation between the rising speed and the width of the glass;
- fifth graders need only a little reflection time to realize that the rising speed in a cylindrical glass and a cocktail glass would be the same, when the widths would be the same;
- fifth graders have an intuitive conception of instantaneous speed, which can be deepened;
- the constant speed in a cylindrical glass may be used to specify the instantaneous speed in a cocktail glass;

- the tangent line signifying the instantaneous speed at a point of a water-height versus time graph, may be developed from the graph of the constant speed of a cylindrical glass with the appropriate width.

In light of the above findings we may speak of a fruitful series of design experiments. Even though the research was exploratory, the findings appear to have significant implications for the way speed is addressed in primary school. Currently the curriculum focuses on average speed, which results in a merely shaky understanding. The research, however, indicates that fifth grade students have an intuitive notion of instantaneous speed, which can be expanded. This result suggests that it might be advisable to shift the focus from average speed to instantaneous speed in primary school. Deepening the students' understanding of instantaneous speed should then be complemented with a more thorough treatment of constant speed than is now common. This challenge to the current school curriculum underscores the power of exploratory design research that adheres to the methodological prescription of "letting the object speak" (Smaling 1992). In concluding this section, we may further point to the central role of the RME instructional design heuristics in supporting the design work of the researchers.

2.3.2 *Structuring Learning Trajectories—An Example Project on Exponential Growth for Grade 10*

Exponential growth is one of the most complex topics in Grade 10, as students must connect all their knowledge about various models and representations for functional relationships (Confrey and Smith 1995). In this section, a design research project is sketched, which focused on a fine-grained analysis of which aspects are to be learned on exponential growth and how they can be structured into a learning trajectory (foundations of the project are given by Hußmann and Prediger 2016; further elaborated by Thiel-Schneider 2018).

The design followed the general design heuristic of emergent modelling (Sect. 2.1), starting from everyday experiences in meaningful contexts and developing the formal connections and their characteristics by horizontal and vertical mathematization (Gravemeijer 1999). However, little was known on how to structure the various aspects in the teaching-learning arrangement.

In the following, the general research framework and selected results from the project are presented. This example shows that although they refer to similar backgrounds, different concrete versions of topic-specific process-focused design research are possible and develop slightly different terminologies. We decided to keep the terminology that was used in the context of the research example under consideration.

Research Framework

The project was conducted within the FUNKEN-model of topic-specific Didactical Design Research that was developed within the FUNKEN-graduate-school for

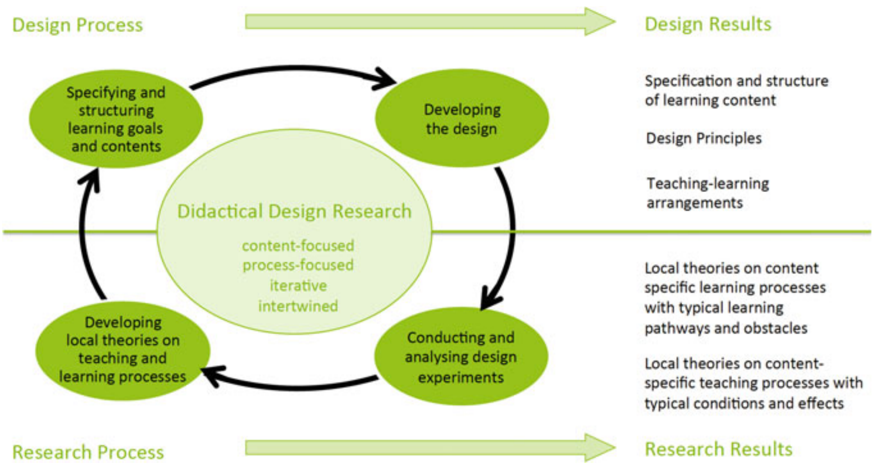

Fig. 2.5 Four working areas for topic-specific Didactical Design Research in the FUNKEN-model (Prediger et al. 2012; translated by Prediger and Zwetzschler 2013)

more than twenty design research projects in nine subject matter didactic disciplines (Prediger et al. 2012; Prediger and Zwetzschler 2013; following main ideas of Gravemeijer and Cobb 2006).

Like other design research approaches, the model relies on the *iterative* interplay between designing teaching-learning arrangements, conducting design experiments, and empirically analyzing the teaching-learning processes. Specific to the FUNKEN-model are the four working areas shown in Fig. 2.5, in which the three typical working areas (developing the design, conducting and analyzing the design experiments, and developing local theories) are enhanced by a forth one, *specifying and structuring the learning content,* which is often too implicit and which is a core focus of endeavour for each topic-specific project (see Sect. 2.4). As the framework is *content-focused* on topic-specific aspects, the specification and structuring of learning goals and content are treated as one of four *intertwined* working areas.[1]

Expected *research outcomes* consist of empirical insights and contributions to local theories on learning and teaching processes of the treated topic (here mainly for identifying fruitful starting points and explaining typical misconceptions on students' learning pathways concerning the topic of exponential growth) and hypotheses on necessary connections to be drawn in the learning trajectories. *Expected design outcomes* comprise the specified and structured mathematical

[1]The FUNKEN-model chooses the term working area instead of phases in order to highlight the iterative interplay and highly intertwined character of these areas, which cannot always be separated chronologically.

content (here exponential growth), the topic-specifically refined design principles (here for emergent modeling) and the prototypic teaching-learning arrangement.[2]

Leading Questions for the Specifying and Structuring the Learning Content

The working area of specifying and structuring the learning content has proven to be crucial also for many other Design Research projects within the FUNKEN graduate school (Prediger et al. 2012; Hußmann and Prediger 2016). The working area developed into a specific way to establish contributions to theory: although there is no schematic recipe for conducting it, the recurring questions listed in Table 2.1 can provide some guidance.

The prospective elaboration can start on the *formal level:* the concepts and theorems relevant for a topic are specified and the logical connections between them are explored for determining logically possible trajectories through the network of definitions and theorems. However, the didactical decision about a suitable instructional sequence of concepts and theorems cannot be determined purely on the formal level. Instead, the priority is on the *semantic level* on which the big ideas and basic mental models are identified and structured into a hypothetical learning trajectory. This work on the semantic level is informed by design principles such as horizontal and vertical mathematization. The semantic level is elaborated iteratively together with the *concrete level* (in which the sequence is realized in a teaching learning arrangement based on suitable contexts and instructional activities) and the *empirical level* which draws upon the design experiments and their retrospective analysis. Hence, the prospective elaboration encompasses the formal, semantic and concrete levels, the retrospective analysis encompasses the empirical, concrete and semantic levels, each tightly interwoven and oriented to the questions in Table 2.1.

Specifying and Structuring on the Formal, Semantic, and Concrete Levels for the Topic of Exponential Growth

For the design research study on exponential growth explored by Hußmann and Prediger (2016), Thiel-Schneider (2018) for Grade 10, characterizing the topic as exponential growth rather than as exponential functions is already a decision on the *semantic level*: Exponential functions should be treated in modelling contexts, following the *big idea* of functions as describing and predicting processes and changes (cf. Schweiger 2006). Hence, the basic mental models contain those of functional relationships, the correspondence model (each x corresponds to a y, e.g., for each year, the stock of a measure can be determined) and the covariation model (asking for the variation in y when x varies, e.g., the change of the measure per month) (Confrey and Smith 1995).

[2]In the FUNKEN-model, the terminology was slightly adapted: as the structured learning content is not always a unidimensional hypothetical trajectory, this term is chosen to distinguish the pure structure of the content from its realization by tasks and support means in the teaching-learning arrangement. The local theory is not called local instruction theory but local theory on teaching and learning processes in order to avoid the misunderstanding that instruction is restricted to teaching.

Table 2.1 Typical questions on four levels for specifying and structuring the content (without assuming completeness) (Hußmann and Prediger 2016)

	Specifying the content (selecting aspects and their backgrounds)	Structuring the content (relating and sequencing aspects, including connecting points for long-term processes)
Formal level	• Which concepts and theorems have to be acquired? • Which procedures have to be acquired, and how are they justified formally?	• How can the concepts, theorems, justifications and procedures be structured in logical trajectories? • Which connections are crucial, which are contingent? • How can the network between concepts, theorems, justifications and procedures be elaborated?
Semantic level	• What are the underlying big ideas behind the concepts, theorems and procedures? • Which basic mental models and (graphical, verbal, numerical and algebraic) representations are crucial for constructing meaning?	• How do the underlying ideas and meanings relate to each other and to earlier and later learning content? • How can the meanings be successively constructed by horizontal mathematization in the intended learning trajectories? • Which trajectories of vertical mathematization have to be elicited in order to initiate the invention / discovery of core ideas, concepts, theorems and procedures? • How can the intended learning trajectories be sequenced with respect to the logical structure?
Concrete level	• Which core questions and core ideas can guide the development of the concepts, theorems, and procedures? • In which context situations and by which problems can the core questions and ideas be treated exemplarily for re-inventing the content?	• How can the meanings be successively constructed in situations in the intended learning trajectories? • How can the intended learning trajectories be sequenced with respect to the problem structure? • Which trajectories of horizontal mathematization have to be elicited in order to initiate the invention / discovery of core ideas, concepts, theorems and procedures?
Empirical level	• Which typical individual perspectives of students (conceptions, ideas, knowledge, …) can be expected? • How do they relate to the intended perspectives (resources vs. obstacles)? • What are origins of typical obstacles or idiosyncratic conceptions?	• Which critical points in students' learning pathways are most crucial (obstacles, turning points,…)? • Which typical preconceptions or previous knowledge can serve as fruitful starting points? • How can the intended learning trajectory be re-sequenced with respect to students' starting points and obstacles?

Prospective Elaboration (arrow downward, formal to concrete level); Retrospective Analysis (arrow upward, empirical to semantic level)

For the realization on the *concrete level,* the bank context of assets and compound interests was chosen, as this context carries main features of exponential growth, is realizable for students and relevant for their later lives.

The complexity of the topic of exponential growth consists in coordinating the different characterizations (here restricted to discrete functions). The function $f: \mathbb{N} \rightarrow \mathbb{R}^+$ is exponential, if it can be expressed in the form

(C1)$f(x+1) = \lambda \cdot f(x)$	(constantly multiplicative growth)
($C2_p$) $f(x+1) = f(x) + p \cdot f(x)$	(constantly proportionally additive growth)
(C3) $f(x) = a \cdot b^x$	(direct determination)

Although on the *formal level*, these characterizations are equivalent and can be easily transformed into each other, they bear huge differences on the semantic level (Confrey and Smith 1995; Thompson 2011): students connect (C1) and (C2) mainly to the covariance model as they characterize the pattern of growth, and they connect (C3) mainly to the correspondence model as the formula can be used for determining *f*(*x*) for a value of *x*. The bank context of interests resonates with (C2), the growth by constantly adding the same proportion (percentage) each year. Deriving (C3) from (C2) requires attention to the correspondence model via (C1) as it builds upon repeated multiplication.

Based on this roughly sketched prospective elaboration, a hypothetical learning trajectory was composed in which students can discover the characterizing features while exploring the growth of assets. The horizontal mathematization was supported by tables as major representation, the vertical mathematization was triggered by prompts to schematize the identified recursive pattern into an explicit formula in order to determine assets after 30 years.

Specifying and Structuring Exponential Growth Iteratively on the Empirical, Concrete, and Semantic Levels

The iterative design experiment cycles with tenth graders were conducted along the developed hypothetical learning trajectory and retrospective analysis on the empirical, concrete, and semantic levels.

Students' knowledge of percentages proved to be a suitable starting point for their learning pathways. In each cycle, the activities were optimized so that more students could discover the main aspects and connect them to each other.

For the *empirical* contribution to structuring the learning content, one empirical finding was most influential (Hußmann and Prediger 2016; other aspects presented by Thiel-Schneider 2018). Although characterization C1 and C2 are easily transformable to each other on the formal level, many students showed a compartmentalized understanding of different characterizations, hence an obstacle on the *semantic level*: For many students, C1 was activated only for integral exponents, completely separate from C2 which was used only for decimal exponents. This compartmentalization produced mistakes such as confusing the growth factor 1.02 (corresponding to 2%) with the constantly doubling growth (corresponding to 200%) (Hußmann and Prediger 2016).

Thus, the restructuring of the learning trajectory had to take more intensively into account the transition between the two characterizations. The iterative refinement of the learning trajectory focused on this transition and how it could be

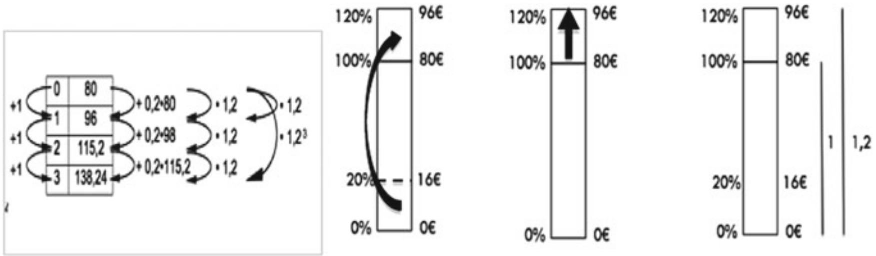

Fig. 2.6 Transition between additive and multiplicative perspective on constant growth in tables and in the graphical representation: In the percent bar, adding 20% is scaling up by 1.2

enhanced by relating different representations (Fig. 2.6). For this purpose, the percent bar had to be included in order to support the transition from constantly proportionally additive growth to constantly multiplicative growth, not only on the formal, but also on the semantic level: In the percent bars, adding 20% can be made visible to be semantically equivalent to scaling by 1.2 (Fig. 2.7), as adding 20% corresponds to scaling to 120%, i.e., scaling by 1.2. For several years, this leads to repeated multiplication with a cumulative factor $(1 + p)^n$ for n years.

Once the graphical representation was introduced, the learning trajectory could be reorganized so that multiplicative and additive perspectives were first adopted separately and then deliberately connected. In a further cycle, it was decided to treat integer growth factors only after the connection of both perspectives.

Although this limited insight into the project cannot account for all findings on students' learning pathways (see Thiel-Schneider 2018 for more details), Fig. 2.7 shows how typical outcomes may appear: By the iterative interplay of all four working areas and levels, the hypothetical learning trajectory (with all the corresponding activities) was enriched and consolidated.

As often appears, the learning trajectory is not a unidimensional one, but takes the character of a multi-facetted landscape, showing the characterizations, representations, core ideas and models to work on in each step. Although there is not the space to explain the details of the compressed, non-self-explanatory Fig. 2.7, it can give an impression of the *kinds* of results. This landscape is a major design outcome, but also a substantially refined analytical tool as it allows the researcher to map students' learning pathways as navigations within and around the structure.

One way of realizing the learning trajectory in a teaching-learning arrangement with all activities, tasks and representations was elaborated into a textbook chapter (Thiel-Schneider and Hußmann 2017), but of course, other realizations are also possible. On the theoretical level, the investigation of learning pathways contributed to the problems of compartmentalization of thinking and the necessity of building bridges.

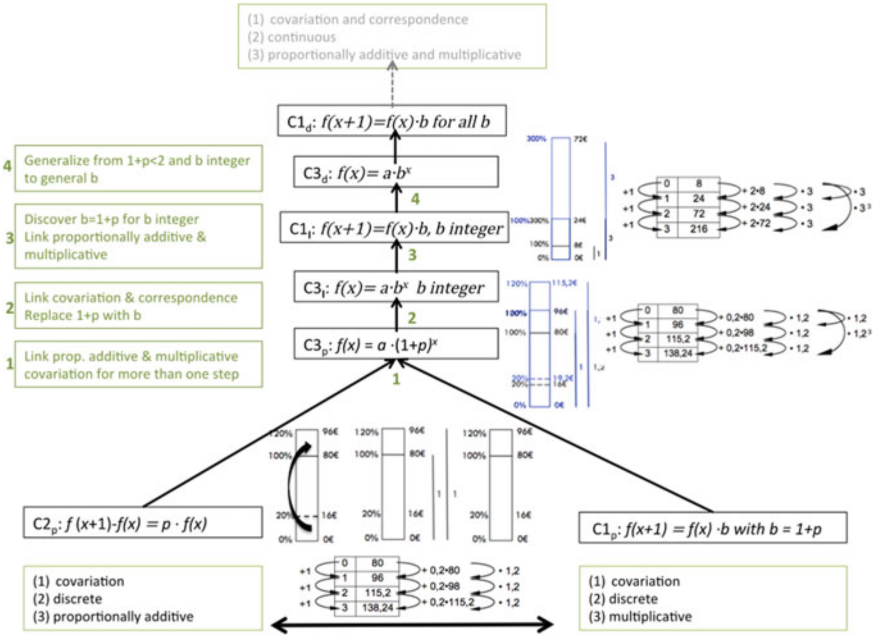

Fig. 2.7 Revised intended learning trajectory for exponential growth (Hußmann and Prediger 2016; more elaborated by Thiel-Schneider 2018)

2.4 Looking Back

The sketched examples of design research Ph.D.-projects in Sect. 2.2 portray what design research may look like, and what is involved in topic-specific design research. In connection with this we may also mention the highly interesting book by Bakker (2018) which offers further insightful advice, especially for young researchers.

2.4.1 *When Is Topic-Specific Design Research a Suitable Methodology?*

Design research is not a panacea for all sorts of research questions. For many educational challenges, other research approaches are better suited. In the following we briefly sketch a series of considerations, using the features of design experiments as described by Cobb et al. (2003) as a framework of reference.

If the aim, for instance, is not to change classroom practices (which is the core of the *interventionist* characteristic), *naturalistic research* approaches such as observation studies of assessment studies might be more suitable for simply analyzing the status quo and background issues.

If the aim is to solve concrete problems of practitioners, but not necessarily to contribute to generating theory (which is the core of the characteristic *theory generative*), research approaches with less methodological rigor such as *action research* might be more suitable. Then the research can be better tailored to the concerns of the practitioners. In contrast, we may note that the practicality of design research that aims at inquiry oriented mathematics may be limited as the goals, the classroom culture, and conceptions of learning that characterize such design experiments, often differ substantially from everyday practice in many mathematics classrooms (Cobb and Jackson 2015).

If the aim is just to explore an existing design and there is no intention of creating conditions for generating and testing theories (*prospective and reflective*), the exploration runs the risk of being atheoretical (but can still be personally interesting for learning about specific designs).

If there is no time for a series of trials and adaptations (*iterative*), it might still make sense to frame the teaching experiment as a first step in a more encompassing design research project, which implies that the teaching experiment should be analyzed as such. Mark, however, that sound design research requires further cycles.

If the aim is to validate a narrow and very clear hypothesis, a *randomized controlled trial* with valid measures for the intended learning gains might be more suitable. Mark, however, that the applicability of the findings in arbitrary classrooms may be limited (Gravemeijer 2016).

If the aim is to validate or refute 'grand theories', a *randomized controlled trial* might again be more fitting. However, the feasibility of judging grand theories in experimental designs might be overrated. Instead, design research aims for more humble, topic-specific, theories that have practical implications (*pragmatic roots and humble theories*).

2.4.2 Meeting Major Methodological Concerns

Critique on the lack of methodological sophistication of design research focuses on issues of generalizability, applicability in everyday practice, and a lack of standardization of methodological procedures. Even though we may argue that methodological approaches must vary because the design researchers' aims vary, there are of course various considerations that have to be taken into account in many variations of design research (see also Bakker 2018).

Background Theories and Assumptions

One of the critiques of design research is that the research question often takes the form of a how-to question, e.g., 'How to shape instruction on topic X?' For many scholars, such a research question is inadequate, because almost any answer would suffice. However, there is always an educational philosophy and a theoretical background against which such a question is posed. In mathematics education, part of the educational philosophy is often that the students should learn with understanding. Additionally, RME requires that the students experience mathematics as an activity, and learn by reinventing mathematics. Such starting points offer the boundary conditions within which the 'how-question' is to be answered. Background theories, such as socio-constructivism or cognitive theory, also significantly influence both the design and the way data are interpreted. The former implies that design researchers should explicate their educational philosophy and their background theories. In a more general sense, it may be argued that researchers have to be clear about their goals, their theoretical stance, and their analysis. The presented example projects show how a concrete project can be embedded in a broader framework, which helps to make the basic assumptions explicit.

Interpretive Framework

An important aspect of the methodological control of the empirical working area concerns the translation of observations of classroom events into scientific data. To make this translation, an interpretive framework is needed. An example of such an interpretative framework is the so-called emergent perspective of Yackel and Cobb (1996), which encompasses norms, practices and beliefs, or Vergnaud's (1996) theory of conceptual fields, which encompasses individuals' concepts-in-action and their relation to the concepts in view (applied, e.g., by Prediger and Zwetzschler 2013). The need for such a framework may be elucidated by observing that it makes a huge difference whether student utterance are to be viewed as a result of the students' own thinking, or as a result of the students efforts to imitate what the teacher has shown. Similarly, RME theory can function as an interpretative framework for interpreting student activity in light of the intended reinvention process.

Argumentative Grammar

Another critique is that design research lacks an argumentative grammar, which offers schemes of argumentation that link data to analysis, and to final claims and assertions (Kelly 2004). In response to Kelly's (2004) call for an argumentative grammar, Cobb et al. (2015) proposed the employment of the following requirements:

- demonstrate that the participants would not have developed particular forms of reasoning but for their participation in the design experiment;
- document how each successive form of reasoning emerged as a reorganization of prior forms of reasoning;
- specify the aspects of the learning ecology that were necessary, rather than contingent, in supporting the emergence of these successive forms of reasoning.

These three components closely relate to our conception of a local instruction theory (and are further explored by Bakker 2018). As the nature of design research is to explore innovative local instruction theories, the first requirement of the argumentative grammar is usually catered for. The second requirement may be linked to the fact that the local instruction theory is meant to function as a framework of reference for teachers, which requires that teachers have to be able to adapt the theory to their situation. We may argue that this is possible only if teachers who want to use the local instruction theory understand how successive forms of students' reasoning emerge as a reorganization of prior forms of reasoning along their learning pathways. The third requirement touches upon the conception of a local instruction theory as encompassing theories about a possible learning process, and theories about possible means of supporting that learning process. However, it asks for a broader description, which also incorporates the specificities of the classroom and what occurred in the classroom during the teaching experiment, and how these aspects influenced the emergence of the successive forms of reasoning.

A Holistic Approach

Most important for us seems to be those considerations that refer to the interplay of experiment and the process of theorizing. Ecological validity requires that the applied theories and the resulting theoretical contributions have to take into account the complexity of classrooms. This aspect requires a different approach than the reductionist approach of the sciences in which phenomena are disassembled in individual variables whose interdependencies can be researched systematically—especially by testing hypotheses.

In this respect, we may refer to Gould (2004) who depicts a complementary way of knowing; the more holistic approach of the humanities, in which, in his view, *concilience* plays a large part: "the validation of a theory by the 'jumping together' of otherwise disparate facts into a unitary explanation" (p. 192). The underlying idea of grasping how things work resonates with Maxwell's (2004) *process-oriented conception of causal explanation*, "that sees causality as fundamentally referring to the actual causal mechanisms and processes that are involved in particular events and situations" (p. 4). We may translate this conception into the recommendation to researchers to search for the underlying mechanisms, and a holistic view that unites seemingly disparate facts.

Summing up, design research provides a research methodology for all who want to combine aims of improving teaching with generating theories which can underpin the teaching. Although the research process can never be easily schematized, procedures and structures have been developed that support the challenging and creative parts of topic-specific design research, also for novices.

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