Chapter 1
Argumentation Analysis for Early Career Researchers

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Abstract Proving processes in mathematics classrooms follow their own peculiar rationale, which raises the question of how to reconstruct and analyse the complex argumentative structures that arise in classroom discussion. In this chapter we describe a method of analysis of argumentation processes in the mathematics class, following the structure of our workshop at the ICME-13 *Early Career Researcher Day*. The method builds on Toulmin's theory of argumentation (Toulmin 1958) and allows the description of both global argumentation structures and local argumentations. A three stage process is followed: reconstructing the sequencing and meaning of classroom talk; analysing local argumentations and global argumentation structures; and finally comparing these argumentation structures and revealing their rationale. The second stage involves two moves, first analysing local arguments on the basis of Toulmin's functional model of argumentation, and second analysing the global argumentative structure of the proving process. We provide an example of the use of the method to analyse a transcript from a mathematics classroom.

Keywords Argumentation · Argumentation processes · Argumentation structures · Proof · Proving · Mathematics classrooms · Model of argumentation

1.1 Toulmin's Functional Model of Argumentation

We make use of Toulmin's (1958) functional model of argumentation which has the important characteristic that it was developed to reconstruct arguments in different fields, such as law or medicine. As Toulmin (1958) investigates the functional

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structure of rational arguments in general, he asks "What, then, is involved in establishing conclusions by the production of arguments?" (p. 97). Toulmin's first answer is that facts (data) might be cited to support the conclusion. He illustrates this by the following example. If we assert that 'Harry's hair is not black', we might ground this on "our personal knowledge that it is in fact red" (p. 97). We produce a datum that we consider as an evident fact to justify our assertion (conclusion). If this is accepted, this very simple step, datum—conclusion, can represent a rational argument.

But this step, its nature and justification, can be challenged, actually or potentially, and therefore it is often explicitly justified. Instead of additional information, an explanation of a more general style, by rules, principles or inference-licenses has to be formulated (p. 98). Toulmin's second answer addresses this type of challenge. A 'warrant' might be given to establish the "bearing on the conclusion of the data already produced" (p. 98). These warrants "act as bridges, and authorize the sort of step to which our particular argument commits us" (p. 98). In the example above the implicit warrant of the argument is 'If anything is red, it will not also be black." (p. 98). While Toulmin acknowledges that the distinction between data and warrants may not always be clear, their functions are distinct, "in one situation to convey a piece of information, in another to authorise a step in an argument" (p. 99). In fact, the same statement might serve as either datum or warrant or both at once, depending on context (p. 99), but according to Toulmin the distinction between datum, warrant, and the conclusion or claim provides the elements for the "skeleton of a pattern for analyzing arguments" (p. 99, see Fig. 1.1). In the following we use "claim" in cases where data and warrants have not yet been provided, and "conclusion" when they have been.

Toulmin adds several other elements to this skeleton, only one of which is discussed here. Both the datum and the warrant of an argument can be questioned. If a datum requires support, a new argument in which it is the conclusion can be developed. If a warrant is in doubt, a statement Toulmin calls a "backing" can be offered to support it.

Figure 1.2 shows a single step in an argument in terms of Toulmin's model. From the data $c^2 = b^2 - 2ab + a^2 + 2ab$ it concludes that $c^2 = b^2 + a^2$. Such a step is typical in algebraic proofs of the Pythagorean Theorem. The warrant for it consists of several standard principles of arithmetic, and these are normally in an algebraic proof left implicit, but can be reconstructed in a straightforward way. Implicit warrants are marked with dotted outlines.
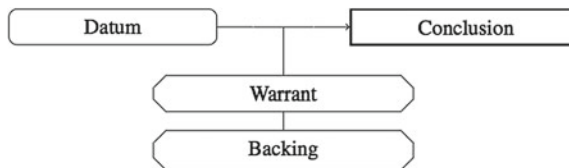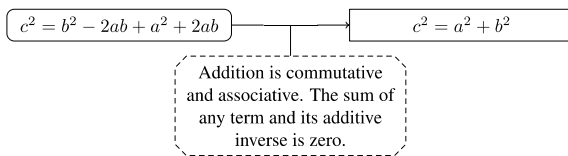

Fig. 1.1 Toulmin Model (In our diagrams Data are enclosed in rectangles with rounded corners, Warrants and Backings in rectangles with angled corners, and Conclusions in plain rectangles.)

Fig. 1.2 Datum, warrant and conclusion for the final step in a proof

$$c^2 = b^2 - 2ab + a^2 + 2ab \qquad\qquad c^2 = a^2 + b^2$$

Addition is commutative and associative. The sum of any term and its additive inverse is zero.
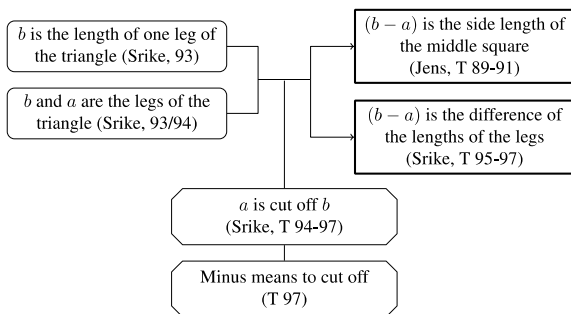
Toulmin states, "The data we cite if a claim is challenged depend on the warrants we are prepared to operate with in that field, and the warrants to which we commit ourselves are implicit in the particular steps from data to claims we are prepared to take and to admit" (p. 100). Therefore careful analyses of the types of warrants (and backings) that are employed explicitly or implicitly in concrete classroom situations, allow us to reconstruct the kinds of mathematical justifications students and teacher together operate on. In particular, the comparison of warrants and backings in different arguments can reveal what sort of argument types are used in proving processes in mathematics classrooms.

For example, in Fig. 1.2, we supply an implicit warrant based on mathematical properties of addition. In a different context the warrant for this argument might have been geometrical, interpreting $2ab$ as the area of a rectangle (or two triangles), or syntactical not interpreting the symbols at all, operating on them purely formally. Any of these types of warrants (and backings) could occur in a classroom and indicate the *field* of justifications in which the students and teacher operate.

Other researchers (e.g., Inglis et al. 2007) make use of other elements in Toulmin's model, including "modal qualifiers" and "rebuttals". Many arguments do not establish their conclusions with complete certainty, and in such arguments we find qualifiers like "probably" and "possibly" as well as rebuttals that identify cases where the conclusion does not hold. Inglis et al. consider the arguments of post-graduate university students in mathematics and find that modal qualifiers play an important role in their mathematical argumentations. In our work in schools, however, we find that the mathematical argumentations produced are often quite different from what advanced mathematics students produce, and as a result we usually do not find it necessary to make use of any elements in the Toulmin model beyond data, conclusions, warrants and backings. We add one element, however, which we call "refutation". A refutation differs from a rebuttal in that a rebuttal is local to a step in an argument and specifies exceptions to the conclusion. A refutation completely negates some part of the argument. In a finished argumentation refuted conclusions would have no place, but as we are concerned with representing the entire argumentation that occurred, it is important for us to include refutations and the arguments they refute, as part of the context of the remainder of the argumentation, even if there is no direct link to be made between the refuted argument and other parts of the argumentation. Aberdein (2006) proposes extending Toulmin's rebuttal element to encompass refutations, but for our purposes we prefer to limit rebuttals to Toulmin's original role, of specifying circumstances where the conclusion does not hold.

It is possible for a step in an argument to make use of several data, and to lead to more than one conclusion. Figure 1.3 shows such a step, in which the two data in

Fig. 1.3 Multiple Data and Conclusions

b is the length of one leg of the triangle (Srike, 93)	$(b - a)$ is the side length of the middle square (Jens, T 89-91)
b and a are the legs of the triangle (Srike, 93/94)	$(b - a)$ is the difference of the lengths of the legs (Srike, T 95-97)

a is cut off b (Srike, T 94-97)

Minus means to cut off (T 97)

combination are used to arrive first at a conclusion about the side length of a square, and later at a conclusion about the difference in the lengths of the legs of a right triangle.

An important way in which we use the Toulmin model that extends it significantly, is our application of it not only to single steps in argumentations, but also as a tool to explore the global structure of an argumentation. In the next section we describe this distinction in more detail.

1.2 Local and Global Arguments

Toulmin (1958) notes "an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one" (p. 94). Toulmin's aim is to explore the fine structure, but in considering classroom argumentations both argumentative forms must be reconstructed. Toulmin's model is useful for reconstructing a step of an argument, which allows us to single out distinct arguments in the proving process (for example as in Figs. 1.2 and 1.3). We call these "argumentation steps" or *local arguments*. But it is also necessary to lay out the structure of the argument as a whole (the anatomical structure), which we call *global argument* or the argumentation "structure" of the proving process.

Between the global argument or the argumentation structure of the entire proving process and the local level of the argumentation steps there is an intermediate level we call an "argumentation stream". An argumentation stream consists of a number of argumentation steps that are connected, and which lead to a final target conclusion. The written proof in the right hand side of Fig. 1.4 provides a simple example. The argument presented on the blackboard, reconstructed as a chain of argumentation steps, is shown in Fig. 1.5 (Additional data and analysis can be found in Knipping 2003). The final conclusion ($c^2 = a^2 + b^2$), a formulation of the Pythagorean Theorem, is the target conclusion of the argument. The argument can be reconstructed as a simple chain of conclusions beginning with a datum "$c^2 = (b - a)^2 + 4rwD$" that has been taken from the drawing on the blackboard. This datum leads to a conclusion: $c^2 = (b - a)^2 + 2ab$, but no warrant is explicitly
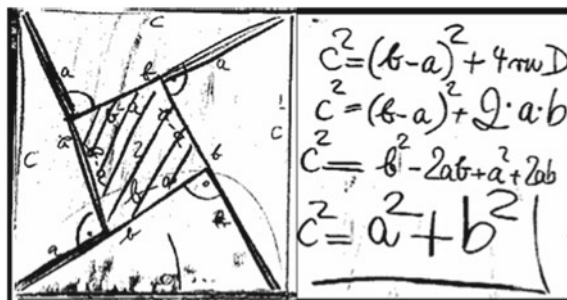
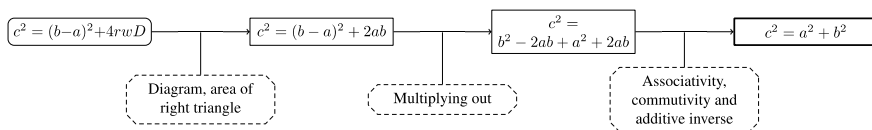**Fig. 1.4** A written proof of the Pythagorean Theorem

Fig. 1.5 Functional reconstruction of the written proof presented in Fig. 1.4 (The target conclusion is outlined with a thicker line than conclusions that are recycled as data for subsequent steps.)

given to support this inference. The information in the diagram (adjacent sides of the right triangle are a and b) and implicit calculations of the area of the four right triangles implicitly support this claim. The next two steps are also based on implicit warrants. In Fig. 1.4 we include reconstructed possible implicit warrants for each step; they are marked by a box with a dashed line. Note that the statement "$c^2 = (b - a)^2 + 2ab$" is not only the conclusion of one step but also the datum of another. Finally the target conclusion: $c^2 = a^2 + b^2$ is established.

This type of argument can be characterised as a chain of statements, each one deduced from the preceding one on logical and mathematical grounds. This has been described by Duval as "Recyclage" (Duval 1995, pp. 246–248) Once a statement has been established as a conclusion it functions as a datum, an established true fact, in the next step. Aberdein (2006) calls this way of combining single steps "Sequential" and he describes four other ways steps could be combined. As we show in the following, our empirical research on classroom argumentation provides examples of Aberdein's ways of combining steps, as well as other ways.

Figure 1.6 shows another argumentation stream, involving verbal argumentation leading up to the written proof shown in Fig. 1.4. It combines the features of multiple data for an argumentation step and the chaining of steps in which the conclusion of one step becomes a datum for the next.

As discussed earlier, the functional model of Toulmin, which is helpful for reconstructing argumentation steps and streams, is not adequate for more complex argumentation structures. Analyzing proving processes in classrooms requires a different model for capturing the global structure of the argumentations developed
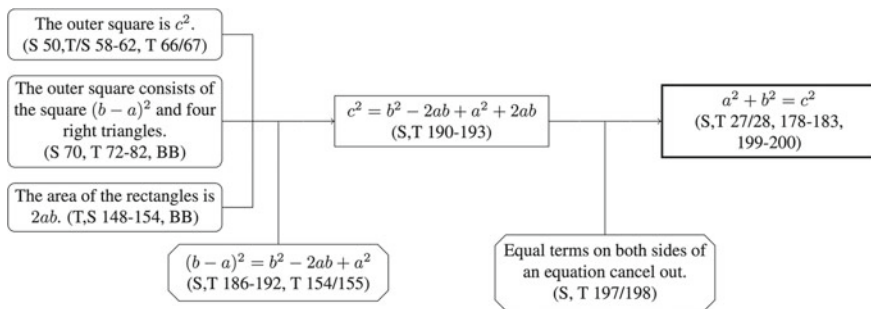
Fig. 1.6 An argumentation stream from verbal argumentation

there. Knipping (2003) developed a schematic representation in order to illustrate the complex argumentation structures of this type of classroom talk.

To address this complexity Knipping developed a schematic representation that allows the description of argumentations at different levels of detail. This approach differs from Aberdein's (2006) as he reduces the complexity of the argumentation by a process of folding that results in a single step that includes all the assumptions (initial data and warrants) of the full argumentation, but which hides the relationships between these assumptions. Knipping's approach also differs from that taken by van Eemeren et al. (1987) who developed two different ways of representing the structures of everyday written argumentations, in that she makes the role of warrants more visible. We illustrate below how Knipping's method makes the global argumentation visible while preserving the relationships in the local steps.

Argumentation streams are combined to make up the global argumentation structure. To represent these structures, it is necessary to lose some information. The function of each element is retained but the details are not. Figure 1.7 shows how the argumentation stream in Fig. 1.6 is reduced to a schematic diagram (shown to the right).

These schematically represented streams are then combined to represent the global argumentation structure, as shown in Fig. 1.8. The shaded area is the argumentation stream shown in Fig. 1.6.
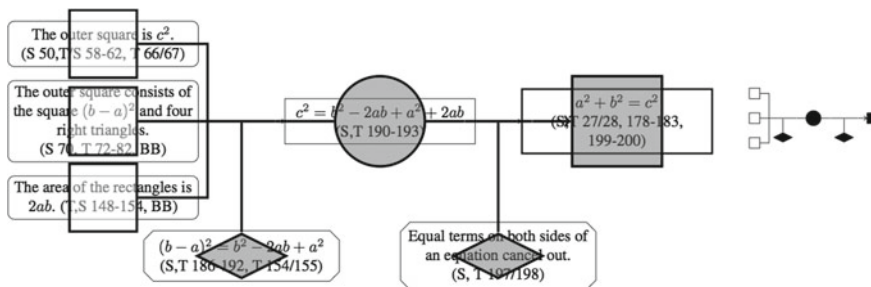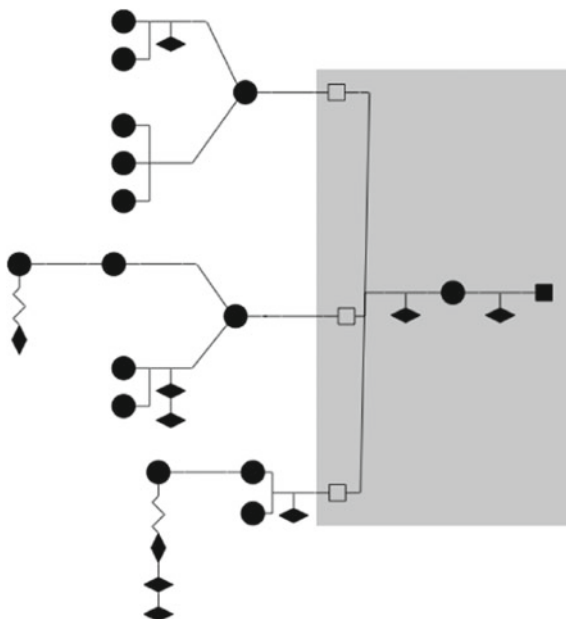

Fig. 1.7 The argumentation stream from Fig. 1.6 reduced to a functional schematic

Fig. 1.8 The argumentation structure of a classroom proving process

1.3 Reconstructing Arguments in Classrooms

To reconstruct arguments in classrooms we follow a three stage process:

1. reconstructing the sequencing and meaning of classroom talk (including identifying episodes and interpreting the transcripts);
2. analysing arguments and argumentation structures (reconstructing steps of local arguments and short sequences of steps which form "streams"; reconstructing the global structure); and
3. comparing local argumentations and comparing global argumentation structures, and revealing their rationale.

We illustrate below each of these stages by discussing episodes of a proving process that occurred in Ms. James' grade 9 (age 14–15 years) classroom in Canada. The class was trying to explain why two diagonals that are perpendicular and bisect each other define a rhombus. The students had discovered and verified empirically that the quadrilateral produced is a rhombus using dynamic geometry software.

1.3.1 Reconstructing the Sequencing and Meaning of Classroom Talk

Reconstructing the sequencing and meaning of classroom talk on proof and proving involves first dividing the proving process into episodes, followed by turn analysis of the transcript, identifying data, conclusions and warrants.

The first step is dividing the proving process into episodes. This means that the general topics emerging in the classroom talk are identified and their sequencing is reconstructed. This allows one to get an overview of the different steps in the argumentation. Proving process in classrooms can occur over long periods of time, from 20 to 40 min or longer. Laying out different episodes of the process helps to make the argumentations in these episodes more accessible to analysis. Once the flow and sequencing of the emerging topics is made visible the reconstruction of the arguments can start. For example, summaries of the episodes in the classroom proving process which we analyse in the following are reconstructed first (see Appendix A, http://www.math.uni-bremen.de/didaktik/ma/knipping/resources_en. html). The full transcript of episodes 1–6 (see Appendix B, http://www.math.uni-bremen.de/didaktik/ma/knipping/resources_en.html) is the basis for further analyses, which are the focus of this paper.

1.3.2 Turn by Turn Analyses

Argumentations in classroom processes are mostly expressed orally and by a group of participants. Generally arguments are produced by several students together, guided by the teacher. As Herbst showed (2002), it is the teacher who mostly takes responsibility for the structure and correctness of the argument, but students contribute to the argument, so there is a division of labour in the class. Argumentations are co-produced; the teacher and the students together produce the overall argument. Their turns are mutually dependent on each other; their public meanings evolve in response to each other. The argument forms in relation to these emerging meanings. So, in order to reconstruct the structure of an argument first the meanings of each individual turn put forward in class have to be reconstructed. As Krummheuer and Brandt state:
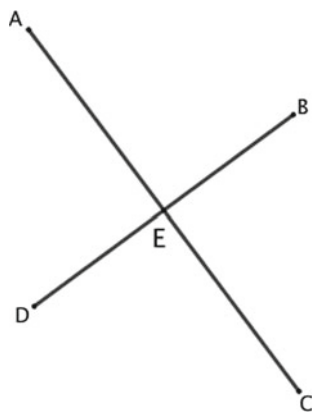
> Expressions do not a priori have a meaning that is shared by all participants, rather they only get this meaning through interaction. In concrete situations of negotiation the participants search for a shared semantic platform. [*Äußerungen besitzen* "a priori *keine von allen Beteiligten geteilte gemeinsame Bedeutung, sondern erhalten diese erst in der Interaktion. In konkreten Situationen des Verhandelns bzw. Aushandelns wird nach einer solchen gemeinsamen semantischen Bedeutungsplattform gesucht"*]. (2001, p. 14, our translation)

Because meanings emerge through interaction, reconstructing meanings necessarily involves some reconstruction of the process by which they emerge. Generally statements of classroom talk are incomplete, ambiguous and marked by deictic[1]

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[1]In linguistics, a deictic term is an expression, for example a pronoun, that gets its meaning from its context. The meaning of "this" depends on what is being pointed to. The meaning of "I" depends on who is speaking. In philosophy the word "indexical" is used to express the same idea.

terms. Deictic terms are replaced as much as possible in the reconstruction of the argumentation. For example, in Episode 1 the following exchange occurs:

Fig. 1.9 The teacher's initial diagram (letters were not included)

Line	
1	T: [I am going to add] to my diagram [see Fig. 1.9] as I go. How do you usually indicate that two things meet at a ninety degree angle?
2	Multiple students: [indecipherable]
3	Teacher: Ok, you put that little square. So I'm going to put that in there [she marks angle AEB with a little square] and that's ninety degrees. Now if that's ninety, due to supplementary angles —
4	Multiple students: they're all ninety
5	Teacher: we know that they're all ninety. Good, thank you

Words like "that", "they" refer to objects that everyone involved knows the reference for, but for us in reconstructing the argument these references must be made explicit, so "that" in "if that's ninety" would be replaced by "angle ABC" or some other precise reference.

This exchange also includes an argument in which a statement is formulated as a question. The teacher's question "How do you usually indicate that two things meet at a ninety degree angle?" is answered by the students in a way the teacher reacts to by marking the ninety degree angle with a little square. We find that such exchanges occur often in classrooms. In the reconstruction of the argument we reconstruct such questions as statements, so that their grammatical form is no longer visible, but their function in the argument is clearer. For example, this exchange is reconstructed as the two statements "To indicate an angle is a 90° angle it is marked with a little square. (T-1-3)" and "Angle AEB is marked with a little square (T-3)".

Because the focus of the analysis is the argumentative structure of the classroom talk the reconstruction of the meanings of statements in the turn by turn analysis must consider the argumentative function of the statements: datum, conclusion,

warrant, etc. These functions are identified in the next step of analysis. Utterances are primarily reconstructed according to their function within the collectively emerging argumentation, not with respect to subjective intentions and meanings as in interaction analyses. The reconstructed turns from the above exchange, and the functions assigned to them, are shown here:

	Actual utterances	Reconstructions	
1	T: [I am going to add] to my diagram [see Fig. 1.9] as I go. **How do you usually indicate that two things meet at a ninety degree angle?**	The diagonals are perpendicular (T-1)	D
2	Multiple students: [indecipherable]		
3	T: Ok, **you put that little square.** So **I'm going to put that in there** [she marks angle AEB with a little square] and **that's ninety degrees**. Now if **that's ninety**, **due to supplementary angles** —	To indicate an angle is a 90° angle it is marked with a little square. (T-1-3) Angle AEB is marked with a little square. (T-3) Angle AEB is a 90° angle. (T-3) "due to supplementary angles" (T-3)	W C D W
4	MS: **they're all ninety**	All angles around E are 90° angles (MS-4)	C
5	T: we know that **they're all ninety.** Good, thank you	All angles around E are 90° angles. (T-5)	C

In Knipping's (2003, 2004) analyses of classroom processes focusing first on conclusions turned out to be an effective step in reconstructing argumentations. It is helpful to begin by identifying what statement the participants are trying to justify, the claim that gains the status of a conclusion by their argument. So, before actually analysing the complete argument we look for conclusions and claims. For example, in the short excerpt above, the statement "they're all ninety" made by the students in line 4 and repeated by the teacher in line 5, is a conclusion. Having identified this conclusion we can now look for the data and warrant leading to it.

In line 3 the teacher says "Now if that's ninety, due to supplementary angles" and then waits for the students to answer. Here we can identify a datum "that's ninety" and a warrant "due to supplementary angles". This can be diagrammed as shown in Fig. 1.10.

It is interesting that in this case both the warrant and the datum are given explicitly. Typically, reconstructed arguments in secondary level classroom proving processes are often incomplete, as was the case with the written proof in Fig. 1.4.
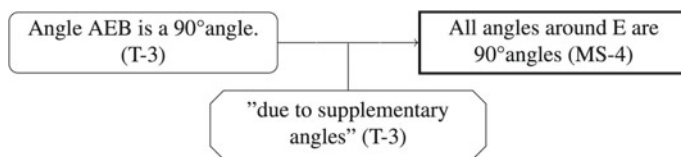

Fig. 1.10 Diagram of argumentation step in transcript lines 3–5

The warrant is often not given, but it can usually be assumed or taken as implicit, as the transition from datum to conclusion must be justified somehow. In our argumentation analyses we usually do not add implicit warrants, but leave them implicit in the reconstruction. This is meant to illustrate the implicitness of both the argumentation and warrant. This allows the comparison of the degree of explicitness in different argumentation structures. In cases where we do want to talk about an implicit warrant we place it in a dashed box (as in Fig. 1.5).

Readers may find it interesting to work through Appendix B at this point and attempt to identify in each turn statements that might be data, conclusions, and warrants.

1.3.3 Analysing Arguments and Argumentation Structures

In the following we describe in detail the moves in the reconstruction of local arguments, then of intermediate argumentation streams, and then of global argumentation structures. This method for reconstructing arguments, argumentation streams and argumentation structures was developed by Knipping (2003, 2008).
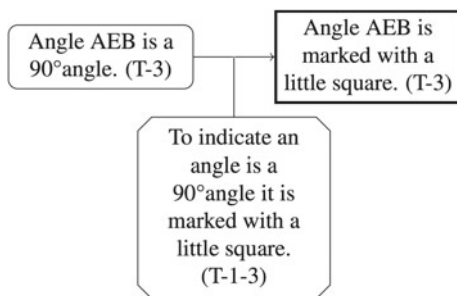
1.3.3.1 Functional Reconstruction of Local Arguments

Having identified statements that might be data, conclusions, and warrants in each turn, the next move is to group these statements together into argumentation steps. As noted above, warrants are often implicit, and it is rare to find backings. We occasionally come across arguments where the datum has been left implicit. In such cases the warrant is present, however, so in the reconstruction the datum is left implicit, and the argument consists of the warrant and the conclusion (see Knipping 2003).

Analysing students' and teachers' utterances in the class according to this functional model allows us to reconstruct argumentations evolving in the classroom talk. In our analyses only utterances that are publicly (in the class) accepted or constituted as a statement are taken into account. The teacher's attention to some utterances and deferment of others can play a major role in this. This is not surprising given Herbst's (2002) findings that in general only the teacher takes responsibility for the truth of statements. Where alternative argumentations or attempts at an argument are publicly acknowledged, they are also considered in our analyses, although the focus is on the main structure of the argumentation.

Episode 1 provides an example where we do not consider some arguments part of the main stream of the argumentation. In Fig. 1.11 the argument relates to a convention for showing mathematical properties in diagrams. In contrast, the argument in Fig. 1.10 includes statements about the properties themselves. Only the argument in Fig. 1.10 is later connected into the main structure of the

Fig. 1.11 Diagram of an argumentation step related to mathematical conventions in Episode 1

argumentation, and even in cases where the same statement occurs in arguments of both kinds (such as "Angle AEB is a 90° angle. (T-3)"), we do not connect them.

Above we indicate how to reconstruct local arguments or argumentation steps, as many researchers in mathematics education have done (see for example Inglis et al. 2007; Krummheuer and Brandt 2001; Fukawa-Connelly 2014). As the literature on reconstructing local arguments is already extensive and easily accessible we do not discuss this further here, but leave the interested reader to look up some of this literature. The reader may also wish to do a functional reconstruction of the local arguments in the transcript given in the appendix for this paper by themselves and we encourage readers to discuss their reconstructions with other researchers or more experienced colleagues. In the next section we move on to describing the process of reconstructing intermediate argumentation streams.

1.3.3.2 Functional Reconstruction of Intermediate Argumentation Streams

Having reconstructed individual steps in the argumentation, the next move is to link these together into streams, but looking for connections between them. Some connections may already have been noticed, for example when the same statement has been identified as both conclusion and as data, because it plays these two roles in two steps of the argumentation. However, it often occurs that these connections are hidden, because two equivalent statements made at two different times play these two roles, and until the equivalent statements are identified the connection remains hidden. For example, in Episode 3, lines 28 and 29, the teacher says "that is the same length as that, is the same length as that, is the same length as that" referring to the perimeter of the figure, and later "AB is equal to BC, is equal to CD, is equal AD" referring to the same segments by name. If the first statement is identified as data leading to the conclusion "Rhombus" and the second is identified as a conclusion of an argument based on congruent triangles, but they are not recognised as being equivalent, then the connection between two steps in the argument may not be made.
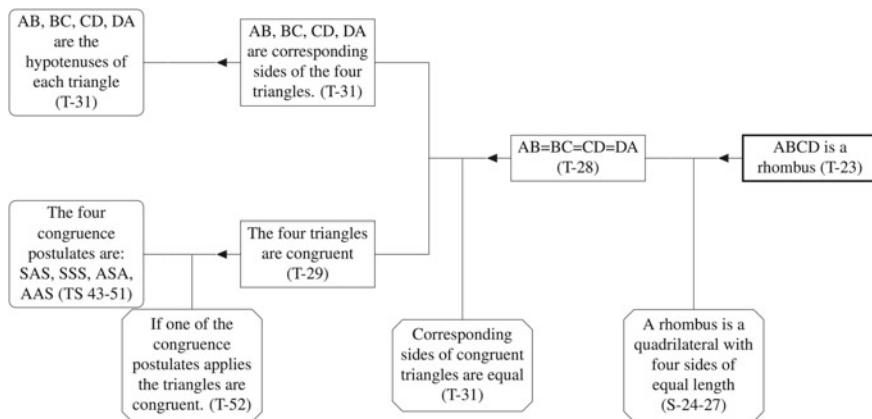
Fig. 1.12 Abductive argumentation stream from episodes 3-5

28	T: four sides of equal length. If I can prove that is the same length as that, is the same length as that, is the same length as that [refers to the perimeter of figure]. If I can prove that, I'm done. Rhombus
29	T: So that's your goal. To prove that and if you want, you can throw in some letters, prove that AB is equal to BC, is equal to CD, is equal AD. Now every single time we did one of these proofs we were looking for congruent triangles. Because if those triangles in that shape are congruent, what would we know about their sides?

These lines also provide an example of a difficulty that emerges in argumentations generally, that of deciding the level of detail that is needed. The statement "AB is equal to BC, is equal to CD, is equal AD" can be interpreted as three statements: "AB is equal to BC", "BC is equal to CD", "CD is equal to AD". Whether it makes sense to do so depends on how this statement is used elsewhere. In our reconstruction of this proving process, we tried fitting both interpretations of the statement into argumentation streams, before deciding that it was used as a single statement in the argumentation.

Figure 1.12 shows the reconstructed argumentation stream that occurs in episodes 3–5. This stream is interesting because the statements that occur first, on the left, were made last, and vice versa. In classroom proving processes such an abductive stream sometimes occurs, where the argument goes backwards from the intended conclusions to the data needed to deduce that conclusion. For a general description of abductive reasoning see Reid (2018) and for references to other representations of abduction using the Toulmin scheme see Papadaki et al. (2019).

In the classroom contexts presented here there are linguistic markers of what is going on. Abduction is indicated by statements that begin for example with "If I can" (see transcript in Appendix B). The phrase "If I can" suggests that the speaker is thinking about establishing a statement that has not yet been established, but which could, if established act as data in the argument. Identifying abductions in argumentation is not simple. In prior analyses of this stream (for example in Reid
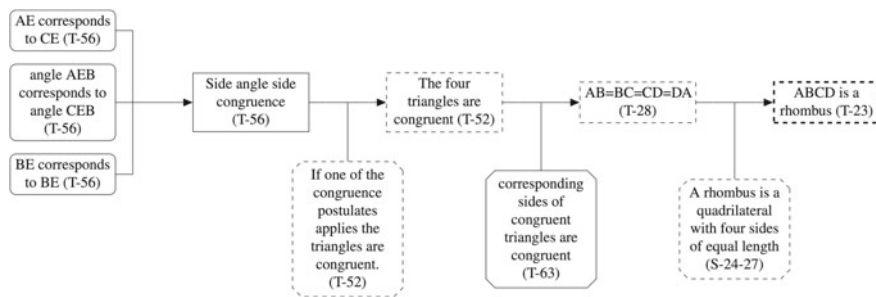
Fig. 1.13 Deductive argumentation stream from episode 6

and Knipping 2010) we conflated this abduction with the deductive stream that follows it, misled by the occurrences of the same statements in both streams. Deductive streams are much easier to understand and to recognise in classroom contexts. In a classroom proving process, where the main flow of the argument is deductive, it makes sense to diagram abductions as if they were deductions, but marking them with arrows showing the flow of the argument in the opposite direction.

Figure 1.13 shows the reconstruction of the deductive argumentation stream that corresponds to the abductive stream in Fig. 1.12. The starting data in Fig. 1.13, from transcript line 56, corresponds to the data hypothesised at the chronological end of the abduction, from transcript lines 43–51. In the deductive stream some steps are left implicit (marked with dashed outlines) as they have already been stated during the abductive stream.

In this section we demonstrate the reconstruction of argumentation streams. Looking for connections between local arguments, i.e., individual steps in the argumentation, reveals connected streams of argumentation. These can be of different types, as we show above. Abduction and deduction are two examples which illustrate that the functional reconstruction of arguments incorporates far more than just the restoration of logically sound deductive arguments. This illustrates the utility of Toulmin's functional model for reconstructing arguments. Because he set out to produce a model of argument that was flexible enough to capture many different types of argument, his model can be used in many different situations.

It can be difficult to observe and represent deductive arguments arising in students' talk or classroom discourses. Revealing and portraying abductions is generally considerably harder as we note above. Both lead to an understanding of different *schemes* of argumentation (as discussed by Godden and Walton 2007; Knipping and Reid 2013; Metaxas et al. 2009, 2016; Metaxas 2015; Aberdein forthcoming; Walton and Reed 2005). In the next section we look at how reconstructing the global argumentation structures of proving processes in classrooms can also lead to understanding different types of argumentation schemes.

1.3.3.3 Reconstructing the Argumentation Structure of Proving Processes in Class

The argumentation streams shown in Figs. 1.12 and 1.13 and those containing the steps shown in Figs. 1.10 and 1.11, as well as those for episodes 7–10, can be combined into a global argumentation structure, shown in Fig. 1.14. Small shapes (rectangles, circles and diamonds) correspond to the shapes used in representing argumentation streams so that corresponding statements can be identified, and more importantly, so that the function of each statement in the argumentation is clear. The colouring of the shapes can indicate statements with special functions, for example, here white rectangles represent initial data and in Fig. 1.8 they represent target conclusions of intermediate stages within the global argumentation.

The two regions marked in grey are the argumentation streams shown in Figs. 1.12 and 1.13. The wavy line connecting the two grey regions shows two equivalent statements discussed above, from transcript line 56 and transcript lines 43–51, where the abduction triggered the deduction.

As is usual for classroom proving processes, this argumentation structure is complex. Three parallel argumentation streams lead to the conclusion "AB = BC = CD = DA" from which the final conclusion is drawn. A fourth argument, proposed by one of the students, leads directly to the conclusion "ABCD is a rhombus", but this faulty argument was refuted by the teacher (shown by a zigzag, see Reid et al. 2011).

Representing argumentation structures, classifying and analysing them is not the end goal of this method. As in Knipping (2003) the goals is to compare
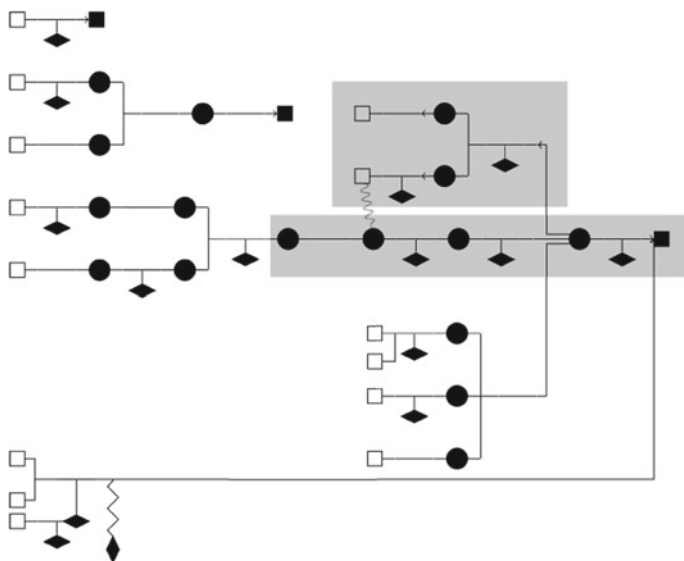

Fig. 1.14 The global argumentation structure in Ms James' lesson

argumentation structures in order to reveal differences in the rationale of proving (and other argumentation) processes in mathematics classrooms. Comparison of these structures can reveal, for example, differences in the goals of the teaching, that might not have otherwise been observed. In the next section we examine several such comparisons of argumentation structures in more detail.

1.4 Comparing Argumentation Structures and Revealing Their Rationale

A goal of reconstructing global argumentation structures is comparing the structures that are observed in different contents in order to reveal differences in the goals of teaching, in the nature of the warrants accepted, or other aspects of the context of the argumentation. In other words, the goal of analysing argumentation structures is not only to understand better the nature of argumentation in mathematics classrooms but also to better understand mathematics classrooms overall.

Krummheuer (2007) considers comparison as a methodological principle that provides a reliable method of revealing characteristics of teaching in mathematics classrooms. As with Glaser and Strauss (1967), for Krummheuer comparative analysis represents a central activity that allows empirical control of the heuristic generation of theory. In this approach comparisons occur continuously, "the comparison of interpretations of different observed parts of reality represents a main activity on nearly every level of analysis: from the first interpreting approach to the later more theoretical reflection" (Krummheuer 2007, p. 71, describing Strauss and Corbin 1990). The aim of these comparisons is "conceptual representativeness" (see Strauss and Corbin 1990), that is, to ground theoretical concepts within the data.

Such comparisons have been carried out by several researchers, and in this section we review their work and findings.

1.4.1 Knipping's French-German Comparison

The first use of argumentation structures for comparative research was Knipping's (2003, 2004) comparison of French and German lessons in which the Pythagorean theorem was proved. She found that two different structures emerged in these two contexts, which she called the source-structure and the reservoir-structure.

1.4.1.1 The Source-Structure

In proving discourses with a source-like argumentation structure, arguments and ideas arise from a variety of origins, like water welling up from many springs.

The teacher encourages the students to formulate conjectures that are examined together in class. In some cases this means that students propose conjectures that are unconnected to the overall structure. More than one justification of a statement is appreciated and encouraged by the teacher. This diversity of justifications results in an argumentation structure with parallel streams in which intermediate statements are justified in various ways. False conjectures are eventually refuted, but they are valued as fruitful in the meantime. In argumentations with a source-structure a funneling effect becomes apparent. Towards the end of the argumentation only one chain of statements is developed in contrast to the beginning where many parallel arguments are considered. The structure has these characteristic features:

- Parallel arguments for the same conclusion.
- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream.
- The presence of refutations in the argumentation structure.

The source-structure is also characterised by argumentation steps that lack explicit warrants or data. While this also occurs in the other types of argumentation structure, it is frequent in the source-structure.

In Fig. 1.15 the typical features of the source-structure are evident. There are parallel arguments for the same conclusion (AS-1 and AS-2; AS-3 and the first part of AS-5), there are argumentation steps that have more than one datum (AS-8), and there are refutations (in AS-3 and AS-6, marked ⚡).
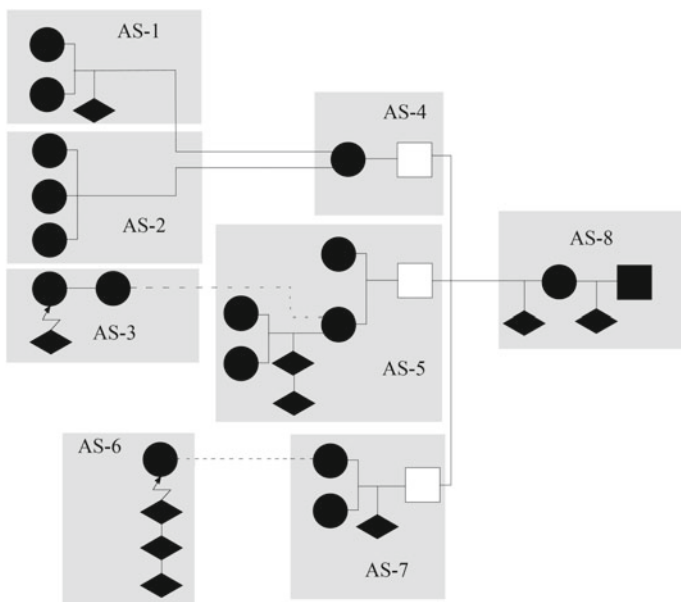

Fig. 1.15 The source-structure in a German classroom (Reid and Knipping 2010, p. 185)

1.4.1.2   The Reservoir-Structure

Argumentations with a reservoir-structure flow towards intermediate target-conclusions that structure the whole argumentation into parts that are distinct and self-contained. The statements that mark the transition from the first to the second part of the proving discourse (shown as rectangles) are like reservoirs that hold and purify water before allowing it to flow on to the next stage. Most of the features listed above as characteristic of the source-structure are missing in the reservoir-structure, with the exception of argumentation steps which have more than one datum each of which is the conclusion of an argumentation stream. Argumentation steps that lack explicit warrants or data occur, but less often than in the source-structure.

   The most important feature of the reservoir-structure, which distinguishes it from a simple chain of deductive arguments, is that the reasoning sometimes moves backwards in the logical structure and then forward again. Initial deductions lead to desired conclusions that then demand further support by data. Through an abduction possible data are identified that, if they could be established, would lead to the desired conclusion (indicated by the dashed line in Fig. 1.16). Once these data are confirmed further deductions lead reliably to the desired conclusion. This characterizes a self-contained argumentation-reservoir that flows both forward towards, and backwards from, a target-conclusion.

   In summary, the reservoir-structure has these characteristic features:

- Abductive steps to identify data from which desired conclusions can be deduced.
- Argumentation steps which have more than one datum.
- Argumentation steps that lack explicit warrants or data (not as common as in source-structure).

   Figure 1.16 shows an example of the reservoir-structure. The class has concluded (in AS-1) that a quadrilateral in the proof diagram is a rhombus. They wish to prove that it is a square, and make an abduction from the desired result that it is a
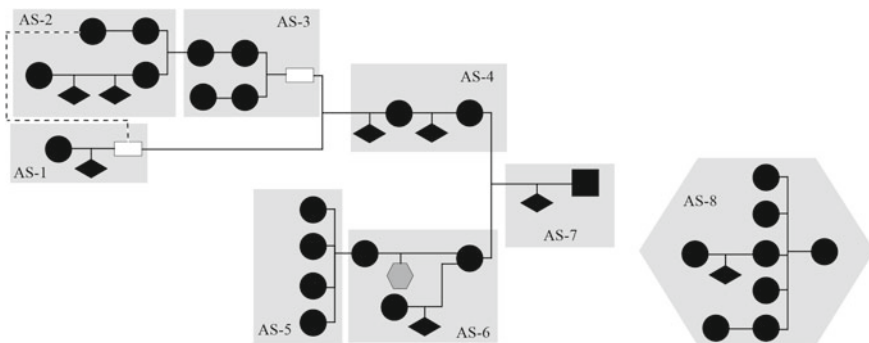

Fig. 1.16 Reservoir-structure from a French classroom (Reid and Knipping 2010, p. 186)

square, the datum that it is a rhombus, and the general rule that if a rhombus has a right angle it is a square, to conclude that the quadrilateral should have a right angle. This becomes the target-conclusion in the argumentation streams AS-2 and AS-3. The three streams AS-1, AS-2 and AS-3 form a reservoir in which the argumentation remains until it is sufficiently clarified to proceed. A closed structure can also be found in the second part of the process, formed by AS-5, AS-6 and AS-7. In contrast to the reservoir in the first part, the argumentation in the second part only flows forwards.

1.4.1.3 Comparison

Knipping found the reservoir-structure in all three of the mathematics lessons she observed in France, and the source-structure in all three of the classrooms she observed in Germany. She concluded that different classroom cultures exist when proving the Pythagorean Theorem. She hypothesised that these differences are due to different cultural traditions of reasoning.

In the German context she characterized that approach to proving as more *intuitive-visual*, in which the teacher does not want to break the proof down into deductive parts, but wants the students to come to their own understanding starting from the figure given. The teacher wants them to *see* the general statement, including its justification, from the given proof figure. In the complete proof discourse she encourages the students to make up their own arguments. This means their conjectures and different arguments are valued as fruitful and discussed in public.

She characterized the proving processes in the French classrooms as more *conceptual*. In these argumentations the conclusions are deduced from concepts. Data can be illustrated and formulated on the basis of figures, but the conclusions drawn from them are arrived at conceptually. The giving of reasons, i.e. warrants or backings, is central in this kind of argumentation. These lead to conclusions that can be recycled, i.e. used in the next argumentation step as data.

1.4.2 Knipping and Reid's Spiral Versus Source Comparison

Knipping and Reid (2013) compared a structure they observed in a Canadian classroom (the same one analysed above) with the source-structure Knipping observed in the French classrooms. They call this new structure a spiral-structure.

1.4.2.1 Spiral-Structure

In a proving process with a spiral argumentation structure the final conclusion is proven in many ways. First one approach is taken, then another and another. Each approach can stand on its own, independent of the others. Students suggest some approaches and the teacher proposes others. Faulty arguments may be refuted by students or the teacher. Some student contributions do not lead to the conclusion and result in disconnected argumentation streams.

The global argumentation structure depicted in Fig. 1.17 shows the spiral argumentation structure from Ms James' class as Knipping and Reid presented it in 2010. Note that there are several changes in comparison to the structure for the same lesson, shown in Fig. 1.14. Most significantly, the abductive stream was not observed in the earlier analysis. In both figures, however, several features characteristic of the spiral argumentation structure are evident:

- Parallel arguments for the same conclusion (AS-B, AS-D, AS-E).
- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (the final conclusions of AS-B and AS-E).
- The presence of refutations in the argumentation structure (AS-D).
- Argumentation streams that do not connect to the main structure (AS-C).

Both the source-structure and the spiral-structure were observed in classrooms where a teacher took a prominent role in guiding the students through a proving processes. Therefore, it is not surprising that these argumentation structures have several similar characteristic features, including parallel arguments, argumentation
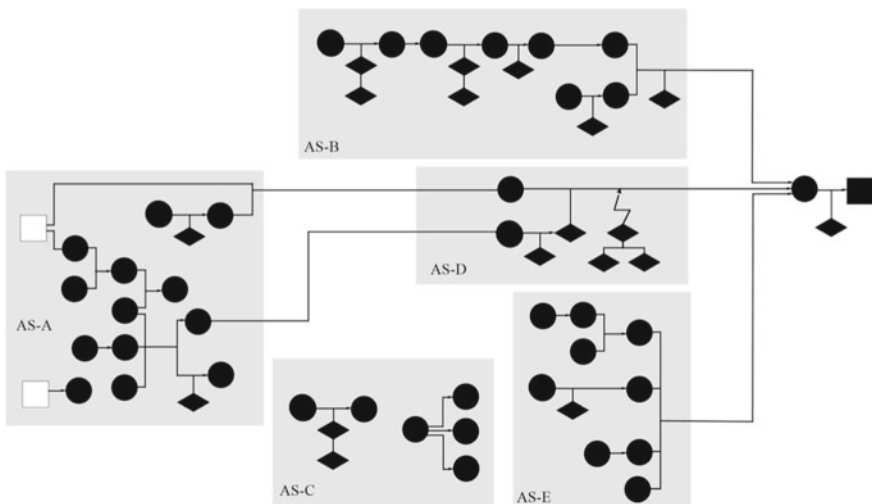

Fig. 1.17 Spiral-structure from Ms James' classroom (Reid and Knipping 2010, p. 188)

steps that have more than one datum, and the presence of refutations. However, a closer comparison reveals that they differ in how these features play out in the global structure.

1.4.2.2   Comparison

One of the main distinctions between the spiral-structure and the source-structure is the location of the parallel arguments. In the source-structure the parallel arguments occur at the start of the proving process (AS-1 and AS-2 in Fig. 1.15). The teacher invites input at this stage, but once the basis for the proof is established, the teacher guides the class to the conclusion through an argumentation that no longer has parallel arguments. In the spiral-structure, however, the conclusions of the parallel arguments are almost the final conclusion in the entire structure. In fact, two of the three parallel arguments in Fig. 1.17 (AS-B and AS-E) could stand alone as proofs of the conclusion. Having proven the result in one way, the teacher goes back and proves it again in a different way. And she values students' attempts to prove the conclusion using other approaches. The source-structure and the spiral-structure differ also in the kinds of refutations they involve and in the inclusion or omission of warrants.

Examining the argumentation structures in these two classrooms allows us to describe their characteristic features, and by comparing them we can understand the different ways these features occur. We see the parallel arguments, refutations and omitted warrants in both, but we see these features occurring differently. Looking more closely at the features of the local arguments helps to explain these differences, and reveals an important distinction between the rationales of the proving processes taking place. In the German classroom, we find in the local arguments a focus on interpreting the given figure. The activity is essentially one of unpacking the data in the figure and expressing it verbally. It is not clear how this could be transferred to proving another theorem, unless a similar complex figure were provided. We suspect this is inevitable in a class focussing on the Pythagorean Theorem.

In contrast, in Mrs James's class the focus is more on proving. The result itself is relatively uninteresting, but the recycling of conclusions as data, the provision of warrants, the fact that the same result can be proven in different ways, and bringing different prior knowledge to bear, are all important. Student contributions are valued, even when flawed, and the argumentation, especially in AS-B, served as a model for the students when proving similar claims in subsequent lessons.

The source structure and the spiral structure are interesting to compare because they have many characteristic features in common, including parallel arguments, argumentation steps that have more than one datum, refutations, and unconnected argumentation streams. There are differences in how these features play out in the global structures, however, and to explain these differences Knipping and Reid (2013) focus again on local arguments, and the goals of teaching.

1.4.3 Abductions in the Reservoir-Structure Versus Ms James' Lesson

As mentioned above, abductions are found in argumentations with a reservoir-structure, such as the one shown in Fig. 1.16. The re-analysis above of Ms James' lesson also revealed an abduction and it is interesting to compare them. In the lessons Knipping (2003) analyses that have a reservoir-structure, the abductions are fairly direct. The datum that is sought leads in a single step to the desired conclusion. However, the argument needed to establish the datum involves several steps. In contrast, the abduction that occurred in Ms James' classroom goes back several steps, to several possible congruence conditions that are quickly established. Once the needed data have been established, the deductions corresponding to the abductive steps are quickly done. In other words, in the French classrooms the abduction does not parallel most of the deduction that follows it. A datum is identified that in needed, and this becomes the target conclusion of an argumentation stream that begins from other, already known, data. In Ms James' lesson, however, the abduction involves several steps, and parallels closely the deduction that follows it. In fact, many statements in the deduction are left implicit because they have already been made provisionally in the abduction.

The abduction in Ms James' class could be seen as an example of what Boero et al. (1996) call 'cognitive unity' in which there is a strong link between an abductive process of conjecturing and a deductive process of proving the conjecture. In the argumentation we describe above, the abductive argumentation stream produces a conjecture: that one of the congruence postulate will apply to the triangles in the diagram. This conjecture is quickly verified, and then used to deduce several intermediate claims made in the abductive argumentation. Because the abduction has traversed the same arguments as the deductive stream, but in reverse order, very little new effort is needed to produce the deductive stream. As noted above, much of it can be left implicit because the statements needed have already been made.

1.4.4 Shinno's Research

Shinno (2017) analysed a lesson sequence on square roots that was intended to introduce irrational numbers in a ninth-grade classroom. Reconstructing the argumentation allowed him to obtain a deeper understanding of the process by which students come to a new concept of number. The global argumentation structure he reconstructed is shown in Fig. 1.18. In AS2 a square of area 10 is constructed geometrically. The conclusion of AS6 is that $\sqrt{10}$ is an irrational number. In AS5 the conclusion is that $\sqrt{10}$ cannot be expressed as a fraction. AS3 and AS4 establish that fractions can be expressed as repeating decimals and vice versa. The global argumentation structure has several features of a reservoir-structure, such as
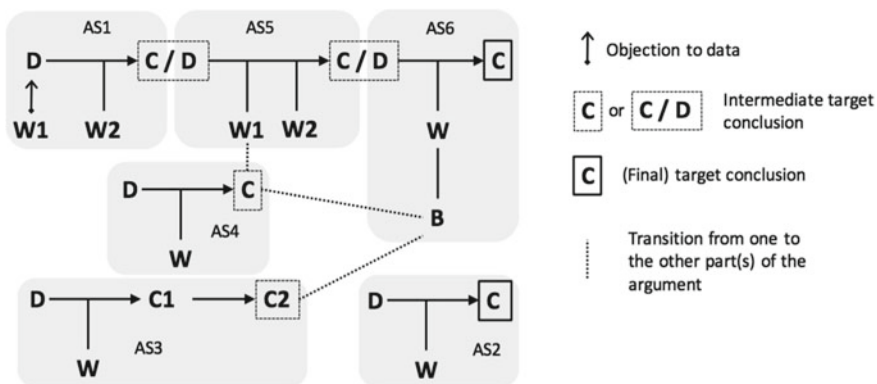
Fig. 1.18 Shinno's global argumentation structure (Shinno 2017, p. 199)

intermediate target conclusions, but also has arguments in support of warrants and backings, a feature only rarely present in the other structures. As Shinno notes "AS3 and AS4 can be considered as mathematical underpinnings for AS5 and AS6. These statements are essential for students to understand the meaning of the proof of irrationality of $\sqrt{10}$." (p. 199). What Shinno calls an "objection to data" in AS1, is a refutation of the datum "the solution of $x^2 = 10$ is 3.1622777" based on a warrant that the product of the final digits must be 0 for the square to be a whole number.

In comparing his global argumentation structure to those described by Reid and Knipping (2010) Shinno observed two novel features: arguments in support of warrants and backings, and the seemingly disconnected AS2, which is nonetheless important to the argument in some non-discursive way (p. 199).

1.4.5 Cramer's Comparisons

Cramer (2018) analyses the argumentations of a group of high school students working in a small group on a wide variety of tasks over a school year. This context allows her to compare the structures of their argumentations between tasks and over time. She categorises their argumentations into several categories, based on the obstacles faced by the students in participating in the argumentation. Her categories include: No real obstacles; Academic language, Rationality, Discourse-ethics.

In the category "No real obstacles" the argumentation diagrams have the following characteristics:

- The reasoning diagrams are rather long and warrants are often made explicit.
- Most learners are involved in the argumentation and the teacher does not play a major role.
- The structure can be characterized as a source-structure.

Argumentation structures in the category of "Academic language" show the following features:

- There are parallel lines of argument similar to a source-structure; however, there are often breaks within lines of argumentation. Refutations are often made. Warrants and backings are explicit in many cases.
- The target conclusions come always from the teacher and the teacher's involvement in the argumentation always consists of simple, scattered utterances. The wording of utterances is difficult to understand.

In the category "Rationality" the argumentation diagrams have the following characteristics:

- In many cases, there are unconnected arguments. Frequently, these occur in connection with refutations. The teacher and the students take responsibility for the argumentation to varying degrees.

Cramer's comparison of the argumentation diagrams in the "Discourse Ethics" category reveals the following properties:

- The arguments are rather short. Often refutations occur. Warrants are often left out.
- Many arguments are strongly guided by the teacher.

Cramer observes that refutations are discernible in most of the argumentation diagrams. She suggests that this can be seen as an indication that there is some confusion about the knowledge base the students had in common. The students evidently started partly from different assumptions. This may have been made the development of collective argumentation more difficult.

The argumentations analysed by Cramer (2018) illustrate in yet another way how argumentation structures can be of very different kinds in mathematics teaching and learning processes. They demonstrate the relevance and value of detailed reconstructions of argumentation structures, using the methods outlined above, based on and extending the Toulmin model.

1.4.6 Potari and Psycharis' Comparisons

Potari and Psycharis (2018) explore prospective mathematics teacher (PMT) argumentation while interpreting classroom incidents and find "different argumentation structures and types of warrants, backings and rebuttals in the process of PMTs' interpretations of students' mathematical activity." (p. 169). They compared PMT's argumentations to the source-structure and spiral-structure and adapted the methods described above to analyze PMTs' interpretations of critical incidents they identified when reflecting on lessons the PMTs observed or taught. These comparisons were enhanced by using Nardi et al.'s (2012) classifications of

different types of warrants. Comparing argumentation structures allowed Potari and Psycharis to reveal potential shifts in PMTs' interpretations of classroom phenomena. They found that some of the teachers' argumentations have "… similar features as the spiral argumentation structure of Knipping and Reid (2015). This is because it involves parallel arguments that could stand alone leading to the final claims, warrants, and backings that adequately justify the claims, and refutations of the main claims." (p. 222). Comparing PMT's argumentations over time they saw different argumentation structures and types of warrants, backings, and rebuttals. The later argumentations were richer and the PMTs could support their claims in different independent ways. So the reconstruction of argumentation structures and types allowed the researchers to describe PMTs' developments in teacher education over time.

1.4.7 Papadaki, Reid and Knipping's Comparisons

Papadaki et al. (2019) compare the role of abduction in the argumentations structure from Ms James' class (described above) to the abductions that occurred in a geometry teaching experiment in Papadaki's research (2017). While in both cases the students involved were using dynamic geometry software to generate conjectures and then engaging in argumentations in a whole class context, the main foci of the lessons were quite different and this is reflected in the argumentation structures. Is Ms James' class, the abduction occurs as part of a mainly deductive proving process. Producing a proof is the goal of the lesson and the abduction is used to identify data needed in the proof. In Papadaki's research, however, the process is primarily one of making conjectures, which can be supported but not proven. The task the students were given asks them to make a hypothesis, and does not provide a framework in which they could prove their hypotheses. The abduction is used to arrive at hypotheses, from which some consequences are deduced. The only proofs that are produced are disproof of false hypotheses. Hence, the overall argumentation is abductive with deductive elements. One consequence of this, visible in the argumentation structures, is that the looping seen in Fig. 1.14, resulting from the backward flowing abductions, does not occur in Papadaki's structures. The flow of the argumentation is all in one direction, whether it is deductive or abductive.

1.5 Concluding Remarks

Proving processes in mathematics classrooms follow their own peculiar rationale, which raises the question of how to reconstruct and analyse the complex argumentative structures that arise in classroom discussion. Toulmin's functional model of argument allows us to reconstruct arguments in mathematics classrooms at the

local level, and above we described how these local arguments can be connected together into a global structure. By comparing these argumentation structures we can describe their characteristic features, and understand the different ways these features occur. Attention to both the local and the global levels are essential to understanding proving processes in the classroom. Examining argumentation structures provides a tool to better understand the different ways in which teachers teach proof in actual classrooms and how students in those classrooms come to an understanding of proof and proving.

The value of looking at the production of arguments in class has been recognised in mathematics education for some time (at least since Krummheuer 1995) as a way of bringing to light the relevance of argumentation for learning processes in general and for mathematics learning in particular. In this chapter, we reference a range of recent research in the area of argumentation analyses in mathematics education that makes use of the methods we describe. This research illustrates the value of reconstructing argumentation and argumentation structures in classroom contexts, as a way of looking at teaching and learning processes. In this chapter we attempted to tease out particularly the value of reconstructing argumentation structures and comparing these structures. The recent research in this area allows us to get deeper insights into a variety of significant fields in mathematics education, for example,

- students' processes of learning and understanding concepts (see e.g., Shinno, Papadaki)
- learning and teaching proving in the mathematics class (Knipping and Reid, Cramer)
- interpretation of classroom phenomena by teachers and their developments in this respect (Potari and Psycharis, Erkek and Bostan).

The focus on reconstructing and comparing argumentation structures in recent years has pushed the research in all these areas forward towards a deeper understanding of students' and teachers' practices. It has also pointed at areas and points that require further attention and research. For example, the relation between abduction and deduction, which has been researched by our Italian colleagues for a long time (see Boero et al. 1996) within the theoretical frame of cognitive unity, can be examined from a new perspective in view of comparative studies of argumentation structures. Also, our comparisons demonstrate that abduction can play different roles in argumentation, which has been clear for a long time in the philosophical and linguistic literature (see, e.g., Eco 1983). Yet it makes a difference to show empirically that significant qualitative differences between roles of abduction occur in classrooms, and that abduction occurs at times when we did not particularly expect it.

Another recent trend in mathematics education has been to look at argumentation schemes (Metaxas et al. 2016). Colleagues call for using theoretical frameworks from other fields, e.g., linguistics, rhetoric, philosophy, and so on, to develop theoretical categorisations of "argumentation schemes" for analysing argumentation processes in the context of mathematics learning and teaching. The examination of

argumentation schemes has been a tradition in rhetoric and philosophy for hundreds or thousands of years. Recently this tradition has been rediscovered in the philosophy of mathematics. Specifically, a branch called the philosophy of mathematical practice has evolved in the last two decades (we are thinking of the work of Bart van Kerkhove, Andrew Aberdein, and others). Working with argumentation theory, communication theory and pragmatics this philosophy of mathematical practice has developed very promising new frameworks to look at mathematical practices. These researchers are also engaging in interdisciplinary work with mathematics educators (see Aberdein and Dove 2013; Inglis and Aberdein 2014; Aberdein forthcoming). This seems to be a very fruitful and promising direction for future research.

References

Aberdein, A. (2019 forthcoming). Evidence, proofs, and derivations. *ZDM Mathematics Education, 52*(4).

Aberdein, A. (2006). Managing informal mathematical knowledge: Techniques from informal logic. In J.M. Borwein, & W.M. Farmer (Eds.), *Mathematical knowledge management* (Vol. 4108, pp. 208–221). Lecture notes in artificial intelligence. Berlin: Springer.

Aberdein, A., & Dove, I. J. (Eds.). (2013). *The argument of mathematics*. New York: Springer.

Arzarello, F., & Sabena, C. (2011). Semiotic and theoretic control in argumentation and proof activities. *Educational Studies in Mathematics, 77*(2–3), 189–206.

Ayalon, M., & Even, R. (2016). Factors shaping students' opportunities to engage in argumentative activity. *International Journal of Science and Mathematics Education, 14*(3), 575–601.

Boero, P., Garuti, R., Lemut, E., & Mariotti M.A. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the Twentieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Valencia, Spain: PME.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014a). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401–429.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014b). Identifying kinds of reasoning in collective argumentation. *Mathematical Thinking and Learning, 16*(3), 181–200.

Cramer, J. (2018). Mathematisches Argumentieren als Diskurs: Eine theoretische und empirische Betrachtung diskursiver Hindernisse. Wiesbaden: Springer Spektrum.

Duval, R. (1995). *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. Bern: Peter Lang.

Eco, U. (1983). Horns, hooves, insteps: Some hypotheses on three types of abduction. In U. Eco & T. Sebeok (Eds.), *The sign of three: Dupin, Holmes, Peirce* (pp. 198–220). Bloomington, IN: Indiana University Press.

Fukawa-Connelly, T. (2014). Using Toulmin analysis to analyse an instructor's proof presentation in abstract algebra. *International Journal of Mathematical Education in Science and Technology, 45*(1), 75–88.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 345–352). Stellenbosch: University of Stellenbosch.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Aldine: Chicago.

Godden, D., & Walton, D. (2007). Advances in the theory of argumentation schemes and critical questions. *Informal Logic, 27,* 267–292.

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*(3), 176–203.

Inglis, M., & Aberdein, A. (2014). Beauty is not simplicity: an analysis of mathematicians' proof appraisals. *Philosophia Mathematica, 23*(1), 87–109.

Inglis, M., Mejía-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66,* 3–31.

Knipping, C. (2003). *Beweisprozesse in der Unterrichtspraxis: Vergleichende Analysen von Mathematikunterricht in Deutschland und Frankreich*. Hildesheim: Franzbecker Verlag.

Knipping, C. (2004). Argumentations in proving discourses in mathematics classrooms. In G. Törner et al. (Eds.), Developments in mathematics education in German-speaking countries. Selected Papers from the Annual Conference on Didactics of Mathematics, Ludwigsburg, 5–9 March 2001 (pp. 73–84). Hildesheim: Franzbecker Verlag.

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. *ZDM Mathematics Education, 40,* 427–441.

Knipping, C., & Reid, D. (2013). Revealing structures of argumentations in classroom proving processes. In A. Aberdein & I. J. Dove (Eds.), *The argument of mathematics* (pp. 119–146). New York: Springer.

Knipping, C., & Reid, D. (2015). Reconstructing argumentation structures: A perspective on proving processes in secondary mathematics classroom interactions. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.) *Approaches to qualitative research in mathematics education.* (pp. 75–101). Berlin: Springer.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale: Lawrence Erlbaum.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *Journal of Mathematical Behavior, 26*(1), 60–82.

Krummheuer, G., & Brandt, B. (2001). *Paraphrase und Traduction. Partizipationstheoretische Elemente einer Interaktionstheorie des Mathematiklernens in der Grundschule*. Weinheim: Beltz.

Metaxas, N. (2015). Mathematical argumentation of students participating in a mathematics–information technology project. *International Research in Education, 3*(1), 82–92.

Metaxas, N., Potari, D., & Zachariades, T. (2009). Studying teachers' pedagogical argumentation. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 121–128). Thessaloniki: PME.

Metaxas, N., Potari, D., & Zachariades, T. (2016). Analysis of a teacher's pedagogical arguments using Toulmin's model and argumentation schemes. *Educational Studies in Mathematics, 93*(3), 383–397.

Nardi, E., Biza, E., & Zachariades, T. (2012). 'Warrant' revisited: Integrating mathematics teachers' pedagogical and epistemological considerations into Toulmin's model for argumentation. *Educational Studies in Mathematics, 79,* 157–173.

Papadaki, C. (2017). Hide and think: Limiting the obvious and challenging the mind in geometry In *CERME 10-Tenth Congress of the European Society for Research in Mathematics Education* (pp. 706–713).

Papadaki, C., Reid, D., & Knipping, C. (2019). Abduction in argumentation: Two representations that reveal its different functions. In Presentation to TSG 1, CERME-11, Utrecht, NL. To appear in the proceedings.

Potari, D., & Psycharis, G. (2018). Prospective mathematics teacher argumentation while interpreting classroom incidents. *Educating prospective secondary mathematics teachers* (pp. 169–187). Cham, Switzerland: Springer.

Reid, D.A. (2018). Abductive reasoning in mathematics education: Approaches to and theorisations of a complex idea. *Eurasia Journal of Mathematics, Science and Technology Education, 14*(9), em1584. https://doi.org/10.29333/ejmste/92552.

Reid, D., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching*. Rotterdam: Sense.

Reid, D., Knipping, C., & Crosby, M. (2011). Refutations and the logic of practice. *PNA, 6*(1), 1–10. HANDLE: http://hdl.handle.net/10481/16011.

Shinno, Y. (2017). Reconstructing a lesson sequence introducing an irrational number as a global argumentation structure, In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193–200). Singapore: PME.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park: Sage.

Toulmin, S. E. (1958). *The uses of argument*. Cambridge, UK: Cambridge University Press.

Tsujiyama, Y. (2012). Characterization of proving processes in school mathematics based on Toulmin's concept of field. In *Pre-Proceedings of ICME12*, (pp. 2875–2884).

van Eemeren, F. H., Grootendoorst, R., & Kruiger, T. (1987). *Handbook of argumentation theory: A critical survey of classical backgrounds and modern studies*. Dordrecht: Foris Publications.

Walton, D., & Reed, C. (2005). Argumentation schemes and enthymemes. *Synthese, 145,* 339–370.

Walton, D., Reed, C., & Macagno, F. (2008). *Argumentation schemes*. Cambridge: Cambridge University Press.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th International Conference on the Psychology of Mathematics Education* (Vol. 1, pp. 9–23). Utrecht: IGPME.