

# Chapter 7

## Theories of and in Mathematics Education



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**Abstract** How far has the didactics of mathematics developed as a scientific discipline? This question was discussed intensively in Germany during the 1980s, with both affirmative and critical reference to Kuhn and Masterman. In 1984, Hans-Georg Steiner inaugurated a series of international conferences on ‘Theories of Mathematics Education’ (TME), pursuing a scientific program that aimed at founding and developing didactics of mathematics as a scientific discipline. Today, a more bottom-up meta-theoretical approach, the networking of theories, has emerged which has roots in the early days of discussing the developmental of mathematics education as a scientific discipline. This article presents an overview of this thread of development and a brief description of the TME program. Two theories from German-speaking countries are outlined and networked in the analysis of an empirical example that shows their complementary nature traced back to the TME program.

**Keywords** Theories · Theory of mathematics education (TME) · Networking theories · Mathematics education as a scientific discipline · Learning activity · Sign use · Semiotic game

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## 7.1 Introduction

This chapter<sup>1</sup> begins with a description of the historical situation of the community of mathematics education in German-speaking countries. The historical development and discussions surrounding the concept of theory related to mathematics education as a scientific discipline are traced from the 1970s up to the beginning of the twenty-first century, in the German-speaking countries as well as internationally. We will describe the main points of the Theory of Mathematics Education (TME) program as introduced by Hans-Georg Steiner. Two theoretical approaches, the theory of Learning Activity developed by Joachim Lompscher and Willi Dörfler's semiotic view of doing mathematics related to diagrammatic reasoning and its semiotic game, are summarized and concretized through the application of them to the analysis of an empirical example, a students' group solution of a mathematical task. Based on this example, we depict the networking of theories and the subsequent contribution to the TME program.

## 7.2 The Role of Theories in Relation to Mathematics Education as a Scientific Discipline: A Discussion in the 1980s

On an institutional and organizational level, the time span from the 1970s until the early 1980s had been a period of considerable change for mathematics education in former West Germany<sup>2</sup>—both in school and as a research domain. The *Institute for Didactics of Mathematics* (Institut für Didaktik der Mathematik, IDM) was founded in 1973 in Bielefeld as the first research institute in the German-speaking countries specifically dedicated to mathematics education research; 1975 saw the inception of the *Society of Didactics of Mathematics* (*Gesellschaft für Didaktik der Mathematik, GDM*) as the scientific society of mathematics educators in the German-speaking countries (cf. Bauersfeld et al. 1984, pp. 169–197; Toepell 2004).

The teacher colleges ('Pädagogische Hochschulen'), being the home of many mathematics educators at that time, were either integrated into full universities or developed into universities of education entitled to award doctorates. The Hamburg Treaty ('Hamburger Abkommen', KMK 1964/71), adopted in 1964 by the Standing Conference of Ministers of Education and Cultural Affairs (KMK), led to considerable organizational changes within the German school system. Although

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<sup>1</sup>This chapter presents the ICME-13 Topical Survey 'Theories in and of Mathematics Education' (Bikner-Ahsbahs et al. 2016) in a shorter, partly reworked version: Sects. 7.1 and 7.2 are slightly revised versions (see *ibid.* pp. 1–9), Sect. 7.3 has been reworked and expanded (see *ibid.* pp. 10–11). Sections 7.4 and 7.5 present a summary and intensified rework of Bikner-Ahsbahs et al. (2016, pp. 13–42).

<sup>2</sup>For an overview including the development in Austria c.f. Dörfler 2013b; for an account on the development in Eastern Germany c.f. Walsch 2003.

the traditional, more vocationally oriented ‘Volksschule’ (a common school covering both primary and secondary education, grades 1–8) was abolished, a new track called ‘Hauptschule’ was instituted for grades 5–8, reinstating a third track besides ‘Realschule’ and ‘Gymnasium’ for the years to come. In 1968, the Standing Conference’s ‘Recommendations and Guidelines for the Modernization of Mathematics Teaching’ introduced profound changes to the content of mathematics education at all ages. Along with this, the traditional designation of the school subject as ‘Rechnen’ (translated as ‘practical arithmetic’) was also abandoned for primary school education in favour of the subject designation ‘Mathematik’ (cf. Griesel 2001; Müller and Wittmann 1984, pp. 146–170).

Likewise, there was a vivid interest in discussing how far mathematics education had developed as a scientific discipline, as documented in both of the German language journals on mathematics education founded at that time: *Zentralblatt für Didaktik der Mathematik* (ZDM, founded in 1969) and *Journal für Mathematik-Didaktik* (JMD, founded in 1980). These discussions mainly addressed two aspects: the role and suitable concept of theories for mathematics education, and the question of how mathematics education was to be founded as a scientific discipline and how it could be further developed. Of course, both aspects are deeply intertwined.

Issue 6 (1974) of ZDM was dedicated to a broad discussion about the current state of the field of ‘Didactics of Mathematics’/mathematics education. The issue was edited by Hans-Georg Steiner. It comprised contributions from Bigalke (1974), Freudenthal (1974), Griesel (1974), Otte (1974) and Wittmann (1974), among others. These articles were focused around the questions of (1) how to conceptualize the subject area or domain of discourse of mathematics education as a scientific discipline; (2) how mathematics education may substantiate its scientific character; and (3) how to frame its relation to reference disciplines, especially mathematics, psychology and educational science. While there was a rich variety in the approaches to these questions, and, likewise, to the definitions of ‘Didactics of Mathematics’ given by the various authors, *cautioning against reductionist approaches* seemed to be a common topic of these papers. That is, the authors agreed upon the view that mathematics education cannot be meaningfully conceptualized as a subdomain of either mathematics, psychology, or educational science alone.

The role of theory was more explicitly discussed about 10 years later in two papers (Burscheid 1983; Bigalke 1984) and in two comments (Fischer 1983; Steiner 1983) published in the ‘Journal für Mathematik-Didaktik’ (JMD). As an example of the discussion about theory of that time, we will convey the different positions in these papers in more detail.

In 1983, Burscheid used the model from Kuhn and Masterman (cf. Kuhn 1970; Masterman 1970, 1974) to explore the developmental stage of mathematics education as a scientific discipline. He justified this approach by claiming that every science represents its results through theories and therefore mathematics education as a science is obliged to develop theories and make its results testable (Burscheid 1983, p. 222). This model describes scientific communities and their development by paradigms. By investigating mainly natural sciences, Kuhn has characterized a

paradigm by four components—symbolic generalizations, models of thought, values and attitudes, and exemplars:

- (1) *symbolic generalizations* as “expressions, deployed without question or dissent [...], which can readily be cast in a logical form” (Kuhn 1970, p. 182) or a mathematical model—in other words: scientific laws, e.g. Newton’s law of motion;
- (2) *metaphysical presumptions* as faith in specific *models of thought* “shared commitment to beliefs”, such as “heat is the kinetic energy of the constituent parts of bodies” (p. 184);
- (3) *values* and attitudes “more widely shared among different communities” (p. 184) than the first two components;
- (4) *exemplars*, such as “concrete problem-solutions that students encounter from the start of their scientific education” (p. 187)—in other words: textbook or laboratory examples.

Masterman (1970, p. 65) categorized these components with respect to three types of paradigms:

- (a) metaphysical or meta-paradigms (refers to 2);
- (b) sociological paradigms (refers to 3);
- (c) artifact or constructed paradigms (refers to 1 and 4).

Each paradigm shapes a disciplinary matrix according to which new knowledge can be structured, legitimized, and embedded into the discipline’s body of knowledge. Referring to Masterman, Burscheid used these types of paradigms to identify the scientific state of mathematics education with respect to four development stages of a scientific discipline (see Burscheid pp. 224–227):

Burscheid described the first stage (Table 7.1) as a founding stage of a scientific discipline where scientists are identifying the discipline’s core problems, establishing typical solutions and developing methods to be used. In this stage, scientists struggle with the discipline’s basic assumptions and kernel of ideas; for instance, with the methodological questions of how validity can be justified and which thought models are relevant. In this stage, paradigms begin to develop, resulting in the building of scientific schools and shaping a multi-paradigm discipline. The schools’ specific paradigms unfold locally within the single scientific group but do not affect the discipline as a whole. In stage three, mature paradigms compete to gain scientific hegemony in the field (Burscheid 1983, p. 226). The final stage is that of a mature scientific discipline in which the whole community shares more or less the same paradigm (p. 226).

Following the disciplinary matrix, Burscheid (pp. 226–236) identified paradigms in mathematics education and features at that time, according to which different scientific schools emerged and could be distinguished from one another, e.g. according to forms, levels and types of schools, or according to reference disciplines such as mathematics, psychology, pedagogy, and sociology. The *constructed paradigms* dealt in principle with establishing adequate theories in a discipline. Concerning building theories, however, the transfer of the model of Masterman and Kuhn was

**Table 7.1** Stage-model of the development of a scientific discipline (p. 224, translated)<sup>a</sup>

Stage	Paradigm characteristics	Core description
1	Non-paradigmatic	Founding phase of the scientific discipline
2	Multi-paradigmatic science	Scientific schools based on paradigms emerge
3	Dual-paradigmatic science	Mature paradigms compete
4	Mono-paradigmatic science	A mature paradigm determines the scientific discipline

<sup>a</sup>Any translation within this article has been conducted by the authors unless stated otherwise

difficult to achieve because symbolic generalizations and/or scientific laws can be built more easily in the natural sciences than in mathematics education. This is because mathematics education is concerned with human beings who are able to creatively decide and act in the teaching and learning processes. Burscheid doubted that a general theory such as those in physics could ever be developed in mathematics education (p. 233). However, his considerations led to the conclusion that “there are single groups in the scientific community of mathematics education which are determined by a disciplinary matrix. [...] That means that mathematics education is [still] heading to a multi-paradigm science” (translated, p. 234).

Burscheid’s analysis was immediately criticized from two perspectives. Fischer (1983)<sup>3</sup> claimed that pitting mathematics education against the scientific development of natural science is almost absurd because mathematics education has to do with human beings (p. 241). In his view “theory deficit” (translated, p. 242) should not be regarded as a shortcoming but as a chance for all people involved in education to emancipate themselves. The lack of impact to practice should not be overcome by top-down measures from the outside but by involving mathematics teachers from the bottom-up to develop their lessons linked to the development of their personality and their schools (p. 242). Fischer did not criticize Burscheid’s analysis per se, but rather the application of a model postulating that all sciences must develop in the same way as the natural sciences towards a unifying paradigm (Fischer 1983).

Steiner (1983) also criticized the use of the models developed by Kuhn and Masterman. He considered them to be not applicable to mathematics education in principle, claiming that even for physics these models do not address specific domains in suitable ways, and, in his view, domain specificity is in the core of mathematics education (p. 246). Even more than Fischer, Steiner doubted that mathematics education will develop towards a unifying single paradigm science. According to him,

<sup>3</sup>Fischer also feared that once mathematics education would develop towards a unifying paradigm, the field of mathematics education were more concerned with its own problems like physics and, finally, would develop separating its issues from societal concerns.

**Table 7.2** Views on mathematics education as a scientific discipline

Discussant	View on mathematics education as a scientific discipline and its development
Burscheid	Theories and theorizing are in the core of mathematics education as a scientific discipline. Taking the development of natural science as a role model, Burscheid assumes that the development of mathematics education advances by a process of maturing and competing paradigms
Fischer	Fischer dismisses to take natural science as role model for scientific development since mathematics education has to do with human beings and it is practice based. It develops from practice bottom-up by the development and emancipation of teachers
Steiner	Steiner dismisses to take natural science as role model for scientific development of mathematics education because not even physics fits this model in all respects. Mathematics education as a scientific discipline is systemic and interdisciplinary at its core. It develops from the inside as a system of interrelations among mathematics, further disciplines and through the relation of theory and practice
Bigalke	The nature of mathematics education as a scientific discipline follows scientific principles. Its theory concept consists of an unimpeachable kernel and an empirical surrounding. From the contextual nature of the scientific knowledge of mathematics education Bigalke infers the necessity to accept multiple principles and theories. This knowledge develops from the inside while theories are inspired by practice and have to prove being successful in research and practice

mathematics education has many facets and a systemic character with a responsibility to society. It is deeply connected to other disciplines and in contrast to physics, mathematics education must be thought of as being interdisciplinary at its core. The scientific development of mathematics education should not rely upon external categories of description and acceptance standards, but should develop such categories itself (pp. 246–247); and, moreover, it should consider the relation between theory and practice (p. 248) (Table 7.2).

One year later, Bigalke (1984) proposed exactly such an analysis from the inside. He analyzed the development of mathematics education as a scientific discipline as well, but this time without using an external developmental model. He proposed a “suitable theory concept” (translated, p. 133) for mathematics education on the basis of nine theses. Bigalke urged a theoretical discussion, and reflection on epistemological issues of theory development. Mathematics education should establish the principles and heuristics of its practice, specifically of its research practice and theory development on its own terms. Bigalke specifically regarded it as a science that is committed to mathematics as a core area with relations to other disciplines. He claimed that its scientific principles should be created by “philosophical and theoretical reflections from tacit agreements about the purpose, aims, and the style of learning mathematics as well as the problematization of its pre-requisites” (translated; p. 142), and he emphasized that such principles are deeply intertwined with research programs and their theorizing processes.

Many examples taken from the German didactics of mathematics substantiate that Sneed's and Stegmüller's understanding of theory<sup>4</sup> (cf. Jahnke 1978, pp. 70–90) fits mathematics education much better than the restrictive notion of theory according to Masterman and Kuhn, specifically when theories are regarded to inform practice. Referring to Sneed and Stegmüller, Bigalke (1984, p. 147) investigated the suitability of their theory concept for theoretical approaches in mathematics education and summarized this theory concept in the subsequent way:

A theory in mathematics education is a structured entity shaped by propositions, values and norms about learning mathematics. It consists of a kernel, that encompasses the unimpeachable foundations and norms of the theory, and an empirical component which contains all possible expansions of the kernel and all intended applications that arise from the kernel and its expansions. This understanding of theory fosters scientific insight and scientific practice in the area of mathematics education. (translated, p. 152)

Bigalke (1984) himself pointed out that this understanding of theory allows many theories to exist side by side providing a frame for a diversity of theories. It was clear to him that no collection of scientific principles for mathematics education would result in a 'canon' agreed across the whole scientific community. On the contrary, he considered a certain degree of pluralism and diversity of principles and theories to be desirable or even necessary (p. 142). Bigalke regarded theories as being inspired by the practice of teaching and learning of mathematics thus providing the link to this practice, founding mathematics education as a scientific discipline in which theories may prove themselves successful in research as well as in practice (Bigalke 1984). Progress of the scientific discipline results from the challenge to overcome the tension between the scientific principles and the values and norms in the practice of teaching and learning mathematics. Theories are the tools to overcome this challenge (p. 159), hence, allowing various forms of theories to be developed.

### **7.3 Theories of Mathematics Education (TME): A Program for Developing Mathematics Education as a Scientific Discipline**

Out of the previous presentation arose the result that the development of theories in mathematics education cannot be cut off from clarifying the notion of theory and its epistemological ground related to the scientific foundation of the field. Steiner (1983) construed this kind of self-reflection as a genuine task in any scientific discipline (cf. Steiner 1986) when he addressed the comprehensive task of founding and further developing mathematics education as a scientific discipline (cf. Steiner 1987c). At a post-conference meeting of ICME-5 in Adelaide in 1984, the first of five conferences on the topic "Theories of Mathematics Education" (TME) took place (Steiner et al.

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<sup>4</sup>We will not further elaborate on the theory concept by Stegmüller and Sneed as we wish to focus on the debate conducted at that time.

1984; Steiner 1985, 1986). This topic is a developmental program consisting of three partly overlapping components<sup>5</sup>:

- “Development of the dynamic regulating role of mathematics education as a discipline with respect to the theory-practice interplay and interdisciplinary cooperation.
- Development of a comprehensive view of mathematics education comprising research, development, and practice by means of a systems approach.
- Meta-research and development of meta-knowledge with respect to mathematics education as a discipline” (emphasis in the original; Steiner 1985, p. 16).

Steiner characterized mathematics education as a complex referential system in relation to the aim of implementing and optimizing teaching and learning of mathematics in different social contexts (p. 11). He proposed taking this view as a *meta-paradigm for the field* (Steiner 1985, p. 11; Steiner 1987a, p. 46), addressing the necessity of *meta-research in the field*. According to Steiner, the field’s inherent complexity evokes reduction of its complexity in favor of focusing on specific aspects, such as curriculum development, classroom interaction, or content analysis. According to Steiner, this complexity also creates a differential classification of mathematics education as a “field of mathematics, as a special branch of epistemology, as an engineering science, as a sub-domain of pedagogy or general didactics, as a social science, as a borderline science, as an applied science, as a foundational science, etc.” (Steiner 1985, p. 11). Steiner required clarification of the relations among all these views, including the principle of complementarity on all layers, which means considering research and meta-research, concepts as objects and concepts as tools (Steiner 1987a, p. 48, 1985, p. 15). He proposed understanding mathematics education as a human activity, hence, he added an activity theory view to organize and order the field (Steiner 1985, p. 15). The interesting point here is that Steiner implicitly adopted a specific theoretical view of *the field* but points to the multiple perspectives in *the field* which should be acknowledged as its interdisciplinary core.

Steiner (1985) emphasized the need for the field to become aware of its own processes of development of theories and models and investigate its means, representations and instruments. Epistemological considerations seemed important for him, specifically concerning the role of theory and its application. In line with Bigalke, he proposed considering Sneed’s and Stegmüller’s view on theory as suitable for mathematics education, since it encompasses a kernel of theory and an area of intended

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<sup>5</sup>This program was later reformulated by Steiner (1987a, p. 46; emphasis in the original):

- Identification and elaboration of *basic problems* in the orientation, foundation, methodology, and organization of mathematics education as a discipline.
- The development of a *comprehensive approach* to mathematics education in its totality when viewed as an *interactive system* comprising research, development, and practice.
- *Self-referent research* and *meta-research* related to mathematics education that provides *information* about the state of the art—the situation, problems, and needs of the discipline while respecting national and regional differences.



applications to conceptualize applicability being a part of the very nature of theories in mathematics education (p. 12).

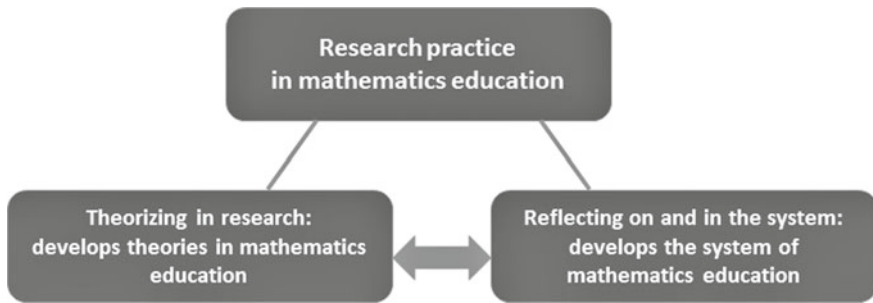
In the first TME conference, theory was an important topic, especially the distinction between so-called *borrowed* and *home-grown* theories. Borrowed theories are taken from outside mathematics education whereas home-grown theories are those developed within mathematics education. With respect to this distinction, Steiner's complementary view made him point to the danger of one-sidedness. In his view, borrowed theories are not just transferred and used but rather adapted to the needs of mathematics education and its specific contexts. Home-grown theories, however, are able to address domain-specific needs but are subjected to the difficulty of establishing suitable research methodologies on their own authority. The interdisciplinary nature of mathematics education requires regulation among the different perspectives but also regulation of the balance between home-grown and borrowed theories (Steiner 1985; Steiner et al. 1984).

So, what is Steiner's specific contribution to the discussion of theories and theory development? Like other colleagues, such as Bigalke, he has pointed to the role of theories as being in the core of mathematics education as a scientific discipline, and he proposed the notion of theory developed by Sneed and Stegmüller (cf. Jahnke 1978; pp. 70–90; see also Bigalke in this article) as being suitable for such an applied science. Steiner proposed complementarity to be a guiding principle for the scientific field and required investigating what complementarity means in each case of the field's topics. In this respect, the dialectic between borrowed theories and home-grown theories is an integral part of the field that allows the discipline to develop from its core and to be challenged from its periphery. In addition, Steiner emphasized that mathematics education as a system (see Steiner 1987b) should reflect on its own epistemological basis, its own theory concepts and theory development, the relation between theory and practice, and the interrelation among all its perspectives. He has added that the specific view of mathematics education always incorporates some epistemological model of how mathematics and teaching and learning of mathematics are understood, and that this is especially relevant for theories in mathematics education.

If we consider the research (practice) in mathematics education as an activity of the discipline, then Steiner has addressed two intertwined sub-activities to be relevant for the foundation and development of mathematics education as a scientific discipline: theorizing in research develops theories, and reflecting on and in the system develops the system of mathematics education; however both activities are related. The following diagram tries to capture Steiner's view on the two activities developing mathematics educations as a scientific discipline (Fig. 7.1):

## 7.4 Post-TME Period

In the following decade, from 1992 up to the beginning of the twenty-first century, the discussion on theory concepts died down in the German community of mathe-



**Fig. 7.1** A tentative diagram of Steiner's view on two necessary and related activities in mathematics education as a scientific discipline

mathematics educators while the theoretical diversity in the field grew. A comprehensive overview of theories used within different (and ever-evolving) strands of mathematics education research in the German-speaking countries over the last 30 years would necessarily go beyond the scope of this paper. We can only give a few cursory examples here, and any selection of such examples is at least prone to subjective selectiveness and personal bias. Therefore, the reader is highly encouraged to consider the other articles in this volume for a more complete picture of the theoretical backgrounds referenced in the respective strands of mathematics education research. Considering the two main scientific journals, we identified scientific contributions from several theoretical communities addressing three topics related to the TME program (without any claim to completeness):

- (1) *Methodology*: *methodological* and thus *theoretical* aspects in interpretative research (Beck and Jungwirth 1999), *interviews* in empirical research (Beck and Maier 1993), *multi-methods* (Wellenreuther 1997); *explaining* in research (Maier 1998), *methodological* considerations on large scale assessments such as e.g. Third International Mathematics and Science Study (TIMSS) (Knoche and Lind 2000);
- (2) *Methods in empirical research*: e.g., two special issues of *ZDM* in 2003 edited by Kaiser presented a number of methodical frameworks;
- (3) *Issues on meta-research* about what mathematics education is, can, and should include: considerations on *paradigms* and the *notion of theory* in interpretative research (Maier and Beck 2001), *comparison research* (Kaiser 2000; Maier and Steinbring 1998; Brandt and Krummheuer 2000; Jungwirth 1994), and mathematics education as *design science* (Wittmann 1995) and as a *text science* (Beck and Maier 1994).

This short list indicates that—at that time—distinct theoretical communities seemed to share the need for methodological and meta-theoretical reflection. In some cases, these theoretical considerations transcended the borders of the distinct theoretical community and led to critical response and discussion:

- Kirsch (1977/2000) and Becker (1978) are some of the rather sparse examples of meta-theoretical reflection on Stoffdidaktik (“subject matter analysis”) from proponents of this traditional strand of mathematics education research in German-speaking countries. Both the notion of “concentration on the mathematical heart of the matter” (Kirsch 1977, 2000) and the sense and purpose of working out mathematically elaborated background theories for school mathematics (Becker 1978) have been questioned from a systems theory perspective in Steinbring (1998) and Steinbring (2011).
- In their discussion of the use of interviews in interpretative research, Beck and Maier (1993) also presented an account of ‘understanding’ in mathematics classrooms (as process and product) developed according to the interpretative paradigm. Weigand (1995) contrasts this view with more traditional, normative accounts of ‘understanding’ developed just within the aforementioned framework of Stoffdidaktik. Weigand raises the question whether interpretative notions of ‘understanding’, originally developed in social science and cultural contexts, can in principle meet the particularities of mathematical thinking and learning, and stresses the complementarity of interpretative and Stoffdidaktik-approaches.
- Knoche and Lind (2000) introduced models of item response theory which were used within the TIMS-Study (Trends in International Mathematics and Science Study) and subsequently were and are used in the Programme for International Student Assessment (PISA) to a broader audience of mathematics education researchers in German speaking countries. Since then, these models have become more widely adopted, and their benefits for assessing and analyzing students’ mathematical competence have been discussed, e.g. in Knoche et al. (2002), Büchter and Pallack (2012) and Leuders (2014). On the other hand, the appropriateness of these models for conceptualizing mathematical learning and the theoretical assumptions related to mathematical learning and student performance underlying these models have been challenged fundamentally, e.g. in Meyerhöfer (2004), Bender (2005), Vohns (2012) and Wuttke (2014)—some of the articles leading to rebuttals and rejoinders.

To reiterate, these are just some cursory examples of theoretical discussions across different strands of mathematics education research, and the reader may again be referred to the other articles in this volume for a more complete and balanced view on theoretical issues that have arisen and been discussed within and between the respective strands.

In order to provide a deeper insight into theory strands of German-speaking countries, we summarize two examples presented during the ICME-13. Both theoretical approaches are then reconsidered and linked in an analysis of an empirical example, as it is usually done in the Networking of Theory strands to show how different theories may be used to better grasp the complexity of teaching and learning mathematics. Referring back to Steiner and his TME program, we will use the insight gained from this exercise to describe how mathematics education as a scientific discipline could reflect on its own epistemological basis, and do meta-research as Steiner proposed to clarify the specificities and roles of its theories and their relations to practice.

## 7.5 Two Theories, Their Origins and Their Purposes

In the survey on theory strands in German-speaking countries (Bikner-Ahsbahs and Vohns 2016), two theories are described in detail and used for the analysis of an empirical example. The first theory, presented by Bruder and Schmitt (2016), is that of Learning Activity, originally developed by Joachim Lompscher in the German Democratic Republic (GDR). The second approach, presented in the same survey by Dörfler (2016), is an example of theorizing mathematics as a semiotic way of doing mathematics by referring to the concept of diagram introduced by Peirce and relating this to the idea of semiotic games by Wittgenstein (1999). For the purpose of this article, we will give a brief overview of both approaches.

### 7.5.1 Learning Activity

Bruder and Schmitt (2016) discuss the theory of Learning Activity developed by Lompscher within the theory culture of activity theory introduced by soviet psychologists, e.g. Vygotski, Leont'jev and Luria (Lompscher 2006). This theory culture takes activities as meaningful, purposeful, culturally and historically coined components of an activity system. Driven by a general motive, an activity brings itself about collectively by actions which are goal oriented and linked to the individuals' psychological development. These actions are influenced by the social and cultural environment in which they are conducted. They are mediated by practical or mental tools available in the cultural environment and directed towards goals; they consist of operations determined by the specific situated conditions (Giest and Lompscher 2006, p. 39), and are often conducted unconsciously (Hasan and Kazlauskas 2014, p. 10). The relation between subject and object is at the core of any activity. This relation, together with actions, goals and available means, structure the activity (Giest and Lompscher 2006, pp. 37–41). Through activities, the subject actively acquires cultural knowledge and knowing, and in the same process this cultural knowledge and knowing is transformed by the individual. Thus, internalisation and externalisation are mutual processes of transformation (Lompscher 1985a, p. 25). Examples of activities are playing activity, learning activity, and working activity (Giest and Lompscher 2006, p. 55).

Lompscher has applied this theoretical view on teaching and learning in school (see Bruder and Schmitt 2016; Lompscher 1985a, b, 1989a, pp. 23–32; Giest and Lompscher 2006, pp. 67–106). Through a learning activity, a student acquires societal knowledge and cognitive competencies by interacting with other individuals and the environmental conditions. Lompscher (1989a) emphasizes that knowledge and competencies are related to “segments of societal experience of the world” (p. 29, translated). The general motive of a learning activity is self-development according to the specific cultural requirements (Giest and Lompscher 2006, p. 83), an aspect that distinguishes learning activity from other activities (p. 93). The teacher is crucial

for constituting a learning activity: he/she arranges the *learning conditions* as tasks and provides the means to solve them. The *learning activity* on a topic is achieved by *learning actions*. These are arranged in steps, building a pathway for a learning trajectory to shape suitable *learning conditions*, providing resources for a sequence of *learning actions* which are supposed to lead to the desired *learning goal*. Sub-tasks are to be arranged in a way that the learner can adopt these tasks and their sub-goals as his/her own. As Lompscher puts it: The outcomes of the individual learning is only achieved by “the intensity and quality of the *learner’s own activity* on and with the learning object, the adequately using resp. shaping or transforming of the learning conditions, the employment of available learning means resp. changes according to adequate aims and conditions” (Lompscher 1989a, p. 32, translated, emphasis in the original).

According to Bruder and Schmitt (2016), Giest and Lompscher (2006) distinguish three parts of a learning action: the *orientation*, the *performance* and the *control* part (p. 197), and three types of orientations a student may be able to conduct (Giest and Lompscher 2006, p. 192; see also Bruder and Schmitt 2016, pp. 16–18): *trial orientation* (driven by some kind of trial and error), *pattern orientation* (a sensitivity to patterns can be followed in a focused area), and *field orientation* (knowledge can be acquired and transferred in a complete knowledge field). A general motive for a learning activity is the development of field orientation, but this is not so easy to achieve. Bruder and Schmitt (2016, p. 16) refer to Davydov’s (1990) idea to start within an initial abstract feature as a means for orientating, exploring and enriching the abstract with the concrete. Ascending from the abstract to the concrete is regarded as a strong approach to reach field orientation as early as possible (see Lompscher 2006, 131–205, 1989b).

Lompscher’s research group has undertaken empirical studies in close connection with the teaching and learning practice in several school domains (Giest and Lompscher 2006). Mathematics was just one of them. The theory of learning activity has been intensively applied, adapted and further developed in research and development for teaching and learning mathematics in various directions (see Bruder and Schmitt 2016): for example, specifying elementary mental operations by Bruder and Brückner (1989), developing a comprehensive model for competence development for modelling, problem solving and argumentation (Bruder et al. 2003), investigating mathematical problem solving (Collet and Bruder 2008; Bruder and Collet 2011), developing learning tasks (Bruder 2010), and difficulties in representing functions (Nitsch 2015), to name just four.

### 7.5.2 *A Semiotic View on Mathematics: Sign Use and Semiotic Game*

The second example, presented by Dörfler (2016), is a specific semiotic view referring to Charles Sanders Peirce and Ludwig Wittgenstein. In the 1990s, Michael Otte

introduced Peirce's semiotics as an important view on mathematics to the German community of mathematics educators (see for example Fischer 2005, p. 375; Dörfler 2016, p. 23; Otte 1997). In the subsequent years, Peirce's theory of semiotics has also been taken up by several researchers for different purposes, for example to develop a semiotic theory on learning (Hoffmann 2001), to illustrate its epistemological nature (Hoffmann 2005), to include the view on diagrams in the mathematics classroom (Dörfler 2006), for analysing chat-communication (Schreiber 2006) or investigating the epistemic role of gestures (Krause 2016).

Dörfler's theoretical view is rooted in a dynamic understanding of mathematics itself (Dörfler 2004, 2006, 2008, 2013a, 2016). Similar to Hoffmann (2005), Dörfler takes the concept of diagram introduced by Peirce as a starting point and describes doing mathematics as *diagrammatic reasoning*. However, the specificity in Dörfler's elaboration is abstaining from the view on mathematical activity as a mental activity building abstract objects in the individual learner.

In Peirce's semiotics, each sign is embedded in a triadic relation between the sign (as a representamen), an object the sign stands for, and an interpretant—which is an effect of the sign allotting meaning to it. For example the interpretant may be produced by an interpreter regarding the sign as standing for an object in some way, like  $\pi$  may be regarded as an irrational number, the limit of a specific infinite sum or as representing the proportional relation between the circumference of a circle and its diameter. The following quote by Peirce (1931–1958) depicts this triadic relation of signs:

A sign, or *representamen*, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the *interpretant* of the first sign. The sign stands for something, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea, ... (CP 2.228, emphasis in the original)

However, an interpretant does not necessarily need to be produced by a human being, it can also be produced in the physical world (Nöth 2000, p. 227). But in any case, the interpretant is the part of the sign that points to meaning. Peirce distinguishes between three kinds of signs in relation to the object: a sign can be an icon, an index, or a symbol. An icon, such as a photo of a person, is a sign that resembles the object: the material person. An index is a sign that refers to another sign because of its direct connection to it, like smoke refers to fire. A symbol is a conventionalized sign or a habitualized sign like the equivalent sign. It links the sign to the object by some kind of regularity or law (Nöth 2000, p. 66).

Referring to Peirce, a diagram, such as an equation, is built by signs of a representation system that provides conventionalized rules. It may include all three kinds of signs described above. A variable in the equation may be viewed as an index referring to another sign, e.g. a measure. The equal sign may represent the rule that two things are regarded the same in a specific manner, as an iconic sign it may refer to a balance scale. In general, a diagram is an inscription representing iconic relations between different signs: that is, it is a complex sign made of other signs and their relations as possibilities to be focused on in an interpretant. While all signs in the

semiotics of Peirce refer to an object, this is not necessarily the case for a diagram (see Dörfler 2016, p. 23). A diagram may just refer to an area of collateral knowledge. Experimenting with a diagram, observing it, and perceiving some new relations may lead “us to suspect that something is true” (CP 5.162). Peirce describes:

All necessary reasoning without exception is diagrammatic. That is, we construct an icon of our hypothetical state of things and proceed to observe it. This observation leads us to suspect that something is true, which we may or may not be able to formulate with precision, and we proceed to inquire whether it is true or not. (CP 5.162)

Diagrammatic reasoning has been worked out more clearly by Bakker and Hoffmann (2005) for mathematics education. As indicated in the quote from Peirce (CP 5.162), they distinguish three steps of diagrammatic reasoning (pp. 340–341): (1) constructing a diagram to represent relations (diagrammatization); (2) experimenting with diagrams based on rules of the specific sign system, rules that tell us what can and what cannot be done with the diagram; and (3) observing the results of the experimentation and reflecting on them (cf. Hoffmann 2005, p. 129). The latter may lead to the discovery of patterns of relations, “which we may or may not be able to formulate with precision, and proceed to inquire whether it is true or not” (CP 5.162).

Dörfler (2016, p. 23) precisely describes how his theoretical view on working with mathematical diagrams represents doing mathematics. He argues that language is a sign system that just mediates between individuals and diagrams. In his view, diagrams are “extra-linguistic signs” (Dörfler 2006, p. 27) with a spatial structure representing relations and providing rules for inventing, exploring and transforming them. As these rules are taken to be without contradictions, mathematical inferences appear consistent and strict. Mathematical meanings are at stake in these transformations as transforming rules. These rules can be exposed linguistically, but their meanings are more directly expressed in the relations of the diagrammatic inscriptions. However, individuals can build a relationship with these diagrams, while exploring, perceiving or talking about them. According to Dörfler, diagrammatic reasoning expresses the nature of doing mathematics, and it is highly creative. Dörfler rejects the existence of mathematical objects as abstract mental objects. Instead, mathematical objects, in his view, manifest in the relations of the diagrams and the rules of their transformations. Thus, “Diagrammatic reasoning is a rule-based but inventive and constructive manipulation of diagrams for investigating their properties and relationships” (Dörfler 2016, p. 26). Hence, it is at the core of the dynamic semiotic view on mathematics, for example when equations are produced they can be transformed into other equations by transformation rules and allow features to be observed and rules to be identified in the diagrams.

Referring to Wittgenstein (1999, according to Dörfler 2016, p. 27), Dörfler strengthens his theoretical view on diagrammatic reasoning by describing mathematics as a semiotic game. This way he sharpens the notion of mathematical meanings: they are in the rules that are the basis for establishing the semiotic game and for building relationships among the signs. Mathematical diagrams are the “essential

and constitutive” (p. 30) means in this game and meanings are in the use of rules. Consequent ensues Dörfler’s view on learning mathematics:

Thus for Peirce, to learn mathematics would be to acquire expertise in diagrammatic reasoning, and for Wittgenstein, it would be to participate in the many various sign games and their techniques. In both cases, which are closely related, it is of great importance to stick meticulously to establish rules... mathematics is thereby fundamentally shown to be a deeply social and socially shared cultural activity and product: sign activity can be executed with others and shown to others in public form. This is very different from imagining mathematics as a kind of abstract and mental activity. (Dörfler 2016, p. 30)

## 7.6 Reconsidering the TME Program by Networking the Two Theoretical Views

In line with the TME-program, we will now present a piece of meta-research to clarify the nature of the two theories above and their relation to inform practice and to raise the awareness of the epistemology on hand. “(...) Steiner (1985) has emphasized the need for the field to become aware of its own processes of development of theories and models and investigate its means, representations, and instruments” (Bikner-Ahsbahs and Vohns 2016, p. 9). This kind of awareness can be achieved by meta-research: that is, research on the research. To do so, we will use the Networking of Theories approach developed since 2006 (see Bikner-Ahsbahs et al. 2014, 2017; Dreyfus 2009). The Networking of Theories approach also emphasizes meta-research. However, it does not explicitly want to advance the field, although this may happen in small steps during the practical process in research. Its main aim is to show a way to solve complex problems for which more than one theory is needed, and reflect on the very process. In order to include meta-research as an additional practice into research, research practices have to be broadened to address also the theories themselves, their methodologies, and the research practices as research objects. The purpose for this kind of meta-research may vary, for example it may be important to obtain methodological or theoretical clarity in a multi-theoretical approach (Kidron et al. 2014; Bikner-Ahsbahs and Kidron 2015), to solve an apparently contradicting problem (Sabena et al. 2014), to clarify the nature of research results or the specificity of the particular epistemology in the study. A nice example for exploring the complementary relation of individual and social processes in an inquiry-based classroom has been presented by Tabach et al. (2017).

In the following section, we will explain what we mean by networking theories. To undertake a networking case, we will present a small piece of data: a students’ group solution for a mathematical problem. This data set will be analysed from Lompscher’s perspective of Learning Activity and from Dörfler’s perspective of semiotic game and diagrammatic reasoning according to the common question of how the process of problem solving yields the result. By comparing and contrasting the two theoretical views rooted in the analyses, and the analyses presented, we want to contribute to the TME program and show the added value of the networking of theories for obtaining





the theories. This step is not always possible; for instance, when theories cannot be combined in a compatible way because this would lead to contradictory results in research. But for instance if the theories address complementary views on the teaching and learning processes (Tabach et al. 2017), the process of improving mutual understanding may progress. The final step is the strategy pair of local integration and synthesizing. Local integration sometimes can be achieved when boundary concepts (cf. Akkerman and Bakker 2011), which can be understood from both theories, are identified (Sabena et al. 2014), or when theoretical concepts of two theories can be integrated into a new theoretical framework (Shinno 2017). As Shinno has shown, the step of integration may have losses and gains: the concepts in the integrated framework may change their notion, but open up new directions of research.

This landscape of networking strategies will now be used to network the two theories, Learning Activity and doing mathematics as a semiotic game of diagrammatic reasoning.

### 7.6.2 *Try to Find a Fraction Representing $\sqrt{2}$*

Figure 7.3 represents the solution of the task to try to represent  $\sqrt{2}$  as a fraction, done by a pair of students at grade 9. The initial task of the lesson before was to construct  $\sqrt{2}$  on the number line by the length of the diagonal of the square with a side length of 1 unit. This was done in a whole class discussion. In the following lesson, the teacher posed the task: Work in pairs and try to represent  $\sqrt{2}$  by a fraction. His aim was to prepare the students for the subsequent proof on the irrationality of  $\sqrt{2}$  as an initial step to expand the rational numbers towards the real numbers.

### 7.6.3 *The Semiotic Game Analysis*

We first theorize the solution process in Fig. 7.3 by applying Dörfler's elaboration on doing mathematics as a semiotic game of diagrammatic reasoning and learning mathematics as gaining expertise therein. To do so, we have to analyse the diagrams as they are transformed step by step, and identify the rules represented explicitly or implicitly in the transformations and relations expressed in the diagrams.

The students begin solving the task with the statement that  $\sqrt{2}$  has to be bigger than 1 but it is not clear where this comes from. They start with the fraction  $\frac{5}{4}$  being bigger than one (line 1), as a kind of tentative true rule that 'this is taken as being equal to  $\sqrt{2}$ '.

In step 1 the tentative equation is transposed by conventionalized transformation rules of equations. The equation obtained is  $32 = 25$ , which is wrong. The inequality is recognized by the students; but their inference 'the fraction is too big' is also wrong (line 2), since the original fraction is smaller than  $\sqrt{2}$ . The implicit rule 'taken as equal' was too vague. This kind of reasoning 'building a tentative equation for  $\sqrt{2}$ ,

Steps:	Solution of the student pair	Translation of the comments
Step 1:	$\sqrt{2}$ muss größer als 1 sein $\sqrt{2} = \frac{3}{4}$ $4 \cdot \sqrt{2} = 3$ $16 \cdot 2 = 32$ $32 = 25$ ohimmi nicht, $\frac{3}{4}$ zu groß	1: $\sqrt{2}$ must be bigger than 1 2: Is not right, $\frac{5}{4}$ is too big.
Step 2	$\sqrt{2} = \frac{6}{5}$ $25 \sqrt{2} = 36$ $50 = 36$ nein zu klein	3: No, too small
Step 3	$\sqrt{2} = \frac{6}{4}$ $\sqrt{2} = \frac{6}{4}$ ein mal $25 \cdot 2 = 36$ $16 \cdot 2 = 36$ noch! $32 = 36$	4: still!
Step 4	$\sqrt{2} = \frac{7}{4}$ $4 \sqrt{2} = 7$ $16 \cdot 2 = 49$ $32 = 49$ <p>naht ein mal noch nicht.                      schlechter als vorher.</p>	5: Still does not fit, 6: Worse than before
Step 5	$\sqrt{2} = \frac{7}{5}$ $5 \sqrt{2} = 49$ $25 \cdot 2 = 49$ $50 = 49$ <p>naht ungefähr                      also ist <math>\sqrt{2} \approx \frac{7}{5}</math></p> <p>Strategie: Bruch passend machen</p>	7: Fits almost, 8: hence $\sqrt{2} \approx \frac{7}{5}$ 9: Strategy: to make [the] fraction fit

Fig. 7.3 A case of diagrammatic reasoning

changing it to remove the square root and interpreting the result’ is repeated in the following steps but with creative changes in constructing arithmetic equations as diagrams.

In step 2, both the nominator and the denominator are changed at the same time by increasing both by 1. Since the nominator is bigger than the denominator, the new fraction has become smaller, but we cannot assume that the students know this. The new inference from the resulting equation  $50 = 36$ , that the fraction is “too small”, is now correct (line 3).

In step 3, the diagram is worked out according to the same rules as before, but this time only the denominator is reduced by 1. If we take this as an interpretant of the previous inference, then the underlying rule is to make the next fraction slightly bigger. From the resulting equation  $32 = 36$ , the students infer now that it is “still” too small (line 4), but this is wrong. The new fraction has indeed become bigger than  $\sqrt{2}$ .

Step 4 reacts to the previous false inference, since the fraction is now made even bigger by increasing only the nominator by 1. The result  $32 = 49$  “still does not fit” (line 5) and it is even “worse than before” (line 6). “Worse” seems to indicate that the difference between 32 and 49 is bigger than the one between 32 and 36 taken from step 3. That the fraction now is bigger than  $\sqrt{2}$  does not appear as an interpretant.

Meanwhile a number of rules have emerged: increasing the nominator of the fraction by 1 and reducing the denominator of the fraction by 1 make the fraction bigger, reducing the nominator by 1 and increasing the denominator by 1 make

the fraction smaller. The tentative rule ‘take the two numbers as being equal’ is a pragmatic rule which can be falsified by the inequality of the result. However, the inferences about the kind of inequality are inconsistent. From step 3 onwards, the rule for changing the fractions seems to be ‘change either the nominator or the denominator by 1 according to the previous result’.

In step 5, we would expect that either the nominator is reduced by 1 or the denominator is increased by 1. Reducing the nominator would reveal the previous fraction, hence, this transformation does not make sense. In fact, the students increase the denominator by 1. Since the manipulation of the equation now leads to the two numbers 50 and 49 close to each other the students’ result is  $\sqrt{2} \approx \frac{7}{5}$ . The approximately-equal sign and doubling the underlining indicate that an approximate result is accepted.

Through diagrammatic reasoning, two kinds of rules are put into effect: (1) if the equation is true, then the manipulation of it will lead to an equation which is also true. Otherwise the result will indicate how to approach the next iteration. (2) Finding an iteration of fractions to box  $\sqrt{2}$  is a quasi-systematic way to determine a fraction close to  $\sqrt{2}$ .

The students’ interpretations are expressed in linguistic terms, taken as inferences or interpretants, which show that they sometimes interact with the diagrams in an ambiguous way (line 2 to line 4). The visible transformations, the rules used and produced, are not precisely expressed. Conventionalized rules for transforming equations are used as routine actions not addressed in the students’ comments. Only the results are interpreted, but partly ambiguously. It turns out that the mistakes in steps 3 and 4 are not relevant because the underlying rule to change either the nominator or the denominator in an opposite direction revealed a result where the mistakes did not harm the process. The final strategy of approximating  $\sqrt{2}$  by boxing it through an iteration of fractions emerged as a heuristic rule that resonates well with the students’ overall strategy “to make [the] fraction fit” (line 9).

#### ***7.6.4 The Learning Activity Analysis***

Let us now add the analysis from the perspective of the theory of Learning Activity. In contrast to Dörfler’s semiotic approach, this theory addresses the complete course of learning, from the teacher’s planning to the goals, whether they are achieved and what comes next. This planning already starts with the question of which cultural-historical knowledge should be learned, whether this knowledge is already accessible, and how the goal should be approached. Specifically the history of teaching and learning in the class has to be considered in the preparation of this course. The teacher in our example has initially constructed  $\sqrt{2}$  on the number line. His next goal is the proof of the irrationality of  $\sqrt{2}$  as a prerequisite for achieving his final goal: the introduction of real numbers. In this teaching course, the task above is a sub-task with the sub-goal yielding the insight that a fraction which exactly represents  $\sqrt{2}$  cannot possibly exist.

Lompscher has emphasized the mutual dependency of the *leaning actions*, the *learning goals*, and the *learning objects* in the *learning conditions* that together provide an arrangement in which the students may constitute their *own learning activity*. Not surprisingly, this open task has produced three more types of solutions in the class. One student pair used their calculator to find an approximate fraction. A second pair tried to find a finite decimal fraction to represent  $\sqrt{2}$  but failed, and therefore showed by the last digits that this does not work: they got stuck. A third pair used the factorizing of prime numbers in a fraction to find a representation for: they also failed, but tried to find a reason why. Given this situation, only the solution above would prepare the teacher's intended proof, although at this stage the proof-lesson could be prepared in a way that also builds on the students' diverse solutions.

In contrast to Dörfler's view, the interplay of the subject and the object is at the core of the learning activity leading to the individual student's personal development. Therefore, we have to ask, what kind of knowledge and competencies have the students previously built, and prospectively are to build in the future. In the solution presented in Fig. 7.3, two elementary acquisition actions (stressed by Lompscher) are shown, *identifying* and *realizing*: The students *identify* a fraction close to  $\sqrt{2}$ , they realize transpositions of equations and build an iteration of fractions for approximating  $\sqrt{2}$ . They use heuristic strategies and transforming equations as heuristic means, and thus realize an argumentation similar to that of a proof of contradiction. In their task solution, the two students show *trial orientation* at the beginning including errors. But through heuristic strategies (equations as heuristic means and systematically changing the starting conditions of the next step) they quickly begin to systematically build an approximation boxing  $\sqrt{2}$  into subsequent fractions, probably not yet conducted quite consciously. However, the way they transform the diagrams systematically depicts their ability of *pattern orientation* in the way boxing is *realized*, based on the interpretation of previous inequality. *Field orientation* does not seem to be touched yet because the theme of irrational numbers has just started to be in the scope of learning.

Can we finally confirm that the students have built their own learning activity through changing the conditions and resources given? We cannot exactly answer this question, but we may find indicators for this outcome. The students used heuristic strategies that are not required, such as equations as heuristic means, and as-if-actions as a heuristic strategy to reveal necessary conditions. They systematically scrutinized the manipulation of equations and checked the results to continue with a slight change of the conditions in the next step. Through heuristics, they constructed conditions which enabled them to proceed in the solving of the problem. In fact they show quite proficient problem solving actions leading to a result that could raise the question as to whether it would be possible to represent  $\sqrt{2}$  by a fraction, and whether or not a final solution could be reached algorithmically. All these aspects indicate that the students really have established their own learning activity yielding *their* solution of the task. However, they might not be aware yet that representing  $\sqrt{2}$  by a fraction is impossible. The theory of Learning Activity would now focus on the teacher's actions of how to systematize all the students' results and provide further tasks and resources to prepare the intended proof.

### 7.6.5 *Undertaking a Networking Analysis*

Networking both theories based on the empirical analyses confirms the results already achieved by the analyses done with another empirical case presented in the ICME-13 survey (Bikner-Ahsbabs 2016).

The semiotic approach elaborated by Dörfler starts from a specific home-grown account of the dynamics of *doing mathematics* and presents an approach which describes this *doing*, however, by adapting the work of two philosophers. The theory of Learning Activity is a more comprehensive theory elaborated for many subjects, borrowed and applied to mathematics to develop students' competencies in *doing mathematics*. It explicitly includes learning goals to be achieved. In terms of the semiotic game view, the students and their mental activities are not at the core of the analysis. The process of diagrammatic reasoning and the transformation rules expressed in the diagrams are addressed rather independently of the students' individual way of interpreting the situation. The inferences can be taken as a mathematical part of the diagrams, thus, of diagrammatic reasoning. We may even state that the relationships shown in the diagrams, in which the next step can be regarded as an interpretant to the previous one, advance the transformation process and constitute the rules. In contrast, the Learning Activity analysis focuses more on the learners, the cultural-historical conditions and the context in the course of teaching and learning in which the students may be able to develop themselves by creating an own learning activity.

Whereas diagrammatic reasoning and the rules obtained belong to the kernel of the theory's identity of Dörfler's semiotic approach, the individual students and their abilities belong to the theory's periphery. In Lompscher's theoretical view, this is the other way round: the students' development is at the core of the theory of Learning Activity, whereas the diagrams are resources belonging to the conditions of this development. This has considerable consequences for research: the research question posed must be interpreted differently by the two approaches, and the methodological and conceptual tools used to gain scientific knowledge in research also differ. However, the two approaches could be used in a complementary way.

This complementarity (see Steiner 1985, 1987a) can be described with the metaphor of "zooming-out and zooming-in" (Prediger et al. 2010, p. 1533, referring to Jungwirth) when looking at the grain sizes of relevant processes. This is possible because both approaches share a certain sensitivity towards acting or doing. Coming from the teacher's long term planning, we would *zoom in* on Dörfler's view to observe and analyze the diagrammatic reasoning on a micro level, in order to reconstruct the rules shown in the semiosis. The students' interpretation may indicate aspects of their development when the Learning Activity theory is considered. We then would have to *zoom out* again in order to take the whole course of teaching and learning as a complementary view into account. If Lompscher's view is considered, we would ask what was learned before the task is posed, what kinds of resources are available, and which resources have to be made available for the students to reach the sub-goals, which conditions are to be met, and how they can be changed to accomplish the

overall goal. Most importantly, the aim would be to construct the course of sub-goals and sub-actions in a way that constitutes a suitable learning activity for reaching the learning goal and revealing field orientation.

## 7.7 Conclusions

What can be learnt from this networking case for advancing the field by meta-research in the sense of the TME program?

The debate surrounding the developmental stage of mathematics education as a scientific field in the 1980s already showed contrasting views. While the analysis of Burscheid based on the model of Kuhn and Masterman indicates that a mono-theoretical view was desirable for advancing the field, Bigalke and Steiner emphasized the multi-theoretical or even the interdisciplinary character of mathematics education as a scientific field with a specific focus on complementarity, in which the practice of teaching and learning of mathematics plays a significant role. If research is used to inform the practice of teaching and learning or to address diverse cultures, multi-theoretical views may be much more useful to grasp the complex nature of the settings in the field. Such an approach could help to gain complementary knowledge to inform practice regarded from different angles, as Steiner has pointed out. In this sense, the networking of theories is a kind of meta-research and a challenging way of research practice, when added to normal research. Its purpose is to contribute to the improvement of solving problems in the field of mathematics education. For that, it is necessary to advance theoretical and methodological clarity on the one hand, and the communication among the theory cultures and among theory and practice on the other. The previously presented networking example shows that the metaphor of ‘zooming in and zooming out’ may guide research with complementary theoretical approaches of different grain sizes heuristically.

In the TME program, Steiner has elaborated a more general top-down view for advancing the field but it does not show how this program can be implemented; that is, how meta-research can be conducted in a way to advance the field. The TME program could rather serve as an orientation scheme, whereas the Networking of Theories regarded as an additional research practice provides examples of concrete meta-research showing how it improves solving problems in the field and why this kind of meta-research is useful. The Networking of Theories approach has been predominantly developed by several European researchers (see Bikner-Ahsbahs and Prediger 2014), but there are forerunners in the theory tradition of German-speaking countries, for example interesting cases of the networking of theories were presented by Bauersfeld (1992a, b) and Maier and Steinbring (1998). Advancing the field as a scientific domain as Steiner has attempted may be the byproduct of such deep and careful case-based meta-research.

The Networking of Theories strand has started to provide concrete examples for such a research practice, pointing to its benefit and being at the same time sensitive

about the difficulties and limits a multi-theoretical approach may bring with it (see for example Bikner-Ahsbahs et al. 2017).

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