Chapter 4
Mathematical Modelling

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Abstract Mathematical modelling plays a prominent role in German mathematics education. The significance of modelling problems in school and in teacher education has increased over the last decades, accompanied by various research projects. In addition, there has been a vivid discussion on the implementation of modelling in schools. This chapter gives on overview on the current state of mathematical modelling in German speaking countries. After a short summary of the development of the past years, a widespread conceptualisation of modelling competence as well as its description within the German Educational standards is presented. Ways of implementing mathematical modelling in classrooms as well as in everyday lessons but also via modelling projects are described and an example of one of these problems worked on by students of grade 9 over two years is given. Furthermore, different modelling cycles are shown and their aims and usage in different circumstances are outlined. In addition, research questions currently being discussed are addressed and an example for a quantitative research project is given.

Keywords Mathematical modelling · Modelling competencies · Modelling cycle · Implementing modelling · Research on modelling · German modelling debate

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4.1 Introduction

Mathematical modelling plays a prominent role in German mathematics education. The significance of modelling problems in school and teacher education has increased over the last decades, accompanied by various research projects. In addition, there has been a vivid discussion on the implementation of modelling in schools.

Thus, this chapter is subdivided into three parts as follows: In the first part, we outline the German discussion of mathematical modelling by presenting definitions, educational standards and modelling competencies. This chapter is based on the topical survey on approaches and developments from German-speaking countries on teaching and learning mathematical modelling (Greefrath and Vorhölter 2016). We also provide an overview on "Implementing mathematical modelling in schools" by presenting several projects of the last two decades aiming at the implementation of modelling in Germany.

In the second part, we provide a classification of modelling cycles that focuses on how these yield greater insights into the cognitive processes of learners when solving modelling problems. We also discuss the role of technology in mathematical modelling in the context of modelling cycles. In the third part, we provide an overview of some important research questions which have arisen in the German-speaking debate on mathematical modelling. In addition, we report an example of findings from a research project conducted in Germany, searching for the "best" learning environment for teaching modelling in a regular classroom.

4.2 Developments in Mathematics Modelling for Teaching in Germany

In Germany, the focus on mathematical modelling has intensified considerably since the 1980s. Earlier, in 1976, Pollak gave a talk at the ICME 3 in Karlsruhe, where he contributed to defining the term "modelling" (Pollak 1977). Different modelling cycles (for example Schupp 1989) were developed and discussed, in order to describe modelling processes and goals, as well as arguments for using applications and modelling in mathematics teaching. After subject-matter didactics (*Stoffdidaktik*) had affected mathematics education with pragmatic and specific approaches in Germany, there was a shift in the last quarter of the 20th century towards a competence orientation, focusing on empirical studies and international cooperation.

4.2.1 Background of the German Modelling Debate

In fact, the discussion of applications and modelling in education has played an important role in Germany for more than 100 years. The background to the German modelling debate at the beginning of the 20th century can be divided into a practical arithmetic approach (*Sachrechnen*) at the public schools (*Volksschule*, primary school and lower secondary school) and an approach of applications supported by Klein and Lietzmann in the higher secondary school (*Gymnasium*).

At the beginning of the 20th century, mathematics education was influenced by the reform pedagogy movement. Johannes Kühnel (1869–1928) was one of the key figures in this movement. Kühnel criticised teaching problems that were basically irrelevant and called for problems that were truly interesting and relevant for students. During this period, applications were considered to be more important for the learning process. They were used to help visualise issues and motivate the students, rather than prepare them for real life (Winter 1981).

In contrast to the practical arithmetic approach at the *Volksschule*, the formal character of mathematics was the centre of attention at the *Gymnasium*. Mathematical applications were mostly neglected. Whereas Kühnel and other educators (representing the reform pedagogy movement) had a greater influence on the *Volksschule*, Klein started a reform process in the *Gymnasium*. At the beginning of the 20th century, a more appropriate balance between formal and material education was requested, due to the impact of the so-called "reform of Merano". The main focus was on "functional thinking". In the context of Merano's reform, a utilitarian principle was propagated "which was supposed to enhance our capability for dealing with real life through a mathematical way of thinking" (Klein 1907, p. 209, translated). Because of the industrial revolution, more scientists and engineers were needed in the economy and society. This is why applied mathematics gained in importance and real-life problems were used more often. Lietzmann (1924) made some important proposals for the implementation of Merano curricula and constituted an implementation of applications in the classroom. Finally, the contents of the Merano reform in 1925 were included in the curricula of Prussian secondary schools.

This trend continued until the 1950s. In the late 1950s, Lietzmann stressed the need for stronger inner-mathematical objectives (Kaiser-Messmer 1986). After World War II, some ideas that had evolved from the progressive education movement and the reform of Merano were picked up again, but with applications losing importance. More emphasis was again placed on subject classification rather than on applications (Kaiser-Messmer 1986).

In 1976, Pollak gave a talk at ICME 3 in Karlsruhe, where he defined the term *modelling*. He pointed out that at that time, people were less familiar with how applications were used in mathematics teaching. To clarify the term, he distinguished between four definitions of applied mathematics (Pollak 1977, s. Fig. 4.1):

- Classical applied mathematics (classical branches of analysis, parts of analysis that apply to physics)
- Mathematics with significant practical applications (statistics, linear algebra, computer science, analysis)
- Single modelling (the modelling cycle is only conducted once)
- Ongoing modelling (the modelling cycle is repeated several times).

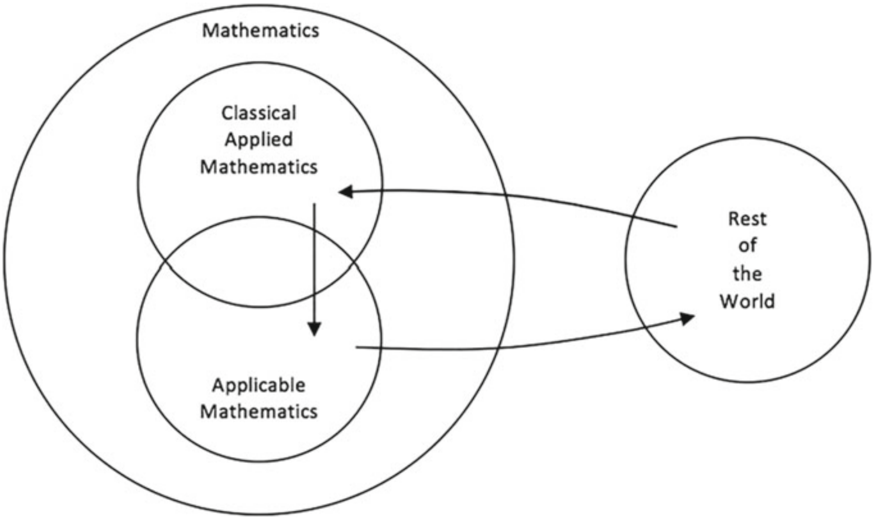

Fig. 4.1 Perspectives on applied mathematics by Pollak (1977, p. 256)

In the 1980s, the so-called New Practical Arithmetic (*Neues Sachrechnen*) evolved at all types of schools in Germany (Franke and Ruwisch 2010). The principles of the reform pedagogy movement were emphasized again and schools started to use applications in mathematics education more frequently. The New Practical Arithmetic aimed at finding authentic topics for students and to conduct long-term projects that were separated from the current mathematical topic and offered a variety of solutions. New types of question, such as Fermi problems, which are formulated as one question only and can be solved by estimating different physical quantities, were used accordingly (Herget and Scholz 1998). At the same time as the development of the New Practical Arithmetic, the term *modelling* became better known in mathematics education, and the two have partially complemented each one another. However, the main focus of New Practical Arithmetic and modelling was on different types of schools (Greefrath 2018).

The development in German-speaking countries was diverse. For example, in the field of stochastics, there were modelling approaches emphazising stochastic aspects (e.g. Eichler and Vogel 2016).

In 1991, the German ISTRON Group was founded by Werner Blum and Gabriele Kaiser. This caused an intensified debate on modelling in Germany. The idea behind ISTRON was that—for various reasons—mathematics education should focus more on practical applications. Students should learn to understand environmental and real-life situations by means of mathematics and develop general mathematical skills (e.g., transfer between reality and mathematics) and become open-minded regarding new situations. They should thereby establish an appropriate comprehension of mathematics including the actual use of the concepts. Learning mathematics should be supported relating it to real life (Blum 1993). A new series established in

1993 and published by Springer since 2014 has enabled the ISTRON Group, having already produced 20 volumes, to be present and visible in mathematics teaching, as well as in the academic community. Their contributions are intended to support teachers in dealing with real-life problems in school. Teachers are considered to be experts in teaching, so that teaching proposals should be modifiable, enabling teachers to adapt them to a specific situation. They should suggest innovative ways of teaching mathematics and support lesson preparation (e.g., Bardy et al. 1996).

4.2.2 Modelling as a Competency and German Educational Standards

Based on results from the Danish KOM project (Niss 2003) and accompanied by international comparative studies, mandatory educational standards for mathematics were introduced in Germany in 2003 (first at the non-university entrance level). Mathematical modelling is now one of the six general mathematical competencies that the education standards for mathematics regard as obligatory for intermediate school graduation. This approach can also be found in the educational standards for primary school, as well as for upper secondary school.

By means of varied mathematical content, students are to acquire the ability to translate between reality and mathematics in both directions. In the work of Blum (Blum et al. 2007), modelling skills are described in a more detailed way as the ability to adequately perform the necessary steps in the process of changing back and forth between reality and mathematics, as well as comparatively analysing and evaluating models.

It is possible to consciously divide modelling into partial processes for reducing complexity for teachers and students, and for creating suitable exercises (see Table 4.1). This view of modelling especially enables training individual partial competencies and establishing a comprehensive modelling competency in the long term. For more information on modelling competencies, we refer to the comprehensive overview by Kaiser and Brand (2015).

The German educational standards for mathematics at the secondary level for 2003—as well as those at the primary level for 2004 and for the higher education entrance qualification of 2012—describe mathematical modelling as a competency. The educational standards for the general higher education entrance qualification, for example, specify the requirements regarding the modelling competency in the three following areas:

Requirement areas of study I: Students can:

- Apply familiar and directly immediately recognisable models
- Translate real situations directly into mathematical models
- Interpret mathematical results in the context to the real situation.

Table 4.1 Sub-competencies involved in modelling (Greefrath et al. 2013, p. 19)

Sub-competency	Indicator
Constructing	Students construct their own mental model from a given problem and thus formulate an understanding of their problem
Simplifying	Students identify relevant and irrelevant information from a real problem
Mathematising	Students translate specific, simplified real situations into mathematical models (e.g., terms, equations, figures, diagrams, and functions)
Interpreting	Students relate results obtained from manipulation within the model to the real situation and thus obtain real results
Validating	Students judge the real results obtained in terms of plausibility
Exposing	Students relate the results obtained in the situational model to the real situation, and thus obtain an answer to the problem

Requirement areas of study II: Students can:

- Conduct modelling processes consisting of several steps and with a few and clearly formulated limitations
- Interpret the results of such modelling processes
- Adopt mathematical models to changing situations.

Requirement areas of study III: Students can:

- Model complex real situations for which variables and conditions need to be specified
- Check, compare, and evaluate mathematical models considering the real situation (KMK 2012, p. 17, translated).

Since 2006, an overall strategy for educational monitoring in Germany has been pursued by the Standing Conference of the Ministers of Education and Cultural Affairs. The aim is to strengthen the competence orientation within the educational system. The general modelling competency plays an important role in mathematics. In addition to international school achievement studies (PISA, TIMSS), there are national achievement studies as well as comparative studies (VERA). These tests are carried out in class in Grades 3 and 8 in all general education schools, in order to investigate which competencies students have achieved at a particular point in time. The comparative studies aim to give teachers individual feedback on the educational standards requirements that students can handle.

Beginning in 2017, a pool with audit tasks for the *Abitur* examination has been be provided for Germany, from which all states can obtain audit tasks for the *Abitur*. This was an important step in improving the quality of audit tasks and gradually adjusting the level of requirements in all states. Tasks are developed based on the educational standards. Thus, by default, some of the tasks for the *Abitur* include modelling as a competency. The use of modelling in examination problems, however, is not unreservedly viewed positively. The fact that in many cases, the relevance of the factual context used is not the focus of examination problems, has given rise to criticism on the part of some expert representatives with regard to modelling in examinations: On the one hand, there is criticism of the fact that "modelling competence" is not examined at all through the used audit tasks. On the other hand, other authors point to the categorical refusal of modelling problems also in examinations. Strong criticism is also directed against the fact that examination problems tend to contain to much text (s. Greefrath et al. 2018).

4.2.3 Implementation of Modelling in Everyday Lessons

Fostering students' modelling competence is compulsory for all mathematics teachers in all grades. But classroom observations regularly reveal only a low proportion of modelling of working on a holistic modelling task in everyday-lessons and class exams in Germany (Blum 2011). Several reasons may apply:

Modelling has been part of the national standards for only about fifteen years. Therefore, many teachers are not trained to teach modelling. Although there are many in-service teacher trainings in modelling, German teachers are not obliged to attend them. Accordingly many simply do not know how to implement modelling in everyday-lessons, how to behave during students' work on modelling problems, and generally how to support their students best. Modelling is a competence that is difficult not only for students, but for teachers as well. Because students are encouraged to develop their own models, teachers can only anticipate what students will do. They therefore have to be able to diagnose and intervene spontaneously, but often, do not feel confident in doing so (Tropper et al. 2015).

In addition, concurrently to the implementation of modelling in the national standards, state-wide comparison tests have been established in Germany. As modelling is one of the six competencies of the national standards, it is included in these tests. Thus, teachers who want to prepare their students for these tests must implement modelling in their classes to a certain extent. But, as Henn and Müller (2013) stated, most of the so-called modelling problems at school and particular in exams, are not modelling at all, according to the description of modelling problems given above. Mostly, not a complete complex modelling task is tested, but only sub-competencies of modelling. Therefore, teachers do not have to tackle entire modelling problems in their mathematics classes to prepare for the central exams.

Furthermore, teachers claim that there is insufficient knowledge about how to foster students' modelling competence best and most effectively. At first glance, this is surprising, as many studies have researched single aspects (for an overview of research results, see Greefrath and Vorhölter 2016). But clearly, until now, these findings have not been integrated in such a manner as to be useful for teachers. However, Böhm (2013) developed a theoretical approach for improving students' modelling competencies systematically and permanently. Furthermore, Blum (2015) presented—based on empirical findings—ten important aspects of a teaching methodology for modelling. In addition, there are various task collections (for example, the

ISTRON series and the collection of tasks by MUED, see Greefrath and Vorhölter 2016) that can be used as a teaching resource for modelling problems.

Summing up, there has been much research on different aspects of fostering students modelling competencies. But until now, this knowledge has not been applied in practice (at least not as much as one would wish). One indication that teachers want to implement modelling, but do not know how to do so is the fact that at least in Hamburg, it is not difficult for researchers to convince teachers to take part in research projects on modelling. Furthermore, teachers in Hamburg often wish to participate in modelling days or even weeks. These projects are introduced in the next section.

4.2.4 Implementation of Modelling via Modelling Projects

Modelling cannot only be conducted during regular mathematics instruction. In Germany, there is quite a tradition of modelling projects, carried out by different universities all over Germany and Austria. They were originally developed at the University of Kaiserslautern by the working group of Helmut Neunzert, an applied mathematician, more than twenty years ago, and their structure has been adopted by different universities. Although the aims and the target group of these projects differ, all modelling projects follow a similar structure. During modelling weeks or days, as these projects are termed, students have to work on one complex problem over a longer period, more or less on their own. The modelling problems often come from research or industry and have been simplified only slightly. Normally, these are introduced in a short presentation. Problems that have been tackled so far include:

- Pricing for Internet booking of flights
- Optimal automated irrigation of a garden
- Chlorination of a swimming pool
- Optimal distribution of bus stops
- Optimal distribution of rescue helicopters in skiing areas.

Often, the students are able to choose between modelling problems, as several problems are offered. Afterwards, according to their particular interests, they are divided into different groups.

The students are supervised either by university teachers or by university students trained as tutors. The supervisors are required to use the principal of minimal help. At the end, the students have to present their solution to an audience. Modelling projects that last roughly one week are referred to as modelling weeks and often take place outside school (usually at a university or a youth hostel), while modelling projects lasting only two or three days are called modelling days and normally take place in a school.

The aim and target group of the modelling days and weeks differ, depending on the host in question. In some cases, as in Kaiserslautern and Aachen, applied mathematicians (originally) carry out those projects. They focus mostly on introducing students to the role of mathematics in other sciences. Often, they offer their

In 2015, ca. 1,5 Mio refugees came to Germany. They are distributed in accordance to the *Königsteiner Schlüssel,* a distribution key, developed in 1949 for distributing money to the federal states of Germany. It regards 2/3 taxes and 1/3 population and is measured every year. These days, it is often claimed that the Köngisteiner Schlüssel is inappropriate for distributing refugees, but till now, no other distribution key has been developed.

Is there a better key for distributing refugees?
Develop an alternative procedure and discuss the pro‘s and con ‘s.

Fig. 4.2 Modelling problem "Distribution of refugees"

modelling projects to highly gifted or at least interested students. In other cases, as in Hamburg, Kassel and Koblenz, carrying out modelling days is only part of a whole programme. Didactical considerations, such as fostering students modelling competencies or increasing their motivation, form the focus. Normally, whole classes, regardless of their mathematical competencies, take part in those modelling days or weeks. Furthermore, in these cases, not only the students working on the modelling problems, but also those supervising them (in- and pre-service teachers) can be considered the target group. Preparation for supervising students during modelling days or weeks not only includes telling them how to behave so as to help students as little as possible, but as much as necessary concerning the special problem, but in general. This includes general knowledge about modelling and diagnosing problems, as well as intervening in such a way that the learning outcome for the students is as high as possible. To convey the general idea, in the following discussion, the procedure of modelling days, as well as student reactions and outcomes concerning a particular task, will be presented.

Since 2001, modelling days are conducted by the working group on Didactics of Mathematics of the Educational Department of the University of Hamburg. Upon consultation with participating schools, they last 2 or 3 days, directly after winter term in February. Every year, whole grade 9 classes from different schools participate, that is over 200 students and about 10 teachers. Teachers meet beforehand, are informed about the modelling problems and trained. Furthermore, student teachers were trained within a didactic seminar that focuses on teaching modelling in general; part of the seminar entails supervising the students during the modelling days. Every year, three different modelling problems are presented to the students who can choose what work they wish to do. In 2016, the problem in Fig. 4.2 was posed.

As there is a shortage of living space in Hamburg, students firstly claim the distribution key is unfair, because some larger counties (which one would assume have more living space) like Mecklenburg-Western Pomerania only had to receive the same number of refugees. They soon decided to consult further aspects for the new distribution key such as area, empty houses and vacancy. They investigated relevant data and considered how to develop a new distribution key. They measured proportions and compared the outcome for the different federal states of "their" distribution key to those of the "Königsteiner Schlüssel". Summarizing, they developed a solution for a highly relevant topic, use proportions in a real context (and not to forget how to calculate a proportion in the future) and form one's own opinion.

In contrast to the implementation of modelling in everyday lessons, there has not been much research on the impact of modelling days or weeks on students, pre-service or in-service teachers. One exception is the study by Stender (2016), that focused on the acting of teachers tutoring students while working on a complex modelling problem. He videotaped the working process (lasting 2.5 days) of 10 groups of students working on the modelling task "Roundabout versus traffic light: Which intersection allows more cars to pass through?". The tutors had been trained before on how to supervise students. The results clearly indicate the trained strategic interventions were used to a considerable extent and were mostly successful. However, tutors had different preferences; some seldom intervened, but their interventions last longer; some intervene more frequently, but gave only very short interventions. Furthermore, the intervention "Explain your work" proved to be very effective and was often used. The results indicate in addition the importance of an accurate diagnosis of the students' situation and their current motivation, as interventions were rather unsuccessful if diagnoses were not accurate. Furthermore, an inadequate understanding of the modelling situation and the mathematical situation by the tutors led to rather misleading interventions.

Although there had not been much research on this issue, modelling weeks and days were evaluated regularly, revealing great approval and good learning outcomes in various types of competencies (for more details, see Kaiser and Schwarz 2010; Kaiser et al. 2013; Vorhölter et al. 2014).

4.3 Modelling Cycles

4.3.1 Mathematical Models

The debate over the term *mathematical model* plays an important role in the research on mathematical modelling in Germany.

As the development of a mathematical model as such is crucial, the term is discussed below. A starting point for the definition of this term can be found in the publications of Heinrich Hertz.

Hertz mentions (logical) *admissibility*, *accuracy*, and *expediency* as criteria. A mathematical model is admissible if it does not contradict the principles of logical thinking. In this context, it is accurate if the relevant relations of a real-world problem are shown in the model. Finally, a model is expedient if it describes the matter with appropriate as well as relevant information. Whether a model proves to be expedient, it can only be judged in comparison with the real-life problem. It can be expressed by means of an economical model or in a different situation by the richness of relations (Neunzert and Rosenberger 1991). A new problem might require a new model, even if the object is the same. Furthermore, Hertz emphasises as a *conditio sine qua non*, that the mathematical model has to correspond to the real-life items (Hertz 1894).

The term *mathematical model* has been described in the German literature in many ways. Models are simplified representations of reality, that is, only reflecting aspects to some extent objectively (Henn and Maaß 2003). For this purpose, the observed part of reality is isolated and its relations are controlled. The subsystems of these selected parts are substituted by known structures without destroying the overall structure (Ebenhöh 1990). Mathematical models are a special representation of the real world enabling the application of mathematical methods. If mathematical methods are used, mathematical models that only represent the real world can even deliver a mathematical result (Zais and Grund 1991). Thus, a mathematical model is a representation of the real world, which—although simplified—corresponds to the original and allows an application of mathematics. However, the processing of a real problem with mathematical methods is limited, as the complexity of reality cannot be translated completely into a mathematical model. This is usually not even desired. Another reason for generating models is the possibility of processing real data in a manageable way. Thus, only a selected part of reality will be transferred into mathematics through modelling (Henn 2002).

4.3.2 Different Modelling Cycles

When looking at the literature on modelling and applications, one can find many different modelling cycles. These cycles are different from one another, as they were developed with different intentions and for different aims. When looking at different cycles, the purposes for which they were developed should always be kept in mind. The following classification shows the different aims and purposes of these cycles for research and practice (Borromeo Ferri 2006, 2018):

- Modelling cycle from applied mathematics
- Didactical or pedagogical modeling cycle
- Psychological modeling cycle
- Diagnostic modeling cycle/modeling cycle from a cognitive perspective.

4.3.3 Modelling Cycle from Applied Mathematics

In almost all books on mathematical modelling from applied mathematics, one can find modelling cycles which have one thing in common: there is a direct transition from a real problem to a mathematical one, which implies that there is no distinction between the real situation and real model, or real model and mathematical model. This has to do partly with the kind of modelling problems which are used in this context. These are mostly "realistic and complex" problems, such as from industry or economics. The complexity of real problems influences the number of phases within the modelling cycle to some extent, because there is no need to make more distinctions.

A well-known researcher in the field of modelling in general, but especially in terms of considering modelling as a way to understand the real world better, is certainly Pollak (1979), whose perspectives on applied mathematics are presented in Fig. 4.1. Pollak's ideas have influenced considerably the development of modeling cycles in research on modelling in mathematics education.

4.3.4 Didactical or Pedagogical Modeling Cycle

Mathematics lessons constitute a different situation in which modelling cycles are used. Here, cycles are used to help students to get to an understanding of the modelling process and to give them a metacognitive tool for overcoming problems. Furthermore, the implementation of a cycle within modelling lessons offers students the opportunity to reflect on what they have done while solving real problems. Furthermore, the students learn the notions of "real models" or "mathematical models" and thus reach a metacognitive level, thus promoting modelling competencies. There are various kinds of so-called didactical or pedagogical modelling cycles. However, what they have in common is that reality and mathematics are seen as two "separated worlds", also in Pollak's model. At first glance, they differ only marginally from the four steps of Ortlieb (Fig. 4.3). However, some of them additionally differentiate between real problems and real models. Furthermore, most of them differ like Ortlieb, between mathematical results and real results, which means that the interpretation of the mathematical result(s) are mentioned as a crucial part of working on a modelling problem (s. Fig. 4.4).

4.3.5 Diagnostic Modeling Cycle: Modeling Cycle from a Cognitive Perspective

During the last few decades, some researchers focused on cognitive processes of individuals during modelling processes (Borromeo Ferri 2007; Blum and Leiß

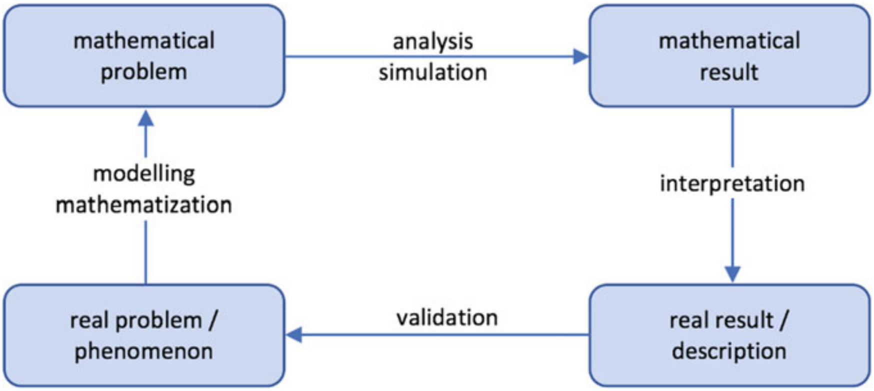

Fig. 4.3 Prototype for modelling cycles from applied mathematics (Ortlieb 2004, p. 23)

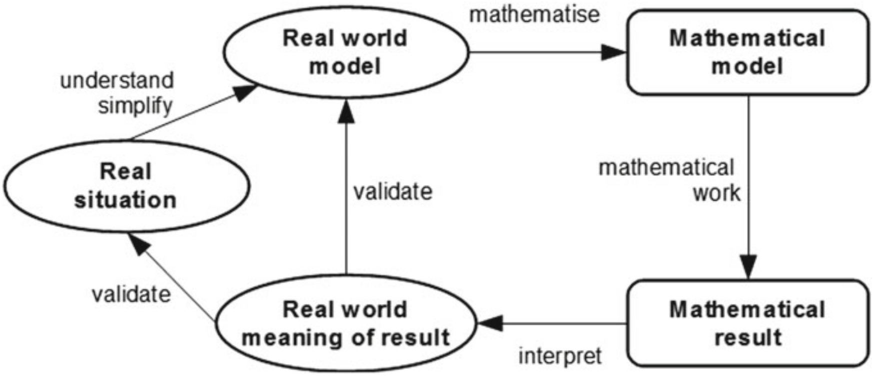

Fig. 4.4 Modelling cycle of Kaiser and Stender (2013, p. 279)

2007). Thus, a situation model was included in the modelling cycle, because researchers assumed that all individuals more or less proceed through this phase during modelling.

Blum and Leiß (2007; s. Fig. 4.5) understand the situation model in their cycle as an important phase during the modelling process. That is because they describe the transition between real situation and situation model as a phase of understanding the task. A similar approach was pursued in the COM^2-project (Borromeo Ferri 2007; see Fig. 4.6). Here, an additional phase was integrated, similar to the situation model of the modelling cycle by Blum and Leiß.

However, Borromeo Ferri used the name “mental representation of the situation” (MRS) instead of situation model, because this term focusses on internal processes through which an individual goes to obtain a corresponding mental picture while/after reading the (complex) modelling task.

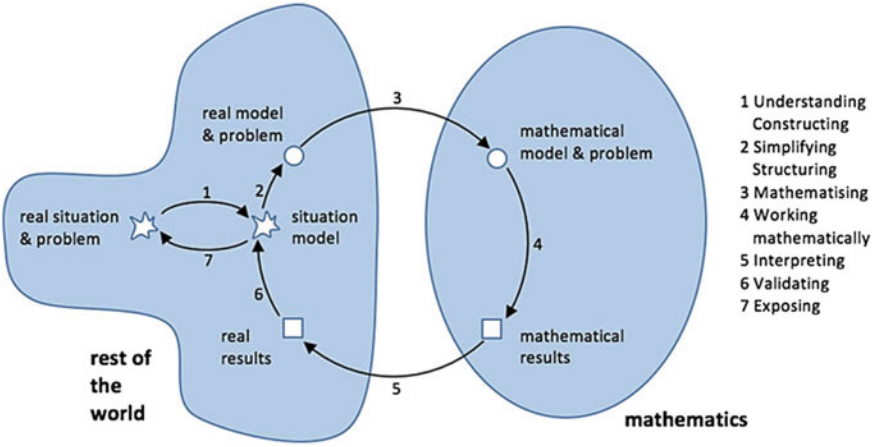

Fig. 4.5 Modelling cycle of Blum and Leiß (2007, p. 225)

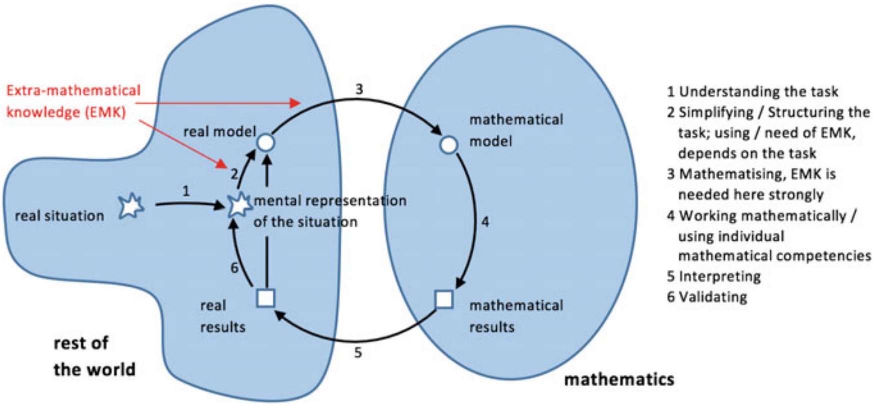

Fig. 4.6 Mathematical modeling cycle from a cognitive perspective (Borromeo Ferri 2007, p. 266)

Through the situation model and the mental representation of the situation, a cognitive view of modeling processes is provided. Thus, for diagnostic purposes, this cycle is a good instrument. If teachers are able to name and to distinguish between steps within the modeling cycle, they can diagnose possible cognitive barriers students encounter while modeling.

The theoretical construct of a situation model comes from text linguistic works and is mainly related to non-complex word problems (see Kintsch and Greeno 1985; Verschaffel et al. 2000).

> The situation model includes inferences that are made using knowledge about the domain of the text information. It is a representation of the content of a text, independent of how the text was formulated and integrated with other relevant experiences. Its structure is adapted

> to the demands of whatever tasks the reader expects to perform. (Kintsch and Greeno 1985, p. 110)

These cycles are not used in school and it was not the developers' intention to do so. However, the relevance of including the situation model in the diagnostic modelling cycle offered new paths for research and practice, and particularly for teacher education and training on mathematical modelling.

4.3.6 Modelling and Digital Tools

Possible modelling activities in mathematics teaching have changed over the last few years, mainly due to the existence of digital tools. Especially when dealing with realistic problems, a computer or an adequately equipped graphical calculator can be a useful tool for supporting teachers and students. Henn (1998), for example, suggested this early on and proposed implementing digital tools, e.g., notebooks with algebra software, because this would enable the introduction of complex applications and modelling into daily teaching (see also Henn 2007).

Currently, digital tools are often used to work on such problems, such as to process models with complex function terms or to reduce calculation effort. Digital tools can perform a range of tasks in teaching applications and modelling. One possibility for using these tools is experimenting and exploring (Hischer 2002). Simulating is very similar to experimenting. Simulations, which are experiments that use models, are intended to provide insights into the real system presented in the model or into the model itself (Greefrath and Weigand 2012).

A common use of digital tools, especially computer algebra systems, is that of calculating or estimating numerical or algebraic solutions (Hischer 2002). Without such tools, students would not be able to make these estimations, at least not within a reasonable time frame. A computer can also be used to find algebraic representations from the information given. In addition, digital tools can perform a visualisation of a subject taught at school (Barzel et al. 2005; Hischer 2002; Weigand and Weth 2002), and also play a useful role in controlling and verifying (Barzel et al. 2005). Therefore, digital tools can, for example, help with control processes for discrete functional models. If computers with an internet connection are provided for mathematics teaching, they can be used to conduct investigations (Barzel et al. 2005), e.g., in context with applications. In this way, real problems can be understood initially and simplified afterwards.

A computer's various functions can be used in mathematics education for a range of steps in the modelling cycle. Control processes, for example, are usually the last step of a modelling process. Calculations are by means of the generated mathematical model, which in analysis, for example, is often represented by a function. Digital tools can be usefully in every step of the modelling cycle (s. Greefrath 2011).

If the steps of calculating with digital tools are considered more precisely, working on modelling problems with digital tools requires two translation processes. First, the

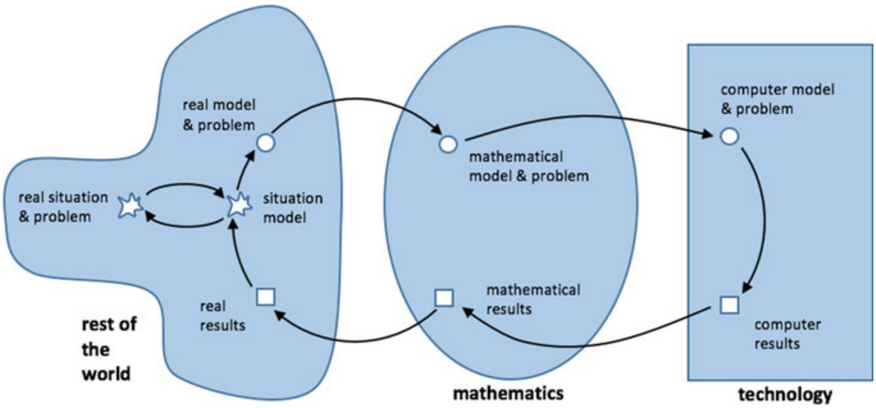

Fig. 4.7 Extended modelling cycle (Greefrath 2011, p. 302; Blum and Leiß 2007, p. 22)

modelling question has to be understood, simplified, and translated into mathematics. The digital tool, however, can only be used after the mathematical terms have been translated into the computer's language. The results calculated by the computer then have to be transformed back again into mathematical language. Finally, the original problem can be solved when the mathematical results are applied to the real situation. These translation processes can be represented in an extended modelling cycle (see Fig. 4.7), which in addition to the rest of the world and mathematics, also includes technology (Savelsbergh et al. 2008; Greefrath 2011). Current studies, however, show that actual modelling activity which includes a computer can be better described by integrating the computer in every step of the modelling cycle.

Currently, there is little empirically established knowledge about the possibilities of teaching modelling with digital tools. Open research questions can be found in the works of Niss et al. (2007). These include the following questions: How should digital tools be used in different grades to support modelling processes? What is the effect of digital tools on the spectrum of modelling problems to be worked on? How is teaching culture influenced by the existence of digital tools? When do digital tools enhance or hinder learning opportunities in the modelling process?

Additional empirical research is required to clarify the above questions, especially considering the extended modelling cycle and the necessary translation processes. Case studies (e.g. Geiger 2011) indicate though, that digital tools could be useful for each and every step of the modelling process. This is particularly true for interpreting and validating.

4.4 Research on Modelling

4.4.1 The Development of Research on Mathematical Modelling in German-Speaking Countries

With regard to Niss et al. (2007), the research activities on mathematical modelling have been characterized by three phases over the last 50 years. The so-called advocacy phase (1965–1975), in which there was little research, because there was no consensus the importance of applied mathematics in school. In the second development phase (1975–1990), there was considerable historical and theoretical research, as well as curricula and material development. The third maturation phase (1990–2005) contains the first (qualitative) empirical studies. At least in Germany—and possibly worldwide—it seems that a new phase has begun, that of consolidation (2005—today). Mathematical modelling is now (or should be) part of everyday school mathematics; research on this issue is respected by other research disciplines, so that it takes place with regard to a variety of methodological approaches. Accordingly, beside theoretical content analysis and qualitative case studies, there is a growing number of projects which use state of the art quantitative methods to obtain answers to research questions.

4.4.2 Research Questions in the Field of Mathematical Modelling

Fifteen years ago, Blum et al. (2002), on the occasion of preparing the 14th ICMI study, pointed out that there are at least nine important topics or research questions on mathematical modelling which yet waiting to be answered:

1. Epistemology: e.g. What is the nature of mathematical modelling?
2. Tasks: e.g. What kind of tasks are needed to teach/learn mathematical modelling?
3. Competencies: e.g. Which subcompetencies can be identified in the process of mathematical modelling?
4. Attitudes: e.g. What is the influence of mathematical modelling on beliefs about mathematics?
5. Curriculum: e.g. What are the main goals of lessons or units with applied problems?
6. Pedagogy: e.g. Does the learning of mathematical modelling require special teaching methods?
7. Implementation: e.g. How can mathematical modelling be implemented in everyday math classrooms?
8. Assessment: e.g. How can the performance of a modelling process be supported by adaptive feedback?

9. Technology: e.g. What implications does the use of technology have for modelling process?

Researchers in Germany followed the call of Blum et al. “Readers are invited to come up with additional relevant issues.” (2002, p. 159) and discovered several new research areas in Germany’s research landscape, for example:

10. Metacognition: e.g. How can metacognitive activities support students’ modelling processes?
11. Strategies: e.g. What is the influence of specific learning tools on mathematical modelling?
12. Language: e.g. What role do language competencies play in the modelling process?
13. …

For example, ten years ago, there were only a handful of projects in German-speaking countries which examined the process of text comprehension during the modelling process. Along with an awareness of the important role of language for mathematical solution processes (Paetsch et al. 2016) there is now a growing community of researchers in this field of interest. Topics range from the role of the mental model of a given task, to the teaching language, language disadvantages or linguistic modifications of tasks (e.g. Leiß et al. 2010; Prediger et al. 2013; Haag et al. 2015). In addition to this thematic extension, there has also been a considerable methodological development. This might be one reason (beside others like national educational standards) for changes in the kind of research projects on mathematical modelling in the last few decades. Research projects in modelling more often used experimental-control-group designs and sophisticated statistical methods for analysing the various research questions.

4.4.3 Multiple Solutions: An Example of Quantitative Research on Modelling

One example of quantitative research on modelling is research project MultiMa (Multiple Solutions for Mathematics Teaching Oriented Toward Students’ Self-Regulation). One starting point of MultiMa project entails expectations about the importance of constructing multiple solutions for learning mathematics. However, there is a lack of empirical evidence on positive effects of this teaching element on student learning in general and on modelling competency in particular. Thus, the aim of the above project was to investigate the effects of encouraging students to find multiple solutions while solving modelling problems and the impact on performance, motivation, emotions and strategies.

The theoretical analysis of student solution processes has enabled distinguishing between three types of multiple solutions for modelling problems: (1) multiple solutions that occur due to different assumptions about missing information and lead

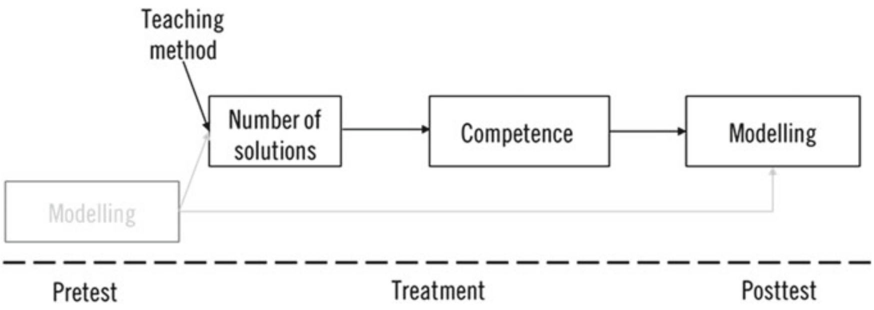

Fig. 4.8 Hypothesised path analytic model (Schukajlow et al. 2015)

to different mathematical results, (2) multiple solutions through applying different mathematical procedures that lead to the same mathematical results and (3) a combination of the first and second type of multiple solutions (Schukajlow et al. 2015). The main part of the first stage of the project was a randomized interventional study aimed at comprising the effects of two teaching methods: encouraging students to find multiple solutions for modelling problems versus encouraging students to find just one solution for modelling problems. On the basis of motivational, emotional and cognitive theories, we assumed that the positive effects of intervention on students affect performance and strategies, and we hypothesised how encouraging multiple solutions affects dependent variables, using path analytic models. We hypothesised, for example, that prompting students to construct multiple solutions will affect the number of solutions they develop and the number of solutions will in turn affect student experiences of competence during teaching units, and their experiences of competence will affect performance (see Fig. 4.8).

In order to examine our hypothesis, we assigned students from 6 classes to two treatment conditions according to a specific procedure. Students in both conditions worked for five lessons long on modelling problems. Before, after and during their lessons, they completed questionnaires. Moreover, before and after this teaching unit, they solved tests on modelling and intra-mathematical performance. In both conditions, we used similar problems and the same type of cooperation script (so-called "individual work in groups"), which was evaluated positively in the DISUM-project. One sample problem is Parachuting (Fig. 4.9).

In the multiple-solution condition, each student was required to find two solutions to the Parachuting problem: "Find two possible solutions. Write down both solution methods." In the one-solution condition, the problem was modified by providing all important data for solving the problem and by changing the question to "What distance does the parachutist cover during the entire fall, if a wind of *medium* power blows? Write down your solution method." In the multiple-solutions condition, all students used the Pythagorean Theorem as a mathematical procedure. Solutions differed, for example, in assumptions about the wind speed in each falling stage.

Contrary to our expectations, we did not find (total) positive effects of encouraging students to construct multiple solutions on student performance. However, we find

Parachuting

When "parachuting," a plane takes jumpers to an altitude of about 4,000 meters. From there, they jump off the plane. Before a jumper opens his parachute, he free falls about 3,000 meters. At an altitude of about 1,000 meters, the parachute opens, and the sportsman glides to the landing place. While falling, the jumper is carried off target by the wind. Deviations at different stages are shown in the table below.

Wind speed	Side deviation per thousand meters during free-fall	Side deviation per thousand meters while gliding
Light	60 meters	540 meters
Medium	160 meters	1,440 meters
Strong	340 meters	3,060 meters

What distance does the parachutist cover during the entire jump?

Fig. 4.9 Modelling problem parachuting

indirect effects of the teaching method on performance, and confirmed the hypotheses that were formulated in hypothesised path analytic model. This result indicated that students who developed multiple solutions and feel competent, benefit from this teaching method.

4.5 Conclusion and Summary

Teacher requests for modelling projects, teacher training on the implementation of modelling as well as a willingness to participate in modelling research projects, all clearly indicate that teachers do attempt to implement modelling in their classrooms. So, the efforts of the last few years do seem to have had a positive effect. Yet, research on how to foster students' modelling competencies best, teacher training on how to support students solving modelling problems, as well as research on the effects of modelling days and weeks, should be intensified. Furthermore, in order to disseminate modelling to a wider audience and a greater extend, modelling tasks should be part of the task pool for the German School leaving examination (Abitur). An initial analysis of the available pool of items shows a rather low proportion of modelling (Greefrath et al. 2018). Thus, developing tasks that are appropriate for testing modelling competences within exams is one of the important open issues for the near future.

As presented above, modelling and applications were and remain an important part of German debate on mathematics education. In the last century, the German debate on modelling focused on conceptual aspects and exemplarily modelling problems. This was an important step in clarifying the content of the concept *mathematical model*. During this time, a discussion on different types of models and modelling examples in the light of a long German tradition of applications in school mathematics took place. An important step in bringing research and school practice closer together, and integrating modelling examples into the classroom, was the establishment of the German-speaking ISTRON group 25 years ago. A new development in integrating applications and modelling in all types of schools started at the end of the 20th century. A much-debated issue is the adaptation of a particular modelling cycle for a particular research question. This development led to a greater internationalisation of German research on modelling, and the integration of modelling as a competency into the curriculum at the beginning of the millennium. Currently, modelling is part of the German educational standards. However, as in most countries, applications and modelling play only a small role in everyday teaching. The presented empirical results reveal the main foci of research on modelling applications over the few last years. At presents, the effective promotion of student modelling competencies is the core of research. Concurrently, instruments for helping students to work on modelling problems independently (and relieving teachers in various ways) are being developed and analysed.

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