Chapter 1
Educational Research on Mathematics—A Short Survey of Its Development in German Speaking Countries

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Abstract In German speaking countries, educational thinking and theorizing on mathematics teaching and learning originated with the establishment of compulsory education for all children and the creation of a school system. Though first efforts go back to the 18th century it does make sense to start this survey with the beginning of the 19th century, with the implication that educational research on mathematics has a history of about two hundred years in German speaking countries. During the 19th century a more and more sophisticated system of publication (journals and books) on mathematics education emerged, the education of mathematics teachers had become more professional and teacher training had developed into one of the main obligations of university teaching. However, didactics of mathematics as an academic discipline is a comparably new achievement. Its establishment began approximately fifty years ago, predominately by creating professorships and opportunities of graduation at universities. After a phase of broad discussion on the identity of the discipline (e.g., in a special issue of ZDM edited by Steiner, 1974), the community of didactics of mathematics steadily expanded, diversified and developed fruitful connections to other neighboring disciplines. This overview intends to outline this development with respect to intuitions, key ideas, research strategies and the connection between research and practice. Selected topics are presented in the following chapters in more detail.

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Preliminary Remark
A word on terminology is in order. We are aware that the term 'didactics of mathematics' can sound strange to a native speaker of the English language. If it is used at all the term provokes rather narrow connotations (such as 'authoritarian lecturing') that do not meet the broad meaning of 'mathematics education' (in research, development and practice) which the German 'Didaktik der Mathematik' is intended to convey. The latter term is deeply immersed into the German system, and in quite a number of other countries a variation of it is used with a meaning similar to the one which is common in German speaking countries. Thus, we decided to use 'didactics of mathematics' in this chapter.

1.1 Early Developments

1.1.1 The Century of Humboldt

In 1810 during the course of the Humboldtian educational reforms, mathematics became a full-fledged university discipline and at the same time a school subject (Jahnke 1990, p. 333ff.; Schubring 1987). The teaching of mathematics at schools would no longer be confined to imparting practically useful arithmetic skills as had been the case in the 18th century. Rather, Wilhelm von Humboldt and other reformers of that period considered mathematics as a constitutive part of 'Bildung', which is the formation of a person's personality. The intellectual activity of an educated human being was not supposed to be determined by prescribed rules. Rather, an autonomous person was expected to be able to guide him- or herself by autonomous ideas. Accordingly, the reformers were strongly convinced that pupils should learn to understand things by their own means and from within themselves ('organic thinking'). In this regard, mathematics was considered an especially important subject.

What resulted from this general attitude was a pronounced *anti-utilitarianism* (Blankertz 1982, p. 95ff.). This becomes clear in some of Humboldt's educational writings where he clearly expressed an attitude against everyday applications and a preference for pure mathematics. According to him, education was to be developed to ensure "that understanding, knowledge and intellectual creativity become fascinating not by external circumstances, but rather by its internal precision, harmony and beauty. It is primarily mathematics that must be used for this purpose, starting with the very first exercises of the faculty of thinking." ["… dass das Verstehen, Wissen und geistige Schaffen nicht durch äussere Umstände, sondern durch seine innere Präcision, Harmonie und Schönheit Reiz gewinnt. Dazu und zur Vorübung des Kopfes zur reinen Wissenschaft muss vorzüglich die Mathematik und zwar von den ersten

Uebungen des Denkvermögens an gebraucht werden."] (Humboldt 1810, p. 261). Elsewhere, he opposed the tendency "… of digressing from the possibility of future scientific activity and considering only everyday life. … Why, for example, should mathematics be taught according to Wirth, and not according to Euclides, Lorenz or another rigorous mathematician? Any suitable mind, and most are suitable, is able to exercise mathematical rigour, even without extensive education; and if, because of a lack of specialized schools, it is considered necessary to integrate more applications into general education, this can be done particularly toward the end of schooling. However, the pure should be left pure. Even in the field of numbers, I do not favor too many applications to Carolins, Ducats, and the like." ["…sich selbst von der Möglichkeit künftiger Wissenschaft zu entfernen, und aufs naheliegende Leben zu denken. Warum soll z. B. Mathematik nach Wirth und nicht nach Euclides, Lorenz oder einem andern strengen Mathematiker gelehrt werden? Mathematischer Strenge ist jeder an sich dazu geeignete Kopf, und die meisten sind es, auch ohne vielseitige Bildung fähig, und will man in Ermangelung von Specialschulen aus Noth mehr Anwendungen in den allgemeinen Unterricht mischen, so kann man es gegen das Ende besonders tun. Nur das Reine lasse man rein. Selbst bei den Zahlverhältnissen liebe ich nicht zu häufige Anwendungen auf Carolinen, Ducaten und so fort."] (Humboldt 1809, p. 194, our translation).

In the beginning of the 19th century, Prussia established a school system consisting of elementary schools ('Volksschulen') for all children and secondary schools ('Gymnasium') in order to prepare students for university studies, civil service careers, outstanding positions in commerce, or industry. On a level between 'Volksschule' and 'Gymnasium', so-called 'Höhere Bürgerschulen' ('Higher schools of citizens') emerged which provided an education of practical skills in craft and commerce. During the entire 19th century, the (male) students attending Gymnasium comprised only 7% of the school age cohort (Jahnke 1990, p. 7). Teachers at gymnasium were trained at universities and had to engage in purely scientific studies, without any didactical elements. Teachers at the elementary schools were trained at newly founded seminars of teacher training ('Lehrerbildungsseminare'). In the 1860s, a separate system of schools with a stronger emphasis on science and mathematics ('Realschulen') in addition and parallel to gymnasium was established. To avoid terminological confusion: These 19th century 'Realschulen' covered like gymnasium grades from 5 to 13, whereas in Germany the modern term 'Realschule' refers to schools covering grades from 5 to 10. In regard to the social functions of the different types of schools the reader may consult Chap. 5 of this book.

Following the Swiss pedagogue J. H. Pestalozzi (1746–1827), outstanding educators as E. Tillich (1780–1807), F. A. W. Diesterweg (1790–1866) and W. Harnisch (1787–1864) advocated didactical ideas according to which pupils at elementary schools should reach a profound and reflected understanding of arithmetic and geometry, and not only be trained by drill and practice. Harnisch coined the concise phrase that pupils should "calculate by thinking and think by calculating" ["denkend rechnen und rechnend denken"] (Simon 1908, p. 22).

Diesterweg stated as a general principle: "Teach in such a way that the self-directed activity of the pupil is developed as far as possible." ["Unterrichte so, dass überall die Selbsttätigkeit des Schülers möglichst ausgebildet werde."] (Diesterweg 1844, III/IV). He favored mental arithmetic in particular, since it is in this domain that students can develop individual and flexible strategies of calculation. "However, in mental arithmetic is much more freedom which allows individual activity, decision and discretion. For this reason mentally agile children love mental arithmetic so much. They like treating a task in multiple ways and in their own manner. … Therefore, by way of exercises in mental arithmetic [the teacher] should strive for the unleashing and liberation of the young mind by as many different solution methods as possible." ["Beim Kopfrechnen dagegen. herrscht viel mehr Freiheit, welche eigene Bewegung, Auswahl und Belieben zulässt. Darum lieben geistig bewegliche Kinder so sehr das Kopfrechnen. Es gefällt ihnen, eine Aufgabe in mannigfacher Art, auf ihre Weise zu behandeln. … Darum strebe man ja an den Kopfrechenaufgaben durch möglichst mannigfache Auflösungsweisen die Entfesselung und Befreiung des jugendlichen Geistes an."] (l.c., IX) For Diesterweg as for Harnisch, mental arithmetic was not a matter of memorizing but of autonomous thinking. Diesterweg coined the artificial word 'Denkrechnen' as a compound of 'thinking' and 'calculating'. Looking at the concrete arithmetic exercises in Diesterweg's handbook the modern reader will easily realize the omnipresence of the 'operative principle'.

Throughout his lifetime, Diesterweg fought against the subordination of elementary schools under the control of the (Protestant) church. In Prussia, this was a completely legal position, but after 1840 conservative forces became dominant and Diesterweg did not hesitate to warn in his handbook against the 'forces of darkness'. "Though, at first, other fields than the teaching of arithmetic will be threatened by the undeniably existing … reaction; but also it [arithmetic] would be touched upon when the reaction against … the self-directed activity of the subject … [i.e.] … in short the school of Pestalozzi … would be victorious." ["Zwar werden zuerst andere Gebiete als der Rechenunterricht von der unleugbar eingetretenen … Reaction bedroht; aber auch er würde an die Reihe kommen, wenn die … gegen die Selbsttätigkeit im Subjecte … kurz: gegen die Pestalozzische Schule gerichtete Reaction den Sieg davontragen sollte."] (l.c., XV)

At the secondary schools (gymnasium and later Realschulen) on the other hand, in accordance with Humboldt's ideas, the teaching of mathematics had a stronger scientific character and teachers considered themselves to be scientists. Standard topics throughout the 19th century were Euclidean elementary geometry, trigonometry, spherical trigonometry and a strand of arithmetic, algebra and elementary analysis. The latter comprised some simple infinite series, but no differential and integral calculus. Within the framework of the general regulations by the state, teachers of mathematics developed multifarious didactical ideas which were published and discussed in textbooks, school programs, journals, and at meetings. Thus, a rich culture of reflection about the teaching of mathematics evolved.

As an important guideline for structuring the arithmetic and algebraic part of the curriculum, the principle of the stepwise extension of number domains from the natural numbers through negative and rational to the real numbers emerged (Jahnke 1990, p. 405ff.). For the first time, this idea was systematically exposed in the 'Perfectly consequential system of mathematics' by Ohm (1822). Later, the underlying principle of adjoining the 'new' numbers to the respective 'old' domain while maintaining the laws of operation was called 'principle of permanence' and received its final formulation in 1867 by mathematician H. Hankel. This principle also served as a guideline for the treatment of the number domains in secondary schools within the new math reform in the 1960s (see Chap. 3).

In the middle of the 19th century, in the wake of the revolutionary events of 1848, the Prussian government was convinced that too much education of the lower classes was politically dangerous. In 1847, the Prussian government removed Diesterweg from his position as director of the Berlin seminar of teacher training, and in 1854 the government officially cut down the teaching of arithmetic and geometry at elementary schools to a simple training of elementary skills ('Stiehl's regulatives').

At gymnasium, the teaching of mathematics experienced a cultural and mathematical loss of meaning which called for new ideas. Also, the emerging system of 'Realschulen' changed the role of mathematics education. Thus, after 1860 a climate in favor of reforming education developed. Specialized journals on the teaching of mathematics and science were founded:

- *Archiv der Mathematik und Physik.* This journal appeared from 1841 to 1920 and was dedicated to the special needs of teachers at higher schools.
- *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht: ein Organ für Inhalt, Methode und Organisation des Unterrichtes in den exakten Wissenschaften an allen Schulgattungen*, (Journal for teaching of mathematics and exact sciences: organ for content, method and organisation of the teaching of the exact sciences at all types of schools). The journal appeared from 1870 to 1940.
- *Unterrichtsblätter für Mathematik und Naturwissenschaften: Organ des Vereins zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts* (Paper for the teaching of mathematics and science: organ of the society for the promotion of the teaching of mathematics and science). The journal appeared from 1895 to 1943.

From the 1880s, German books on methods of teaching mathematics at gymnasium began to appear. These included *Die Methode des mathematischen Unterrichts: nebst Proben einer schulmäßigen Behandlung der Geometrie* by T. Wittstein (1879); *Anleitung zum mathematischen Unterricht an höheren Schulen* by F. Reidt (1885); and *Didaktik und Methodik des Rechnens und der Mathematik. Zweite, umgearbeitete und vermehrte Auflage* by M. Simon (1908).

Beyond that, articles on mathematics education appeared in general journals of pedagogy.

In regard to mathematical topics covered in school, two important developments emerged by the end of the 19th century. First of all, Euclidean elementary geometry was questioned from two sides. Some teachers and mathematicians recommended the introduction of elements of transformational and projective geometry (c.f., Krüger 1999, Sects. 3.2 and 5.3). A famous textbook representing this trend was Henrici and Treutlein (1881–1883). P. Treutlein spoke for the introduction of a course on propaedeutic geometry comprising an intuitive study of figures and solid bodies. Only after such an intuitive phase Euclidean geometry was to be systematically studied

(Treutlein 1911). In 1877, a second development started with a talk by the famous physiologist E. Du Bois-Reymond (1818–1896) who pleaded for the introduction of analytic geometry into gymnasium. And in fact, analytic geometry became a topic in the gymnasium syllabus in 1892.

1.1.2 'Reformpädagogik' and Its Influence on the Teaching of Mathematics

'Reformpädagogik' was an educational theory favoring the promotion of the child's creativity. It originated at the beginning of the 20th century and resulted in a new attitude towards the child as well as a new understanding of 'Bildung', which in part continued Humboldtian ideas. From its very beginnings, this reform movement was international (Blankertz 1982, p. 255ff.). The Swedish pedagogue E. Key proclaimed the *century of the child*, the Italian M. Montessori considered the child as a *constructor of its self*. Teaching was to be based and focused on the child and its creative powers, thus letting the child evolve its individual personality.

The new attitude towards the child was furthered by the emergence of cognitive psychology. Gestalt psychology (M. Wertheimer) and later, from the 1920s onward, the research of J. Piaget and L. S. Vygotsky considered the thinking of children a quality of its own.

The reform movement influenced the teaching of arithmetic at elementary schools. Mathematics educators like J. Kühnel (1869–1928) (cf. Schmidt 1978) and J. Wittmann (1885–1960) created concepts that replaced the supposedly predominant style of teaching based on passive reception and drill by teaching environments in which pupils could independently work with adequate material and discover number relations by themselves.

In a similar vein new pedagogical ideas emerged for students of the upper grades. Especially prominent was the movement in favor of the so-called 'Arbeitsschulen' ('working schools'). The meaning of this term, however, differed considerably among its proponents. On the one hand, Georg Kerschensteiner (1854–1932) was of the opinion that self-determined *manual* activity be the most important means for educating self-confident personalities, whereas Hugo Gaudig (1860–1923) based his educational philosophy on the ideal of a 'free *intellectual* activity' (Führer 1997, pp. 20–45; Oelkers 1996). With their emphasis on the concepts of 'activity' and 'self-confident personalities' both authors pursued ideas similar to those of the earlier Humboldt time. Kerschensteiner and Gaudig had developed their pedagogical ideas in the decade after 1900, but the movement in favor of 'Arbeitsschulen' became important only in the 1920s when their writings influenced also teachers of mathematics and lecturers at teacher training institutions and universities.

1.1.3 New Ideas on the Teaching of Mathematics at Gymnasium

By the end of the 19th century and into the beginning of the 20th, some professors of mathematics began to offer special courses on school mathematics for future teachers at gymnasium. Among them were R. Baltzer (1818–1887), H. Weber (1842–1913) and F. Meyer (1856–1925). Above all, the activities of F. Klein (1849–1925) had a strong influence, which persists until today: He proposed the concept of function as a new guiding principle of school mathematics and engaged himself in the introduction of infinitesimal analysis into school mathematics. In 1911, Klein made possible the habilitation[1] of the mathematics teacher R. Schimmack, which was the first such qualification in the field of 'Didactics of mathematics' in Germany. A second habilitation in 'Didactics of mathematics' was that of Hugo Dingler at the Technical Institute of Munich in 1912. In general, habilitation in didactics of mathematics at university remained controversial up to the 1980s. Beyond this, Klein was also influential in developing international relations between people who were interested in the teaching of mathematics which, among others, resulted in the foundation of the International Commission on Mathematical Instruction (ICMI). These international initiatives, however, were severely set back by the two world wars and the political development in Germany during the fascist period.

In 1925, the year of Klein's death, infinitesimal analysis was officially introduced into the teaching of mathematics at gymnasium and Realschule, thus finally realizing Klein's reform initiative. During the period between the wars and for a long time after the Second World War, didactical conceptions of mathematics at gymnasium were heavily influenced by W. Lietzmann (1880–1959), a doctoral student of D. Hilbert, then teacher of mathematics and physics, later head of a so-called 'Oberrealschule' in Göttingen (today 'Felix-Klein-Gymnasium'), and since 1910 a close collaborator of Klein.

Above all, Lietzmann's influence on the teaching of mathematics at gymnasium was established by his book, *Methodik des mathematischen Unterrichts. 2. Teil: Didaktik der einzelnen Unterrichtsgebiete* (Lietzmann 1916), which appeared in several editions until the 1960s (for an analysis see Kaiser-Meßmer 1986a). Another textbook on the teaching of mathematics at gymnasium was *Didaktik des mathematischen Unterrichts* written by the Austrian philosopher and pedagogue Alois Höfler (1853–1922) (Höfler 1910).

In the 1920s and in the tradition of Klein, mathematicians O. Toeplitz (1881–1940) and H. Behnke (1898–1970) engaged in improving the teaching of mathematics at gymnasium. Toeplitz worked on a concept of genetic teaching (see Toeplitz (1927), translated into English in Fried and Jahnke (2015)); Behnke created a seminar for furthering the cooperation between universities and schools. In 1932, the two of them founded the journal 'Mathematisch-physikalische Semesterberichte'.

[1]The term 'habilitation' is used for a formal post-doctoral qualification for university teaching, and is an essential key towards professorship in many European countries.

With regard to elementary education the 1920s brought two important innovations. The new constitution of Germany, proclaimed in 1919, stipulated that all children, independent of their social and religious background, had to attend elementary school ('Grundschule') for the first four years of their school career (Blankertz 1982, p. 232). This was the first type of comprehensive school in Germany and for the teachers at these schools it implied an enormous strengthening of their professional position. Consequently, it was felt that teachers at elementary schools should receive better training. Some of the German states reacted to this requirement, but not all. For example, in 1925, Prussia abandoned the former seminars for training teachers of elementary schools and replaced them by so-called 'Pädagogische Akademien' (pedagogical academies) which required as entrance qualification the 'Abitur'. In 1930 Friedrich Drenckhahn, who for one year had been the personal assistant of Felix Klein, was the first to get a professorship at such an academy with the dedication 'Didaktik der Mathematik' (didactics of mathematics) (Schubring 2016, p. 9). Later, in the 1950s, Drenckhahn was to play a considerable role in further upgrading the Pedagogical Academies to 'Pädagogische Hochschulen' ('Colleges of Education'). In regard to Drenckhahn's ideas on didactics of mathematics see Hefendehl-Hebeker (2016) and Chap. 2 of this book.

1.2 The Establishment of Didactics of Mathematics as an Academic Discipline

With the expansion of the university system about 50 years ago, the establishment of didactics of mathematics as an academic discipline began to develop. Different factors were involved, in particular, the 'sputnik-shock' (i.e. the unexpected success of the Soviet Union to launch a satellite) mobilized political efforts directed towards the educational system. They were based on the assumption that a scientific modernization of the subject matter, especially in gymnasium, could contribute to the enhancement of the competitive ability of a nation. This development encouraged tendencies in many of the states of the Federal Republic of Germany to provide training courses in mathematics education, even for teachers at gymnasium, and to establish corresponding chairs at universities.

In general, teacher education began to develop towards academic standards. At around 1980, many teacher-training institutions (Pädagogische Hochschulen) were integrated into the universities. Those teacher-training institutions, which remained autonomous, gradually developed into universities of education with corresponding research and graduation opportunities. At the same time, the development of a community of people in mathematics education was supported by the foundation of a society of German-speaking mathematics educators in 1975 (Gesellschaft für Didaktik der Mathematik, GDM) and a research journal (Journal für Mathematik-Didaktik), which first appeared in 1980.

Thus, basic research included different types of schools and linked different institutions. Furthermore, international cooperation of math educators became possible. The works of J. S. Bruner, J. Piaget and L. S. Vygotsky initially defined a common frame of reference for the international discussion. In this context, the Institute for Didactics of Mathematics (IDM) in Bielefeld, which was founded in 1973, gave essential impetus.

The further development of didactics of mathematics in Germany from these origins was determined by an interaction between the efforts of educational policies and internal impulses. Within the scientific community of didactics of mathematics, research approaches that have been developed and used in related academic domains played an important part. They led to a progressive widening of perspective upon all determinants, which influenced the process of teaching and learning, including the individual mental states of the teachers and learners.

In the German Democratic Republic, the development of research on mathematics education was strongly influenced by activity theory developed in the Kharkov School of Developmental Psychology (e.g., Leont'ev 1978; Galperin 1989). Particular important is the work of J. Lompscher in the fields of educational psychology and psychology of learning activities, which emphasizes "the psychic regulation of the learning activity" (Lompscher 1984, p. 40f) conceptualized as the interrelation of learning goals, cognitive and social learning motives, and the process structure of learning actions and learning outcomes. Although activity theory has become canonical in international research on the teaching and learning of mathematics, Lompscher's elaboration of activity theory and learning activities has received only minor attention within German-speaking countries after 1990 (but cf. Bruder and Schmitt 2016).

1.3 The New Math Reform and Its Consequences

In the middle of the 1960s, the New Math movement led to a comprehensive conceptual system of school mathematics with a high level of formalization.

The 'modernization of mathematics teaching' was significantly influenced by Bourbaki, a group of French mathematicians who, as is well known, pursued to reorganize the entire field of mathematical knowledge on a structural-logical basis. In Bourbaki's approach the concepts of set, relation and composition became fundamental. Some mathematics educators and influential mathematicians adopted this approach to school teaching, and elements of logic and set theory were implemented in mathematics teaching from the primary level upwards. This process was part of an international phenomenon (Kilpatrick 2012).

An important motive of the reform was the idea that mathematics education should be science-oriented from the very beginning. Thus, at the secondary schools the treatment of the number domains was consequently organized according to the principle of permanence, and enriched by structural aspects and concepts (for example the concept of the 'ordered field of rational numbers'). There was a clear succession, which more or less continues to the current day: natural numbers in grade 5, fractions in grade 6, negative numbers in grade 7, and irrational numbers in grade 9. Algebra and functions were integrated into this setting. The algebra of equations experienced a rigorous extension and reorganization by concepts of logic and set theory, variables were uniquely conceived as placeholders. This reorganization had conflicting effects. On the one hand, it achieved a modernization by considering equations and systems of equations with different cases of solvability (empty, finite or infinite sets of solutions). On the other hand, the logical and set theoretical frame required additional efforts, which were experienced as superfluous complications by the learners (Andelfinger 1985). On the whole, the reform underestimated the epistemological problems entailed by the high level of formalization (Damerow 1977).

Basic elements of logic and set theory were also implemented in the primary schools, which entailed vigorous public discussions. Thus, in the beginning of the 1970s the guidelines for primary schools were revised by the Standing Conference of the Ministers of Education and Cultural Affairs (Kultusministerkonferenz—KMK), and thus the focus of the teaching of mathematics at primary schools shifted. The considerable influence of H. Winter (1975) strengthened the awareness that mathematics teaching should meet requirements from different perspectives, from the science of mathematics, the demands of society, the dispositions of the learners, and from their right of self-realization.

With respect to content, weights were shifted: set theory was no longer considered as an appropriate foundation for the development of number concepts and number operations at school, rather the number concept was considered and taught in its entire complexity (Müller and Wittmann 1984, s. 154). The predominance of arithmetic was restored, however, in a different shape than in traditional teaching, and enriched by further topics (combinatorial counting, algebraic and number theoretical considerations). Geometry as an extended study of spatial phenomena, such as multiple applications (magnitudes, applied arithmetic, stochastic situations), also played an important role. With respect to the learning process, Winter pointed out that mathematics teaching should not only impart content knowledge to the learners in some way or another, but combine contents and general objectives of mathematics teaching by stimulating typical mathematical activities such as exploring, ordering, systematizing, generalizing, formalizing and reasoning. In this context he emphasized the importance of heuristic strategies and discovery learning (Winter 1989). He also introduced the concept of 'productive exercises' related to substantial mathematical problems, which could stimulate exploring, discovering and exercising at once (Winter 1984).

Initially, the newly established concept of mathematics teaching at secondary schools was mainly subject-orientated. 'Didactically oriented content analysis' was developed as a tool for research in didactics of mathematics, resulting from the ambition for solid foundations and conducted with the aim to present the contents in a way that is compatible with the standards of the field and at the same time appropriate to the learners and the requirements of teaching (Griesel 1974).

To begin with, the emphasis was on the lower secondary school level, especially in the domain of algebra and arithmetic, complemented by an analysis of the concept of function (Vollrath 1974). A special problem was the attempt to find exact foundations for the relations between numbers and quantities in primary and secondary teaching. The term ‘domain of magnitudes’ was coined to denote the structural basis of popular reckoning (Kirsch 1970; Griesel 1997).

Kirsch also presented far-reaching analyses of the foundations of proportional reasoning as well as of linear and exponential growth (Kirsch 1969, 1976a). He pointed out that the structural basis of proportional reasoning was a simple mapping between two divisible domains of quantities following an elementary rule: to the n-fold quantity in the one domain is assigned the n-fold quantity in the other domain. From this starting point he developed further basic rules, for example ‘the sum is assigned to the sum’, and demonstrated that the mathematical foundations of proportional reasoning can be considered as a small axiomatic theory and its application. In subsequent stages of reasoning and by making use of the isomorphism mentioned above, he finally developed the concept of linear transformation.

The focus in geometry was determined by the ambition to organize geometry by a unifying concept corresponding to the concept of function in arithmetic and algebra (Struve 2015). Thus, didactically oriented content analysis in geometry was mostly centered on transformation geometry (see for example Holland 1974/1977; Schupp 1968; Bender 1982). The supporters of an implementation of transformation geometry in school curricula tried to find convincing arguments for its superiority: a dynamic approach, which was considered to have more clarity and conformity to the disposition of young people than the traditional Euclidean style; the possibility to achieve a clear structure within the geometric school material and to progress stepwise from elementary kinematic considerations to logically strict reasoning, better opportunities to apply heuristic strategies. However, teachers remained reluctant and discrepancies persisted between the official syllabuses and the teaching practice. In a detailed analysis, Bender detected discrepancies between the elementary concept of moving figures based on kinematic experiences, and the abstract concept of mapping (Bender 1982). Towards the end of the 1970s, the general orientation towards logic and set theory was reduced and classical considerations of geometrical figures returned to the center of attention.

After the initial concentration on the primary and lower secondary levels, the didactically oriented content analyses were extended to the domains of upper secondary school teaching. Here the contents already had a solid scientific foundation and the problems were mostly opposite to those with respect to the lower stages. The question was how mathematical theories and concepts could be simplified and made accessible without falsifying the central mathematical content. W. Blum and A. Kirsch suggested more intuitive approaches (at least for basic courses) with the original naïve ideas of function and limit and sequential steps of exactitude, which could be achieved according to the capacity of the learners (Blum and Kirsch 1979; Kirsch 1976b).

A general goal was to develop concepts with which to represent mathematical knowledge in a way that corresponds to the cognitive ability and personal experience of the students, while simultaneously simplifying mathematical material without distorting it from its original form, with the aim of making it accessible for learners

(Kirsch 1977). The simplifications introduced into mathematical subjects should be 'intellectually honest' and 'upwardly compatible' (Kirsch 1987). That is, concepts and explanations should be taught to students with sufficient mathematical rigor in a manner that connects with and expands their knowledge of the subject.

As these analyses initially were conducted without the accompaniment of systematic empirical research, an increasing critical discussion arose, where the German word 'Stoffdidaktik' (material related didactics) was coined in order to denote the limitations of this approach.

1.4 Widening the Perspective

The didactically oriented content analysis as a research method was strongly related to the teaching methods of the 1970s and 1980s, which were primarily based upon instruction and supported by the implicit belief that, in an appropriate ready-made setting, knowledge could be handed over or transmitted from the teacher to the learners—an attitude which was entitled by its critics as 'broadcast metaphor' (Seeger and Steinbring 1992). Among others, this criticism was supported by carefully documented teaching experiences showing that even sophisticated concepts of teaching frequently led to unsatisfactory results with respect to the pupils' learning (Hefendehl-Hebeker 1988, 1991).

Freudenthal (1973) had given an influential impetus through the idea that learning mathematics should become an active process in the construction of knowledge and that substantial mathematics, geared to the preconditions of learners, should be made the foundation of teaching. Following this idea, the development and research project 'mathe 2000' (Müller et al. 1997; Wittmann 2002) was founded, culminating, among other didactic material, in a textbook that is still based on the concepts of 'productive exercise' (Winter 1984) and 'substantial learning environment' (Wittmann 2002). It also promises potential for advanced levels of teaching and learning. Wittmann (2012) characterized this approach to didactics of mathematics as 'design science' with a focus on 'mathematics education emerging from the subject'. In this concept, mathematics curricula are organized around 'substantial learning environments', where children can gain mathematical experience, recognize patterns and solve problems. To construct such environments requires a 'structure-genetic empirical analysis' (ibid.), which comprises content-related analyses of the traditional type and in addition the analysis of the cognitive preconditions of the learners, mathematical practices of exploring patterns and the objectives of teaching.

Empirical studies of interaction in mathematics classrooms contrasted and complemented these developments. The analyses of interpretative research studies (Maier and Voigt 1991) sharpened the awareness that knowledge cannot be transferred in a simple manner, but is developed within the social interaction between a learner, the teacher and other learners in the group (Steinbring 2009). This awareness was accompanied by the conviction that a good structure of an optimal representation of ready-made mathematics does not automatically provide a good structure for teaching, but that the learning process should be oriented towards natural conditions of knowledge acquirement. It is still a matter of debate how ethnographies of the teaching and learning of mathematics on the one hand and design-based developments of didactic artifacts on the other hand can be related in a productive way (Gellert 2003); productive both for the development of theory in mathematics education and for the improvement of mathematics teaching and learning.

Parallel to this development, the didactics of mathematics in the Federal Republic of Germany experienced an increasing international orientation, which was supported by the foundation of the IDM in Bielefeld (1973). An indication for this internationalization is the Third International Congress on Mathematical Education (ICME-3) in 1976, during which more than 2000 interested educators from all over the world gathered in Karlsruhe. Subsequently, in the 1980s and the early 1990s, several international research groups including German researchers emerged. Among these an influential group dealt, and still deals, with a theoretical foundation of didactics of mathematics, with an outline of a socio-political perspective on mathematics and mathematics education, with learning of mathematics by modeling and application, or with the construction of a micro-sociology of classroom interaction.

1.4.1 Theoretical Foundation of Didactics of Mathematics

As any other scientific discipline didactics of mathematics needs a reflection on its theoretical foundations as well as its paradigmatic problems and its basic methods. Increasingly, such meta-theoretical considerations gained recognition. Works stemming from the IDM had a high influence on this process, especially those of M. Otte and H.-G. Steiner. Otte (1993) worked on the philosophy of mathematics by focusing on the teaching of mathematics in regard to its historical, cultural, social, technical, and ecological status—and was influential within an international research group on basic components of mathematics education for teachers (BaCoMET; e.g., Christiansen et al. 1986; Keitel and Ruthven 1993). Steiner's (1985) main interest lay in systematizing fundamental theories and methodologies in mathematics education with respect to research, development, and educational practice. Furthermore, he aimed to identify or draw interdisciplinary connections to other disciplines as well as to analyze its relationship of theory and practical doing. In addition to that, Steiner initiated the foundation of an international research group, Theory of Mathematics Education (TME), which came together five times in the years between 1984 and 1991. This early research on meta-theoretical questions about the didactics of mathematics still has an impact on today's research, and the broad variety of research areas of didactics of mathematics in particular can be traced back to these early beginnings. The works and findings of the TME are regarded as the basic foundation of working groups at meetings such as CERME and ICME. By recognizing the complexities of the field of research in mathematics education, attempts to coordinate and combine theories and methodologies have been systematized and promoted. One characteristic of these attempts is their meta-theoretical depth. In contrast to pragmatic strategies, such as a use of theories or a combination of theories in the form of a 'bricolage' (Cobb 2007), the basic principles of theories, the degree of their coherence and contrast and of their commensurability and incommensurability receive systematic attention from working groups at CERME and ICME (Bikner-Ahsbahs et al. 2016; Prediger et al. 2008).

1.4.2 Socio-Political Perspectives on Mathematics and Mathematics Education

As one consequence of the internationalization of didactics of mathematics, it became common practice to see it well distinguished from its former tight and traditional link to teaching and learning processes at schools. From this emancipated point of view, parameters such as the cultural relevance, political guidelines of governmental institutions, the socio-historical perspective on the education of mathematics, and the mathematically based technical progressing of society became the focus of attention. Awareness to these parameters became obvious in 1988 at ICME 6 on the topic of Mathematics, Education, and Society. The respective report was published by UNESCO and edited by C. Keitel and P. Damerow, together with A. Bishop and P. Gerdes (1989). Internationally, the focus on this research program was, among others, integrated into the conference set Political Dimensions of Mathematics Education (PDME), which in 1998 was re-founded as Mathematics, Education, and Society (MES); nine MES meetings have taken place so far with MES 6 having hosted 2010 in Berlin (Gellert et al. 2010). Substantial influence on the organization of mathematics education as an international scientific discipline had C. Keitel who, first as Convenor of the International Organisation of Women and Mathematics Education (IOWME), then as Vice-President and subsequently President of the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques (CIEAEM; an ICMI-affiliated organization), directed the research community towards a socio-critical understanding of theory and practice in mathematics education (e.g., CIEAEM 2000; Keitel et al. 1993).

1.4.3 The Teaching and Learning of Applications and Modelling

The basic idea to understand mathematical modelling not only as a specific scientific research area, but also as a part of the school curriculum and thereby acknowledging its didactical value, originated from Great Britain. Since the beginning of the 1990s, mathematics educators from Germany also contributed substantially to the internationalization, development and spreading of the idea to focus the teaching of mathematics on applications. Under the strong influence of W. Blum and G. Kaiser, a new curricular concept was developed. Summarized by the keyword 'modelling', it was later defined as one of the basic mathematical competences and integrated into the educational standards ('Bildungsstandards') and curricula ('Rahmenlehrpläne'). Many of the 17 conferences of the International Community of Teachers of Mathematical Modelling and Applications (ICTMA; an ICMI-affiliated study group), which were organized from 1983 onwards, or the ICMI Study 14: Applications and Modelling in Mathematical Education (Blum et al. 2007), were under the guidance of German colleagues (e.g., conference proceedings Blum et al. 1989; Kaiser et al. 2011). During the last 30 years or so, research on modelling and applications developed empirical research on cognitive aspects of modelling, such as ability (Kaiser-Meßmer 1986b) and competencies (e.g., Maaß 2006; see Schukajlow et al. 2018 for an overview of empirical research during the early years of modelling and recent developments). Furthermore, in 1991, the study group ISTRON was founded. This study group included members from universities as well as from schools in order to connect theoretical research with practical observations and findings. Its main goal was to discuss and develop suggestions for integrating modelling as well as fitting references to reality into the teaching of mathematics at schools.

1.4.4 Construction of Meaning Within Classroom Interaction

While the cooperation of a study group led by H. Bauersfeld in Bielefeld and colleagues with a study group led by P. Cobb at the Purdue University in Indiana did not reach an institutionalized status comparable to one of the above-mentioned study groups with their regularly held conferences, their influence on the development of the research on mathematics education is still of considerable importance. Due to this cooperation, the practical ongoings of the teaching of mathematics were considered a non-negotiable part of the research on teaching-learning processes. Furthermore, it systematically interconnected socio-cultural as well as individual-based psychological perspectives of learning mathematics to their theoretical foundation. Cobb and colleagues began with the description of the learning of mathematics as an individual process of constructing a mental concept. The group of researchers in Bielefeld, on the other hand, focused on the social and interactional processes of learning mathematics within the school environment (Krummheuer 1995; Voigt 1995). As a result of this cooperation, learning mathematics from then onwards was considered a socially situated process of emergence based on an individual's interpretation and construction (Cobb and Bauersfeld 1995). The micro-ethnographic methodology of the work of G. Krummheuer and colleagues has been recognized by educationalists as a promising way to connect classroom research in educational science and in mathematics education.

One effect of the widening of the perspectives on the research object during the last decades is the co-existence of a variety of languages of description in the field. As in many parts of the world, mathematics education in Germany as a field of scientific inquiry is characterized by a horizontal knowledge structure with specialized and, to a certain degree, localized languages and research methods. The spread of research on mathematics education in Germany was further augmented when large-scale assessment studies entered the field. The upcoming psychometric influence in the field broadened the spectrum of research methods in German mathematics education research substantially.

1.5 Development of Empirical Research and Recent Stimuli

1.5.1 Educational Research on the Learning and Teaching of Mathematics

During the 1980s and 1990s empirical research in German speaking didactics of mathematics further developed and diversified. Researchers strived to establish research areas that were subject-specific. U. Viet (Viet et al. 1982), for example, criticized that empirical research on mathematics education was dominated by case studies, by pre-post-evaluation of interventions and by achievement testing that regarded mathematical abilities as a rather global construct. Viet and colleagues therefore conducted research on content specific learning processes, drawing flexibly on qualitative and quantitative methods and on diverse theories of cognition. Many groups started similar endeavors, thereby shaping mathematics education as distinct research area. This was acknowledged by the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) by funding many studies from didactics of mathematics, like e.g. Individual differences in the mental representation of term rewriting (E. Cohors-Fresenborg), Solution strategies of students in primary school during work on picture-text-integrating tasks (M. Franke), Categories of students' mathematical thinking processes (K. Hasemann), Formats of collective argumentation (G. Krummheuer), Problem solving strategies in spatial-geometric tasks (K. Reiss), Understanding of probability of children in kindergarten and primary school (B. Wollring) and many others.

1.5.2 Large-Scale Research

In the international comparisons delivered by large-scale studies (particularly TIMSS 1997 and PISA 2000), German students unexpectedly showed only average achievements. This triggered repercussions in society and politics that can be compared to the 'sputnik-shock' (see above). Several activities in education policy and education research ensued that were mutually interconnected. Education research experienced a boost in political awareness and a demand for 'evidence-based policy'. This also led to increased funding in educational research and had an impact especially on those branches of research in mathematics education that used quantitative approaches.

Roughly beginning in 2000 education in schools was regulated by defining 'outcome standards' rather than 'content standards' (Klieme et al. 2004). For this purpose, a system of continuous, centrally administered assessment procedures and examinations was developed. These developments also resulted in newly accentuated goals in mathematics teaching (although this shift is only gradually recognizable in the classroom): Referring to the concept of 'mathematical literacy' (Neubrand 2003; Niss 2003), the use of mathematics in everyday and professional life received more attention, strengthening competence areas like modelling or using data, while classical topics like plane geometry receded. A debate on the alleged overemphasis of aspects of usefulness of mathematics—incorrectly attributed to Heymann's (1996) analysis of the educational purposes of mathematics—quickly subsided and scholars compromised on Winter's (1995) suggestion of balancing mathematics as a useful tool in describing reality, as deductively structured domain and as an opportunity for intellectual and heuristic activity. Also, after 2003, German education standards throughout the 16 states began to emphasize competences related to mathematical processes (such as modelling, reasoning and problem solving) on the same level as competences for the dimensions of content.

On the part of research, the interest of education policy in a continuous monitoring of the outcome have led to a broad development of assessment measures (predominantly based on the measurement approach used by PISA), coordinated by a centrally founded Institute for Educational Quality Improvement (IQB). Researchers in mathematics education were regularly consulted as domain experts. Furthermore, the video studies conducted within the TIMSS-assessment (Stigler et al. 1997) inspired researchers from Germany and Switzerland to investigate the quality of teaching and its impact on learning by means of video analysis of mathematics lessons as seen in regular classrooms (Rakoczy et al. 2007). The relationship between general and subject specific features of mathematics classroom is still an object of interest and debate.

1.5.3 Interdisciplinary Research Programs with Collaboration of Mathematics Education

The stimuli that influenced education policy also had an impact on the research in mathematics education in several areas. It was characteristic for this research that it was conducted cooperatively in interdisciplinary groups connecting their respective research interests and domains of expertise. Some outstanding examples include the following:

- A federal priority research initiative on 'Educational quality in schools' (BiQUA, Prenzel and Schöps 2007) initiated many joint projects between educational psychology and mathematics education. Researchers connected their expertise to investigate questions of mathematics learning and teaching, such as how to learn

proof by heuristic worked examples (Reiss and Renkl 2002), or how to foster problem solving and self-regulation (Bruder et al. 2007).
- Several research initiatives were launched in order to develop innovative approaches to educational assessment, e.g. in the federally funded priority program 'Competence models for assessing individual learning outcomes and evaluating educational processes' (Koeppen et al. 2008) or the international study of the IEA on teacher competences (Blömeke et al. 2014). Many projects were based on the cooperation between researchers from educational psychology and mathematics education (e.g., Leuders et al. 2017).

A similar development can also be found in education policy and education research with respect to other domains (e.g., Neumann et al. 2010). The impact was especially strong in mathematics, science education and in the domain of reading. Furthermore, the mentioned impulses and cooperations are accompanied by a certain emphasis on quantitative research strategies used in educational psychology. Other research strategies, which are highly relevant in mathematics education, such as the development of local (topic specific) learning theories by means of qualitative analysis of case studies or the cyclical development of learning environments in a design research approach receive considerably less funding within the mentioned programs and initiatives. This may bear the risk of splitting up the research community in opposing factions.

1.6 Institutional Structure of Mathematics Education in German-Speaking Countries

The development described above led to the situation of mathematics education in Germany that we encounter today (as of 2018). Pre-service mathematics teacher education in Germany for all school types is located at universities (and universities of education, 'Pädagogische Hochschulen'), each endowed with up to six full professorships and further research and teaching personnel. In most universities mathematics education is located within a faculty of mathematics, sometimes within a faculty of education. Similarly, there are teacher training institutions without university status (also termed 'Pädagogische Hochschulen') in Switzerland and Austria.

Newly appointed professors have usually completed a PhD thesis in mathematics education, accomplished further research and have a certain amount of practical teaching experience in schools. During the last decade, the group of young researchers has grown considerably since more universities offer funded doctoral programs. Nowadays, fewer researchers have experience in mathematics research and more researchers have experience in educational research than in former times. However, there are concerns that this may be accompanied with a loss of experience in mathematics and a reduction of discipline specificity in research.

German mathematics educators meet on a regular basis on national conferences and special interest groups. A large part of their publications is in the German language, some of them can be found in two peer-reviewed journals (Journal für Mathematik-Didaktik, mathematica didactica). The number of international publications is steadily increasing. The 'Zentralblatt der Mathematikdidaktik', founded in 1969, has developed into the international journal 'ZDM—Mathematic education'.

German mathematics educators are also engaged in the organization of international conferences. Forty years after ICME-3 in Karlsruhe, the ICME-13 again took place in Germany at the University of Hamburg. With about 3500 participants from 107 countries it was the greatest congress in the world congress series so far. The proceedings (Kaiser 2017) provided a comprehensive overview on the state-of-art-discussion in mathematics education.

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