Chapter 4
The Italian Didactic Tradition

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Abstract Starting with a historic overview highlighting the increasing interest and involvement of the community of mathematicians in educational issues, the chapter outlines some of the crucial features that shaped Italian didactics and, more specifically, the emergence of research studies on mathematics education. Some of these features are related to local conditions, for instance, the high degree of freedom left to the teacher in the design and realization of didactic interventions. The specificity of the Italian case can also be highlighted through a comparison with the reality of other countries. The fruitfulness of this comparison is presented by reporting on collective and personal collaboration experiences between the French and Italian research communities. A final contribution, coming from East Asia, puts the Italian tradition under the lens of a completely new eye, and invites reflection upon historical and institutional aspects of the Italian tradition.

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4.1 Introduction

This chapter presents the key features of what we consider to be the Italian didactic tradition.

A short historic overview begins the chapter, showing different voices that illustrate specific aspects of an Italian trend in mathematics education (*didattica della matematica*) from both inside and outside the community of Italian didacticians.

In the next section, a historic overview highlights the increasing interest and involvement of the community of mathematicians in educational issues, in particular, the role played by special personalities in the emergence and the development of mathematics education as a scientific and autonomous discipline.

Paolo Boero and Maria G. Bartolini Bussi then outline crucial features that have shaped Italian didactics and, more specifically, the emergence of research studies in mathematics education; some of these features are related to local conditions, for instance, to the high degree of freedom left to the teacher in the design and the implementation of didactic interventions.

The specificity of the Italian case can also be highlighted through a comparison with the realities of other countries. The fruitfulness of this comparison is presented in a section by Nadia Douek and Bettina Pedemonte, who report and comment on collective and personal collaboration experiences between French and Italian research communities.

A final contribution, coming from East Asia, places the Italian tradition under the lens of a completely new eye. Xu Hua Sun from Macau recently came in contact with the Italian tradition in collaborating with Maria G. Bartolini Bussi on the organization of the 23rd ICMI study.

4.2 Mathematicians and Educational Issues: A Historical Overview

Similar to other European countries such as France, we find great interest and intense engagement from the community of mathematicians in educational issues, starting from Italy's unification and through the 20th century.

The modern state of Italy does not have a long history: The unification of the different regions into one single nation under the monarchy of the Savoy family dates back only to 1861, and only in 1871 was Rome included in the Kingdom of Italy. Immediately after the constitution of the Kingdom of Italy, when the new state began to reorganize the central administration and in particular the educational system, we find eminent mathematicians involved in the Royal committees that were nominated to elaborate curricula for the different schools and engaged in writing textbooks.

In fact, the varied historical and cultural experiences in different parts of the country had created a great variety of textbooks and a substantial dependency on non-Italian books, mainly translations or re-elaborations of French books such as the classic *Elements de Géometrie* by Legendre or of the more recent books by Antoine Amiot, Josef L. F. Bertrand and Josef A. Serret. The flourishing of new textbooks between the end of the 19th century and the beginning of the 20th century testifies to the active role distinguished scholars such as Luigi Cremona and Eugenio Beltrami began to play in supporting the position that mathematics began to assume within the curriculum (Giacardi, 2003). A key issue concerned the recruitment of teachers, since at the time there were no regulations, and people without any specific mathematical competence were allowed to teach. It was only in 1914 that legislation was approved regarding the legal status of mathematics teachers. Many factors determined the active engagement of mathematicians on the social and political front, one of which being the indubitable influence of Felix Klein, which is of interest considering the topic of this chapter (Giacardi, 2012, p. 210). Some of the Italian mathematicians had the opportunity to visit Germany, and they took advantage not only of Klein's scientific training but also of learning about Klein's ideas on reforming secondary and university mathematics teaching. A tangible sign of their appreciation of Klein's project is the early translation into Italian of Klein's Erlangen Program by Gino Fano and Corrado Segre (Giacardi, 2012). Under the influence of the German experience, a wide debate about the structure and the content of the new schools for teachers (*scuole di magistero*) involved many professional mathematicians such as Alessandro Padoa, Gino Loria and Giuseppe Peano. The core of the debate concerned the tension between the pedagogical and disciplinary stances.

An important occasion of international comparison between different experiences in the field of mathematics education was the IV International Congress of Mathematicians which took place in Rome in April 1908. During this congress the International Commission on the Teaching of Mathematics was founded.[1]

However, in spite of what we could define a very promising start, in the 1920s, the Minister of Education, Benedetto Croce, abolished the *scuole di magistero*, so that any kind of teacher training for the secondary school level was cancelled. In the same trend, after the advent of Fascism and with the nomination of Giovanni Gentile as Minister of Education, the role and the influence of mathematics, and generally speaking of scientific disciplines, decreased in the curriculum. The community of mathematicians fought against this trend, defending the cultural value and the fundamental contribution of mathematics and scientific education in general. The most prominent figure in this cultural fight was certainly Federigo Enriques, but other mathematicians such as Guido Castelnuovo, the father of Emma, played a fundamen-

[1] Details of the history of this institution can be retrieved at: http://www.icmihistory.unito.it/timeline.php.

tal role, defending the cultural and educational dimension of mathematics, opening specific university courses and producing material for teacher preparation (Marchi & Menghini, 2013). Didactical essays were written discussing original didactical principles, such as dynamic teaching, elaborated by Enriques (1921), and the need for synthesis between intuition and logic argued by Severi (1932, p. 368).

Noteworthy is the cultural project on the elaboration of elementary mathematics content; Enriques (1900, 1912) was the editor of a huge six-volume text entitled *Questioni riguardanti le mathematiche elementari* that collected articles about elementary topics that the authors felt were fundamental for any secondary school mathematics teacher to know.

After the long and traumatic period of the Second World War, the Italian educational system saw the formulation of new curricula for junior high (*scuola media*) and high schools (*licei*). However, in spite of some methodological suggestions inspired by Enriques, the general structure of the curriculum remained unchanged, and a well-settled textbook tradition contributed to crystallise mathematics teaching in schools. Though we can say that Italy was not touched by the New Math 'revolution,' we can find an echo of the international debate in the active discussions that took place in the mathematicians' community. Two particularly interesting occasions were meetings that took place in Frascati (1961–62) leading to a document known as 'Programmi di Frascati', which contained the outline of a curriculum that the Unione Matematica Italiana, the Italian Association of Mathematicians, offered to the Ministry (Linati, 2011). Unfortunately, however, the negative inheritance of Gentile's reform was to last far longer than the new wave, and the Italian upper secondary school had to wait until the academic year 2010–11 to see a reform, both of its structure and of its curricula. However, in spite of the lack or the episodic character of any reform of the curricula or possibly precisely because of this phenomenon, didactics in the Italian school lived a very peculiar development: Never pressed by strict obligations or official curricula, teachers have always had a certain freedom, leaving the possibility of being open to innovations; they could be influenced by new ideas and they could experiment and develop new approaches to math teaching (see the following section for more details).

Though she can certainly be considered unique, it would have been impossible for Emma Castelnuovo to realize the innovative projects that made her so famous if the context of the Italian school had been different.[2] This aspect of the Italian school also explains the influence that, in spite of the immobility of the official curriculum, mathematicians were able to have both in promoting innovations and in developing new research field of mathematics education.

In order to understand the role of the mathematicians' community in this process, it is important to consider the foundation and the role played by the associations of mathematicians. The roles played by individuals become stronger when they are played by representatives of professional associations: This is the case in Italy, where

[2] https://www.youtube.com/c9918297-edab-4925-9b95-1642ddbaa6a2; The legacy of Emma Castelnuovo remains in institutions such as the Officina matematica di Emma Castelnuovo (http://www.cencicasalab.it/?q=node/23).

there have been two associations that in different ways have played and still play key roles in promoting mathematics teaching. The first is the Association Mathesis, founded in 1895, and the second is the Unione Matematica Italiana (UMI) founded in 1920 by Vito Volterra as part of the International Mathematical Union. A common characteristic of these associations is the presence of both university and secondary school teachers and the nomination of a permanent committee whose specific role has been to treat educational issues both at the national and at the international levels.

The commitment of the community of mathematicians to educational issues became extremely evident at the beginning of the 1970s, when stable groups of researchers in mathematics education emerged, as will be explained in the following section.

In this respect, we can see a great similarity between the Italian and the French traditions, though for different reasons: The first generation of Italian researchers in mathematics education was made up of professional mathematicians belonging to mathematics departments and often distinguished scholars active in mathematics research fields.

4.3 The Development of an Italian Research Paradigm

In the 1970s in Italy there was a consolidated tradition of cooperation between universities and schools. Famous mathematicians and mathematics teachers were involved in this process: Among the former, it is worthwhile to mention the late Giovanni Prodi, Lucio Lombardo Radice and Francesco Speranza, who engaged in the renewal of the teaching of mathematics in schools with teaching projects, collective documents, and pre- and in-service teacher education activities. Among the latter was Emma Castelnuovo, to whom the ICMI award for outstanding achievements in the practice of mathematics education is dedicated.

Research projects for innovation in the mathematics classroom at different school levels have included mixed groups of academics and teachers at many universities who have developed a cooperative attitude on a basis of parity (Arzarello & Bartolini Bussi, 1998). For more than two decades (1975–1995), the Italian research funding agency, the Italian National Research Council, under the pressure of some mathematicians who were members of the Committee for Mathematical Sciences, funded research groups with the aim of contributing to the renewal of school teaching of mathematics. Research fellowships were allocated to young people engaged in such groups (in particular, at the end of the 1970s).

The simultaneous presence of active school teachers (who later were called teacher-researchers; see Malara & Zan, 2008) and academics allowed the creation of a very effective mixture of practical school needs and an academic tradition: The constraints of the local contexts had to meet international research trends. For instance, the teacher-researchers highlighted the importance of whole-class interaction (beyond the more popular studies on individual problem solving and small-group cooperative learning), the teacher's role as a guide (beyond the more popular focus on learners' processes), long-term processes (beyond the more popular studies on short-term processes) and the manipulation of concrete artefacts (e.g., abacuses, curve drawing devices, and tools for perspective drawing) without overlooking the theoretical aspects of mathematical processes.

During the last four decades, specific conditions of the Italian institutional context of mathematics instruction have demanded and allowed the engagement of university mathematicians in the renewal of school teaching of mathematics through the elaboration and experimentation of long-term educational projects:

- Programs and (more recently) guidelines for curricula have not been strictly prescriptive in comparison with other countries; moreover, teachers and textbook authors feel free to interpret them in rather personal ways. The interpretation often follows a transmissive, traditional paradigm, but innovative experiences, even beyond the borders of the ministry texts, have been proposed, thus preparing further evolution of programs and guidelines for curricula.
- The quality of national programs and (more recently) guidelines for curricula have been rather good: This also depends on the engagement of researchers in mathematics education, mathematics educators and mathematicians in the elaboration of these guidelines (within the Ministry commissions that prepare them). In particular, the Guidelines for Primary and Lower Secondary School Curricula (from 2003 to 2012; MIUR, 2012) were strongly influenced by the previous work of a commission that included mathematicians, statisticians, researchers in mathematics education, teacher-researchers and delegates of the Ministry of Education (see MIUR, SIS & UMI-CIIM, 2001).

This was the context where the present research in didactics of mathematics arose in Italy, with some original features that were later appreciated at the international level.

4.3.1 The Early International Presentations of the Italian Didactics of Mathematics

In the 1980s, when Italian researchers started to take part in international conferences on mathematics education (e.g., ICME and PME), the difference between what was happening in Italy and elsewhere appeared quite evident. In Italy there was not a consolidated tradition of cognitive studies such as those developed in the US and other countries, but there was a tradition of epistemological and historical analysis of content and a growing pressure on active teachers to become an important part of the innovation process.

In 1987, the first session of the Italian National Seminar on Didactics of Mathematics (later dedicated to Giovanni Prodi; see http://www.airdm.org/sem_naz_ric_dm_3.html) took place. In those years, the Italian community started to publish reports on mathematics education research in Italy for the ICME (ICME 9 and ICME 10). They were collective volumes and had a strong effect not only on the external presentation of the Italian traditions, but also on the internal construction of the identity of the Italian community.[3]

This situation was summarized in the 1990s by an invited contribution to ICMI Study 8 (Arzarello & Bartolini Bussi, 1998). The paper summarized a discussion that had been going on in those decades and presented a systemic view, where research for innovation was related to the most popular trends of epistemological analysis and of classroom teaching experiments and to the new trend of laboratory observation of cognitive processes. This choice aimed at pointing out the specific positive features of our action research in the mathematics classroom and answering criticisms coming from other traditions, where the tendency towards more theoretical studies was prevalent.

4.3.2 A Retrospective Analysis 30 Years Later: Some Examples from Classroom Innovation to Theoretical Elaborations

This systemic approach is still fundamental, although, on the one hand, additional components have strongly emerged and are well acknowledged at the international level (e.g., for studies on beliefs and affect issues, see Di Martino & Zan, 2011, and for multimodality in the teaching and learning of mathematics, see Arzarello & Robutti, 2010) and, on the other hand, the landscape has changed with, for instance, the impact of technologies, the attention to students with special needs, the major issue of teacher education, and the awareness of multicultural aspects.

In this section we report on some examples of mathematical education research chosen from those focused upon during specific sessions of the National Seminar and reconstruct their roots. These projects and others (e.g., the ArAL project on the early approach to algebraic thinking; see http://www.progettoaral.it/aral-project/) have been developed by groups of academic researchers and teacher-researchers in many parts of the country.

The field of experience construct

Teaching and learning mathematics in context was an initial aim for a five-year project for elementary school (now referred to as primary school) and a three-year project for lower secondary school planned during 1976–80 and gradually implemented in the 1980s in more than 200 classes (Boero, 1989). The initial roots of these projects were ideas that were circulating both in other countries (e.g., the School Mathematics Project in the UK) and in Italy (e.g., *Mathématiques dans la réalité* or "mathematics in reality"; Castelnuovo & Barra, 1980) together with general educational aims related to education for citizenship. The projects aimed at preparing students to deal with outside-of-school problem situations and horizons and

[3] http://www.seminariodidama.unito.it/icme9.php; http://www.seminariodidama.unito.it/icme10.php.

at establishing links between learning aims and opportunities offered by students' outside-of-school experiences. Gradually, the researchers involved in the projects realized that such didactic and cultural-educational choices needed suitable frameworks in order to establish a relationship between outside-of-school culture(s) and classroom activities. The field of experience construct, proposed by Boero (1989) and then implemented in other studies (see Bartolini Bussi, Boni, Ferri, & Garuti, 1999; Dapueto & Parenti, 1999; Douek, 1999) was one of the answers to that need. A field of experience concerns areas of outside-of-school culture and actual or potential experiences for students, such as in the 'sun shadows' and 'money and purchases' examples, and also geometry that is intended not as school geometry but as geometry that applies to outside-of-school culture, including the geometry of the carpenter example, and not only mathematicians' geometry. A complex relationship can be established between outside-of-school culture, the goals of school teaching and classroom activities through the mutual influence between the three components of a field of experience: teachers' inner contexts, including their knowledge, conceptions, experience and intentions related to the field; the students' inner contexts, including their conceptions and experience; and the external context, particularly the artefacts related to the field of experience, such as material objects and external representations together with customary practices.

Within a given field of experience, classroom activities are conceived as opportunities offered to the teacher to intervene on the students' inner contexts through practices and tools belonging to the external context in order to develop students' everyday concepts into scientific concepts. Specifically, concepts used as tools to deal with typical problems of the field of experience may be further developed as objects belonging to disciplinary fields of experience such as geometry or arithmetic. The field of experience construct was used as a tool to plan and manage long-term classroom activities according to a methodology (for field of experience didactics, see Boero & Douek, 2008) that was essentially based on students' individual or small group productions that become the object of classroom discussions orchestrated by the teacher and are then brought towards progressive phases of institutionalization.

By the end of the 1990s, another need emerged from experimental activities: to develop a cultural framework to deal with cultural diversities in the classroom. Indeed, as far as outside-of-school culture and practices enter the classroom as objects of experimental activities, cultural approaches to outside-of-school realities (e.g., random situations and natural and social phenomena) may be quite diverse and different from how they are dealt with in the classroom. Moreover, within mathematics and between mathematics and other disciplines, knowing, acting and communicating show specificities that must be taken into account in order to develop specific learning activities. As an example, we consider mathematical modelling and proving in mathematics: criteria for the truth of statements and strategies and ways of communicating results are only partially similar, while differences concern relevant aspects, for instance, how models fit the modelled reality versus how the truth of statements derives from axioms and already proved theorems. In order to cope with all these aspects, Habermas' (1998) construct of rationality, which deals with knowing, acting and communicating in discursive practices in general, was adapted to specific discursive practices in mathematical fields of experience and in other fields of experience (Boero & Planas, 2014).

While dealing in our projects with the development of proving skills in Grades 7 and 8, teachers and researchers identified a phenomenon that could result in the choice of suitable tasks to ameliorate students' approach to proving: the cognitive unity of theorems (Boero, Garuti, & Lemut, 2007; Garuti, Boero, Lemut, & Mariotti, 1996), which consists in the possibility (for some theorems) of establishing a bridge between conjecturing and proving. During the conjecturing phase, students produce ideas and develop ways of reasoning that can be resumed and reorganized in the proving phase as components of proving and proof. Other researchers (in particular, Fujita, Jones, & Kunimune, 2010; Leung & Lopez Real, 2002; Pedemonte, 2007), using the theoretical construct of cognitive unity, investigated obstacles and opportunities inherent in the transition from conjecturing and proving in different learning environments.

The theory of semiotic mediation

A pair of projects chaired by one of the authors of this contribution was briefly reported on during PME 22 (Bartolini Bussi, 2009), where a rich list of references is contained. The two projects are complementary and address different school levels. They both have produced papers published in scientific journals and volumes and, in the last decade, they have converged into a larger project on cultural artefacts: semiotic mediation in the mathematics classroom.

The first project concerns a Vygotskian approach to social interaction, presented for the first time in 1991 (in a plenary speech at PME 15). In this project the theoretical construct of mathematical discussion orchestrated by the teacher was elaborated with teacher-researchers (Bartolini Bussi, 1996). A mathematical discussion is

> a polyphony of articulated voices on a mathematical object, that is one of the leitmotifs of the teaching-learning activity. The term voice is used after Wertsch (1991), following Bakhtin, to mean a form of speaking and thinking, which represents the perspective of an individual, i.e., his/her conceptual horizon, his/her intention and his/her view of the world. (Bartolini Bussi, 1996, p. 16)

The teacher utters many different voices, among which the voice of mathematics as it has been developed within the culture where the teaching is being realized. The idea of mathematical discussion orchestrated by the teacher was shared by another research group (see the field of experience didactics mentioned above).

The second project concerns the mathematical laboratory approach centred on the use of concrete instruments. The concrete instruments are, in this case, mathematical machines (e.g., abaci, pair of compasses and curve drawing devices, and perspectographs) taken from the history of mathematics and reconstructed and used for didactical purposes. They are true cultural artefacts (Bartolini Bussi, 2010).

Although the groups of teacher-researchers belonged to different school levels (primary school and secondary school), from a theoretical point of view the two projects were consistent with each other and both referred to two crucial aspects of the Vygotskian tradition: language in social interaction and cultural artefacts in

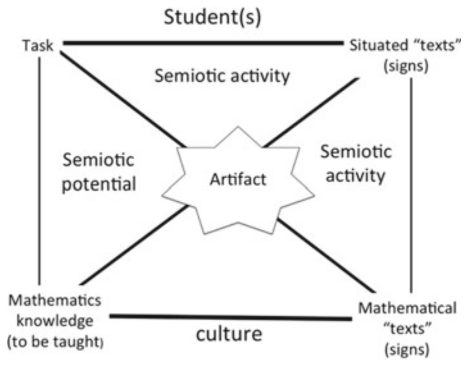

Fig. 4.1 Semiotic mediation

mathematical practices. In the following years, thanks to the contribution of Mariotti (2012), new technologies also entered this joint project, adding information and communication technologies to concrete artefacts and exploiting the effectiveness of mathematical discussion. In this way a comprehensive approach to studying semiotic mediation by artefacts and signs in the mathematics classroom (Bartolini Bussi & Mariotti, 2008; Mariotti, 2009, 2012) was developed. The theory of semiotic mediation using a Vygotskian approach aims to describe and explain the process that starts when students use an artefact to solve a given task and then leads to the students' appropriation of a particular piece of mathematical knowledge. The semiotic potential of an artefact refers to the double semiotic link that may be established between an artefact, the students' personal meanings (emerging from its use when they try to accomplish a task) and the mathematical meanings evoked by its use and recognized as mathematics by an expert. The process of semiotic mediation consists of developing initial situated signs, which evoke the artefact, into mathematical signs referring to mathematical knowledge. In this process, pivot signs, hinting at both the artefact and the mathematical knowledge, appear and are exploited by the teacher. This process is shown in Fig. 4.1. On the left is the triangle of the semiotic potential of the artefact. The teacher plays two different roles in this scheme in the development from artefact signs to mathematical signs: (1) the role of task design (on the left) and (2) the guidance role (on the right). This development is promoted through the iteration of didactical cycles (Fig. 4.2), where different categories of activities take place in the unfolding of the semiotic potential of the artefact: individual or small group activities with the artefact to solve a given task; individual production of signs of different kinds (e.g., drawing, writing or speaking, and gesturing); and collective production of signs, where the individual productions are shared. The collective phases are realized by means of mathematical discussions.

These projects were all originally developed for classroom innovation but later influenced the curricula prepared by the Italian Commission on Mathematical Instruction and adopted by the Ministry of Education; they also were included in handbooks for teachers (e.g., in the Artefatti Intelligenti series: https://www.erickson.it/Ricerca/

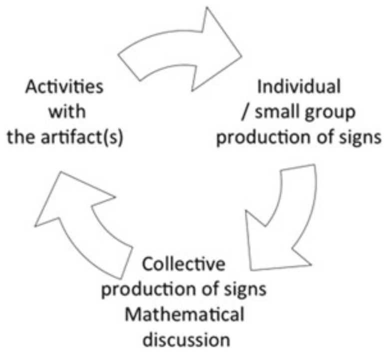

Fig. 4.2 Didactic cycle

Pagine/results.aspx?k=artefatti%20intelligenti). Hence, they proved to be effective at the institutional level.

4.3.3 The Impact of Italian Research at the International Level

At the international level, Italian research activities such as those exemplified in the previous sections produced dozens of research papers published in the proceedings of international conferences and in major scientific journals and volumes by international publishers. The two main foci of the Italian contribution were the historical-epistemological dimension and original didactical implementations, which were carried out by mixed groups of academics and teacher-researchers following the steps of well-known personalities of the previous generation (e.g., Emma Castelnuovo). This contribution had the potential to support the dialogue with other dimensions that were more prevalent in other research traditions (e.g., cognitive, sociological and affective). Thus, in spite of the fact that in Italy there has been and still is a structural difficulty related to the discontinuity of financial support and available jobs at the university level, international collaborations have been developed and Italian scholars have reached important positions within various international boards (ICMI, ERME and PME).

Some of the ideas carried out by Italian scholars (e.g., the teacher's role as a guide and a mediator in mathematical classroom processes and the focus on historical-epistemological issues) were not aligned with some widespread assumptions and systems of values shared by other communities who put more emphasis on non-mathematical issues (e.g., sociological and psychological). For example, we may consider the debate between the constructivist positions very popular in US and the historical-cultural trend developed by scholars following the Vygotskian tradition (e.g., Bartolini Bussi, Bertolini, Ramploud, & Sun, 2017; Vianna & Stetsenko, 2006; Waschescio, 1998).

As examples of leading positions that are different from those elaborated by Italian researchers, let us consider the constructs of negotiation of meanings and of socio-mathematical norms (Cobb & Yackel, 1998) that have resulted from a relatively radical vision of constructivism, where the focus is on the learners as comparatively isolated subjects, that has a more or less explicit separation from the historical-cultural context. In those approaches, the very formulation of the teacher's actions is only weakly linked to the specific mathematical task at stake. For instance, Cobb and Yackel (1998), while discussing the solution of a problem in the observed classrooms, state that

> the teachers rarely asked whether anyone had a more sophisticated or more efficient way to solve a problem. However, their reactions to students' solutions frequently functioned as implicit evaluations that enabled students to infer which aspects of their mathematical activity were particularly valued. . . . These implicit judgements made it possible for students to become aware of more developmentally advanced forms of mathematical activity while leaving it to the students to decide whether to take up the intellectual challenge. (p. 169)

This short excerpt indicates that teachers should be cautious to avoid any form of explicit teaching, such as introducing from outside (e.g., from the textbook or a story about a different classroom) a more effective strategy, even though it is part of the historical development of mathematics.

Drawing on this literature, in the Italian tradition we have stressed the necessity for a clear focus on the specificity of the piece of mathematics knowledge at play, on the particular task and the artefacts proposed to the students, on the cultural heritage to be exploited in the classroom, and on the related cultural roots of mathematics teaching and learning.

Yet our background choices seem to be consistent with the systems of values and cultural traditions developed in other large regions of the global world that have appeared only recently in the forum of mathematics education (e.g., East Asian regions and developing countries). It is worthwhile to mention the very recent paper by Bartolini Bussi, Bertolini, Ramploud, and Sun (2017), where some assumptions of the theory of semiotic mediation are integrated into the Chinese approach to lesson study and exploited to carry out teaching experiments in Italian classrooms. Chinese lesson study is linked to but less known than the Japanese lesson study (Huang et al. 2017). Both are robust methodologies aimed at improving classroom practice by means of a collective design of a lesson, a collective observation of the lesson implemented by a teacher of the design group, and a collective analysis and re-design of the lesson. In the Chinese model, the focus is on the content and on the instructional coherence of the lesson (Mok, 2013), which highlights the implicit or explicit interrelation of all mathematical components of the lesson. Hence from the epistemological perspective, the focus is consistent with the theory of semiotic mediation, where the mathematics knowledge is highlighted. In spite of this epistemological consistency, however, differences appear when the didactical aspects of classroom organization are considered; for instance, experiments carried out in Italy introduced teaching methodologies developed in the West (e.g., cooperative learning for small group work and mathematical discussion) and observation of the learning process highlighting the development from artefact signs (including drawing, verbal utterances and gestures) to mathematical signs. This was possible thanks to a careful cultural transposition (Mellone, Ramploud, Di Paola, & Martignone, 2017) ‘where the different cultural backgrounds generate possibilities of meaning and of mathematics education perspectives, which in turn organize the contexts and school mathematics practices in different ways’ (Mellone & Ramploud, 2015, p. 571). This difference appears not only in the choice of artefacts (in the above case, cultural artefacts) but also in the choice of the analytical tools, and related theoretical elaborations may be related to the cultural context (Boero, 2016).

The issue of the cultural contexts is coming to the foreground in the international community: It was chosen as a specific focus in the Discussion Document of the ICMI Study 23,[4] it was addressed at a plenary forum in CERME 9 (Jaworski, Bartolini Bussi, Prediger, & Novinska, 2015) and it was focused on by Barton (2017) in his plenary address, “Mathematics, education and culture: a contemporary moral imperative” in ICME 13.[5] It was acknowledged also by Clarke (2017) during his plenary address at PME 41 in Singapore:

> Within any educational system, the possibilities for experimentation and innovation are limited by more than just methodological and ethical considerations: They are limited by our capacity to conceive possible alternatives. They are also limited by our assumptions regarding acceptable practice. These assumptions are the result of a local history of educational practice, in which every development was a response to emergent local need and reflective of changing local values. Well-entrenched practices sublimate this history of development. In the school system(s) of any country, the resultant amalgam of tradition and recent innovation is deeply reflective of assumptions that do more than mirror the encompassing culture: They embody and constitute it. (Clarke, 2017, p. 21)

The same author (Clarke, personal communication) observed that in several languages (such as Japanese, Dutch, Russian, Chinese and Polish[6]) some communities have acknowledged the interdependence of instruction and learning by encompassing both activities as one process and, most significantly, labelled with a single word, whilst in English, we seem compelled to dichotomise classroom practice into teaching and learning.

4.3.4 Mathematics Teacher Education and Professional Development in Italy

In spite of the rich, decades-long development of the Italian research, the optimistic illusion of a gradual expansion to schools throughout the country was overcome

[4] http://www.mathunion.org/fileadmin/ICMI/docs/ICMIStudy23_DD.pdf.

[5] https://lecture2go.uni-hamburg.de/l2go/-/get/v/19757.

[6] Japanese: *gakushushido*; Dutch: *leren*; Russian: *obuchenie;* Chinese: *xue;* Polish: *uczyć.* Also in ancient Italian, *imparare* had both meanings.

and the problematic issue of mathematics teacher education and development came into the foreground. The story of teacher education and development in Italy is recent. The model for pre-primary and primary school dates back to 1998, with the creation of a university program (with a 4- to 5-year master's degree) that has a balance between university courses, workshops and practicums (now 600 h). In nearly 20 years, a good balance between the different components has been reached thanks to the strong coordination of a national conference of all the deans of the university programs and to the realization of specific scientific national seminars. In particular, for mathematics there are, in all the universities where the program is implemented (about 30 throughout the country), 22 European credits[7] for mathematics. The story of secondary school teacher education is more discontinuous. After a start in 1999, with nine cycles of two years each (specialization after a master's degree in mathematics or in other scientific subjects), the process was interrupted and has only recently started again. The most problematic issue seems to be the real collaboration between the general educators and the subject specialists within the discussion about a correct balance between 'pedagogical' issues and specific content. The situation is in progress now.

In recent years, the development of pre-service teacher education at the university level has stimulated Italian research on this subject. In particular, we think that teachers' cultural awareness of the reform processes happening in mathematics should be developed in countries (such as Italy) in which teachers' autonomy is great in implementing national programs and guidelines for curricula. These considerations suggested the elaboration (Boero & Guala, 2008) and implementation (Guala & Boero, 2017) of the construct of cultural (epistemological, historical and anthropological) analysis of the content (CAC). CAC is a competence that should be important to develop in mathematics education courses, as a way of looking at the mathematical content presented.

Pre-service teacher education is only a part of the professional development of teachers: In-service teachers are required to continue their education, thus there is a large number of in-service teachers in development programs. In the past, in-service education (i.e., teacher development) was usually left to individual choices or the local choices of some private institutions, with a few but important exceptions. As an example, we can quote the national program for in-service primary teacher education developed after the reform of national programs for primary schools. The new programs were issued in 1985; in the following years primary teachers were involved in compulsory teacher education initiatives promoted by the regional school supporting institutions, with a strong engagement in many regions with university researchers in mathematics education and teacher-researchers. In recent years, the Ministry of Education has promoted a set of initiatives (MIUR, 2010) aimed at making the image of mathematics and sciences more attractive for pre-university school students in order to try to counter the decrease in the number of students in mathematics and sciences at the university level. Among the initiatives of the Ministry, the so-called laboratory was an original kind of in-service teacher education

[7] https://ec.europa.eu/education/sites/education/files/ects-users-guide_en.pdf.

activity: University researchers (in particular, researchers in mathematics or science education) joined with teacher-researchers and teachers to plan innovative classroom activities on important content in various disciplines. Planning was followed by classroom implementation of the activities designed by the teachers. A team analysis of what happened in the classroom was produced and reports on those innovative experiences were disseminated.

Since the school reform in 2015 (the so-called *la buona scuola*, 'the good school'; see https://labuonascuola.gov.it), in-service teacher development has become compulsory, systematic and funded for the first time in Italy's history.

4.4 Collective and Personal Experiences of Collaboration Between French and Italian Researchers

In this section, a rich collective experience will be presented from the perspective of two researchers: Nadia Doueck and Bettina Pedemonte. These contributions reflect aspects and exemplar modes of collaboration between French and Italian researchers.

4.4.1 The Séminaire Franco Italien de Didactique de l'Algèbre

The Séminaire Franco Italien de Didactique de l'Algèbre (SFIDA) was founded in 1993 by Ferdinando Arzarello, Giampaolo Chiappini and Jean-Philippe Drouhard and was held until 2012.[8] It was a full-day seminar that took place alternatively in Nice and Genoa or Torino twice a year. During the seven hours of the seminar, six presentations were programmed around a main theme. The allotted time was sufficient to develop deep discussions, in French and Italian, and if needed, in English. A French-Italian vocabulary was displayed to help understand terms that were quite different in the two languages (such as *élèves*-*alunni*). Abstracts were shared before the meeting to foster interactions. Work documents have been collected in three volumes up to the present.

The stable group of participants (including the promoters and Assude, Bagni, Bazzini, Boero, Douek, Malara, Maurel and Sackur) did not form a research team, but collaborations flourished now and then between the participants and with others who would occasionally attend the seminar. Researchers whose work related to the theme of the meeting were often invited to attend the seminar. Invited researchers included Duval, Hanna, Radford, Tsamir and Vergnaud.

[8]The reader will find the presentation of SFIDA, the French-Italian seminar of algebra's didactics, written by the late Jean-Philippe Drouhard in 2012, in the introduction of a special issue on Algebra published by RDM: (REF).

Openness was the main and constant characteristic of our meetings, in terms of the choices of themes, the variety of theoretical frameworks and the arguments used within the debates: Among theoretical references that framed presentations and backed the arguments exchanged and discussed were conceptual fields, experience field didactics, local knowledge and triple approach, embodiment, semiotic registers, semiotics, theory of didactic situations, anthropological theory of didactics and rational behaviour.

There was also openness in the wide extension to connected areas such as epistemology and history of mathematics, philosophy, and cognitive sciences.

Debates were not bound to remain in one particular framework nor at a particular theoretical level. This openness and variety stimulated creativity, favoured advancements and often led to more precise productions. More than 20 PME research reports were inspired and supported by the work done in SFIDA.

SFIDA has also had an important impact on international collaborations: It not only prepared some of the participants to take on responsibilities within PME and ICMI, but it also had a special role in the foundation of ERME. J. Philippe Drouhard and Paolo Boero were among the founders of ERME and were the first two presidents. In fact, the type of collaboration used in SFIDA deeply inspired the organization of CERME's working groups.

4.4.2 Encounter with Various Intellectual Traditions and Methods in French-Italian Ph.D. Projects

A personal testimony by Nadia Douek

I attended various seminars in France and in Italy, specifically some of the regular meetings of Genoa's research group. The span of contents and the styles of interaction I was exposed to were quite different but all very stimulating. It was in this rich frame that my Ph.D. project started, under the supervision of Gerard Vergnaud (Université Paris V Sorbonne) and Paolo Boero (Università di Genova). Grounded in the French and Italian education traditions, my thesis developed through experiencing different types of relations between theory and class activity and between a priori/a posteriori analysis and the flow of class experiments. I learned to combine theoretical clarifications and quick adaptations (improvising!) in classroom contexts in close relation to the teacher's steps. It also brought me to combine references from the French didactics (works of Vergnaud and influences from Douady and Duval) with Italian didactics, in particular Boero's construct of experience fields.

Vygotsky's *Thought and Language*, a strong reference for several Italian colleagues, and for Vergnaud as well, was also crucial to my intellectual development. Philosophers such as Deuleuze and epistemologists such as Feyerabend that I discovered in French seminars influenced my ways of studying.

Class experiments were central to my thesis work. They took place in Italian classes within the long-term projects developed by the Genoa research team led by

Boero. I had the opportunity to explore and co-develop experimental situations in relation to my participation in the Italian seminars. I was introduced to the Italian colleagues' way of intertwining theoretical with very pragmatic questioning about ongoing class activity developed by the teachers within their research for innovation (see Sect. 3). In this method, research problems are scrutinized by the team under the lens of teaching and learning difficulties in classes. While the theoretical frame is a tool to understand, predict, design didactical settings and so forth, it is questioned through long-term and repeated experimentations. Theoretical components are developed on the basis of the analysis of experienced situations. The support of the teachers—who are involved as researchers in the team—was crucial in my scientific development.

I do not know to what extent my scientific development depended on national scientific trends or on scientific personalities. To characterise some of the methodological or cultural differences between my two supervisors, I can say that working with Vergnaud allowed me to explore the limits of various theoretical frames and to elaborate fruitful ideas even when these were not mature. Working with Boero helped me to mature these ideas into more organized ones and to develop a consciousness of their theoretical limits or validity.

Experimental work in the classroom fostered my creativity and improved my attention to pupils' activity, freed from a priori expectations. Most importantly, it made me understand the fruitfulness of slowing down the pace of class activity and deepening it, but also it allowed me to share intellectual ambitions with the teachers and with the pupils. This is generally difficult to achieve in French classes where curricula are more closely followed with stricter time organization through the year and with a stricter epistemological perspective on mathematics.

The research for innovation methodology was combined with Vergnaud's influence, and the methods of organizing class activity, as practiced within Boero's team, had a very harmonious resonance with my philosophical interests nourished by the Parisian seminars I attended. In particular, they had a great effect on my creativity and conception of knowledge as dynamic, always intertwined with activity and swinging between focusing and questioning.

The experience with Boero's research group also affected my vision of teacher training. Research for innovation implied a specific human organization and composition of the team. Today I can compare it with Engeström's elaboration on expansive learning: Teachers' learning and professional development as research team members relied on collaborative relations, tending to be horizontal. The team members were teachers and university researchers involved in matters of various disciplines. They were all productive. Their oral interventions, transcriptions of class discussions, analysis of students' productions, critical analysis of didactical settings, cooperation to transform them or produce new ones and other activities modulated the global evolution of the group's activity. Their discussions and reflections had an impact on productions involving various questions from very practical to highly theoretical or philosophical ones.

Following their activities and collaborating with Italian researchers until the present, I can still see the richness and openness I enjoyed during my studies. However, I think I can also see that it has not appeared to spread. Therefore, a question arises: Is this methodology, reflecting a leading motive in Italian research, still operating as strongly and productively as in the past?

Two factors may have weakened this trend: a favouring of theoretical development per se (less contrasted by long-term class experiments and by multilevel and multi-spectrum discussions) and the need to systematize class activities and designs under the pressure of seeking visible, stabilized utilitarian products.

If this is the case, I fear that it might restrain the evolution of didactic research into more narrow paths that are more tightly managed and that it would weaken the collaboration with teacher-researchers, especially if the spirit of the research is to stabilize results in too firm a manner. I do however remain confident in its powerful potential at both the scientific level and the social and human levels.

A personal testimony by Bettina Pedemonte

In 1997, I obtained my BS degree in Mathematics in Italy at Genoa University. My supervisor, Paolo Boero, presented the results of my study at the 21st PME Conference and invited me to participate to the congress. It was my first experience as a participant of a didactical conference and I was immediately fascinated by this new world. I decided to pursue my studies to earn a Ph.D. in mathematics education. At that time, there was not a specific program in this field in Italy. So I went to France, and after earning my degree, a *Diplôme d'etude approfondi* (DEA), I continued my studies within a joint agreement between the universities of Grenoble, France, and Genoa, Italy. I developed my Ph.D. project under the co-supervision of Nicolas Balacheff from the National Centre for Scientific Research and Maria A. Mariotti from the University of Pisa. I defended my Ph.D. dissertation in June 2002, obtaining two titles: Doctor of Mathematics Education from Grenoble University and Doctor of Mathematics from Genoa University.

The main purpose of my Ph.D. project was to study the relationship between argumentation and proof in mathematics from a cognitive point of view and to analyse how students can be supported in understanding and constructing mathematical proofs.

The hypothesis of my research was that there would usually be continuity, called cognitive unity, between these two processes (Boero, Garuti, & Mariotti, 1996; Garuti, Boero, Lemut, & Mariotti, 1996). The theoretical framework of reference, however, did not provide the tools to validate this hypothesis. I made a scientific contribution to this field by constructing a specific frame for mathematical argumentation with the aim of comparing it with proof. On the basis of contemporary linguistic theories, I designed a theoretical framework that allowed me to 'model' a mathematical argumentation, to compare it with the proof and to describe the cognitive processes involved. This cognitive analysis was based on a methodological tool, Toulmin's (1958) model integrated with the 'conception' model (Balacheff, 2013). This tool and the results obtained using it in different mathematical fields, geometry and algebra, were the innovative aspect of this study.

During the work on my Ph.D. thesis, I learned to study, compare and integrate educational studies that were deeply different from each other from methodological, cultural and scientific points of view. I learned to distinguish and select materials from different cultures and merge them perfectly. My two supervisors were very helpful during this stage. Nicolas Balacheff taught me to always have a solid basis for expressing my opinion and point of view. He taught me the importance of constructing a theory to support my ideas. Maria A. Mariotti helped me to order and organize my thoughts. When I was lost and felt I had too many ideas in my mind, I would talk with her and she would guide me and help me find my focus based on the purpose of my thesis. They both helped me to gain confidence and they always supported me, pushing me to be creative and innovative.

The interconnection between Italian and French cultures was essential for my Ph.D. project. The content of my thesis stems from an Italian notion: that of cognitive unity between argumentation and proof. However, it was in France that I developed the methodological competences to set this notion inside a theoretical framework. The creative and innovative aspects of my thesis probably come from Italian culture, but without the structural frame provided by the French methodology, the results of my work would probably not be so impressive and well accepted by the educational community.

During my Ph.D. work, I designed and implemented experimentations in French and Italian classes. What I observed in these classes not only confirmed my research hypothesis but also provided new insights and clues for future perspectives.

The three geometry problems I used in my experiment during my Ph.D. were solved differently by Italian and French students. In the eighth and ninth grade curricula, geometry proofs in France focus on transformations (translation, rotation and symmetry), while in Italy they are oriented toward the congruence theorems for triangles. Furthermore, the notion of proof for Italian and French students is not completely the same at these school levels. In Italy, when asked for a proof, students know that the proof needs to be deductive (*dimostrazione*). In France, the justification of the conjecture is also accepted when it is not deductive (*preuve* instead of *démonstration*).

Aside from these important differences, the results of my experiments were very similar. In both countries, I observed that sometimes students were unable to construct proofs because they were not able to transform the structure of their argumentation into a deductive structure (for example, some students often construct abductive proofs if they have previously produced abductive argumentations). This aspect was observed not only in France but also in Italy, where students were aware that deductive proofs were expected. Italian students were more reluctant to write proofs that were not deductive. In some cases, they preferred not to write anything at all. Instead, in France, although deductive proofs were explicitly required by the teacher, when students were not able to construct a deductive proof, they handed in informal justifications.

Despite the different curricula and the different conceptions of proof students had in France and in Italy, in both experiments I observed how hard it was for students to construct a deductive poof when they had previously constructed an abductive argumentation. This strong result, confirmed in two different cultures, pushed me to broaden my research. I decided to investigate whether these results could be generalized not only to other mathematical domains but also to other cultural environments (Martinez & Pedemonte, 2014; Pedemonte & Buckbinder, 2011; Pedemonte & Reid, 2010) in my studies concerning students' conceptions in the construction of proofs (Pedemonte & Balacheff, 2016). This research, bridging results from the sciences of language, epistemology and mathematics education, has been going on for more than 10 years. I believe that without my intercultural experience between Italy and France this research would not have been developed to this extent.

During my Ph.D. work in France, I was constantly in contact with other students. I had many discussions with them. We regularly had meetings where we compared our results, helped each other to find appropriate methods to construct our experiments and anticipated possible critical comments from external researchers. It was in this environment that I learned to accept negative feedback as constructive for my research, learning to become a strict reviewer of my own research.

I was very lucky to have two supervisors, but I needed to convince both of them about my new ideas. This taught me to strongly justify my research before sharing it with others. In a broader sense, my research needed to be accepted by two different research communities—Italian and French—with different research methodologies and theoretical frames. This taught me to always consider multiple points of view, to anticipate possible feedback and to integrate research characteristics (theoretical frameworks, methods and experimental research) from different cultures. Even now that I am living in the US, I have realized how important that experience was for my professional career. It would probably be harder to integrate myself into this new community if I had not learned in France how to manage this kind of situation.

Even if this experience was probably one of my best in my entire life, it was not easy to take the decision to move to France to earn my Ph.D. Beyond the difficulties coming from my poor French language skills, the most difficult aspect was to interconnect two different cultures, the Italian inside and the French outside. During the first six months, I refused the new culture: I did not like the French language because it was a real obstacle to me and I did not like the French bureaucracy, which I thought was even worse than the Italian. Also, people appeared strange to me even if they were nice. However, little by little this perception of the French world changed in me and this new culture became part of me, until I recognized both cultures inside me. I finally understood that this experience not only gave me a professional career but also a more flexible mind and an open view on what research consists of.

4.5 The Italian Tradition From a Chinese Cultural Perspective

A personal testimony by Xu Hua Sun

It is not easy for me to clarify what the Italian tradition is due to my limited knowledge. My presentation will comment on some impressive aspects of the Italian tradition from my personal point of view and from a Chinese cultural perspective. My considerations are based on my experience in collaborating with Mariolina Bartolini Bussi on the organization of the ICMI Study 23 (Bartolini Bussi & Sun, 2018; Sun, Kaur, & Novotna, 2015) and aim to foster a reflection on institutional and historical aspects of the Italian tradition.

4.5.1 On the Historical Aspects of the Italian Tradition

There is a long history of relationships between Italian and Chinese people. Marco Polo is perhaps the most famous of Italian visitors. When he visited ancient China in the 13th century, he was not the first European to reach China, but he was the first to leave a detailed chronicle of his experience, and his book opened communication between the East and West. His *Book of the Marvels of the World*, also known as *The Travels of Marco Polo*, c. 1300, was a book that described to Europeans the wealth and great size of China.

Less known to the majority, but well known to mathematicians, is the Italian Matteo Ricci, who in the 17th century collaborated with Xu Guangqi on translating the first six books of Euclid's *Elements* (Volumes 1–6; Chinese title: 幾何原本) that were published and printed in 1607.

This translation offered the Chinese people a first look into the Western tradition of logical deduction. Euclid starts with a list of definitions and postulates in order to reach some clearly stated theorems, providing the details of the necessary steps explicitly. This did not happen in ancient China as '[the Chinese] propose all kinds of propositions but without proof', as Ricci stated (from Dai & Cheung, 2015, p. 13). Most Chinese traditional mathematics concerned arithmetic and algebra that was not supported by deductive reasoning. In the ancient mathematical texts, a typical situation is as follows:

'Very often the presentation starts with a *particular* problem (*wen* in Chinese) being stated in words. Then an answer (*da* in Chinese) is given immediately after that, and from time to time, when it is necessary, the technique or algorithm (*shu* in Chinese) will be outlined' (Dai & Cheung, 2015, p. 13).

Chinese scholars (e.g., Guo, 2010) credit Matteo Ricci as having started China's enlightenment. Deductive reasoning, introduced by the translation of Euclid, had important influences on Chinese mathematics (Sun, 2016a, b). It is considered one of the most important events in Chinese mathematics history, changing Chinese mathematics teaching. The concepts of axioms, theorems, assumptions and proving appeared in the field of mathematics education for the first time. The Chinese terms for points, segments, straight lines, parallel lines, angles, triangles and quadrilaterals were created for the first time and have remained until the present, afterwards having spread to Japan and other countries.

4.5.2 On the Characteristic Aspects of the Italian Tradition

The impact of the Italian geometry culture became evident to me in 2011 when I visited the Laboratory of Mathematical Machines in Modena. The rich collection of more than 200 copies of ancient mathematical instruments stored and displayed in a large room in the Department of Mathematics was impressive to me. Most of these instruments were geometrical instruments (e.g., curve drawing devices and perspectographs) and were evidence of an advanced technological development together with a geometrical knowledge that did not seemed comparable with that of China in the same period. The cultural value of geometry, and in particular its relationship with artistic productions of famous painters such as Leonardo da Vinci, clarified to me the strength of geometry in the Italian didactic tradition and gave me a reason to understand the weakness of a lack of thoughtfulness in the Chinese tradition. This impression was reinforced by some talks that Mariolina Bartolini Bussi and Ferdinando Arzarello gave during the ICMI Study 23 in Macau and by the passionate discussions that followed.

In particular, I was very interested in the didactical setting of the mathematics laboratory as described in the curricula proposed by the Italian Mathematics Association.

Mathematics laboratory activity involves people, structures, ideas and a Renaissance workshop in which the apprentices learn by doing, seeing, imitating and communicating with each other, namely, practicing. In these activities, the construction of meanings is strictly bound both to the use of tools and to the interactions between people working together.

The experimentation-oriented approach that is promoted in the mathematical laboratory setting is different from exercise-oriented learning processes, which are dominated by the manipulation of symbols and a general lack of interest in the real world and are so common in Chinese school practice; at the same time, the experimentation-oriented learning process supported geometric perceptual intuition, for which concrete manipulation seems so fundamental (Zhang, 2016).

A critical comparison of Italian and Chinese culture (Bartolini Bussi & Sun, 2018) highlighted another key element: While the use of the number line can be considered a common feature and a familiar teaching aid in Italian classes throughout the different school levels, Chinese traditional practice nearly ignores it. In fact, the Chinese and Italian traditions are based on different ways of representing numbers. On one hand, representing a number on the number line can be traced back to the geometric roots of Euclid's *Elements* and can embody its geometric continuum as cultural heritage (Bartolini Bussi, 2015). On the other hand, in Chinese culture, numbers are related to the old practice of the abacus and to the verbalization of procedures based on numbers' names embedding a positional representation (Sun, 2015).

4.5.3 On the Institutional Aspects of the Italian Tradition

I had the opportunity to take part in the early implementation of a model of mathematics teacher development in Italian schools inspired by Chinese lesson study (Bartolini Bussi, Bertolini, Ramploud, & Sun, 2017) already mentioned above. This cooperation allowed me to understand many differences between the Italian and Chinese school systems from an institutional perspective.

First, the Italian school system is completely inclusive, whilst students with special needs in China attend special schools. Though it could be easier to plan lessons for homogeneous groups of students, in Italian classrooms, activities must be designed in order to have a positive effect on all students, including low achievers. This requires great care to design lessons for students with very different cognitive levels; nevertheless, the possibility for low achievers to be supported is important for an equitable society from a long-term perspective.

Second, the lesson duration in mainland China is strictly 45 min, without any flexibility, whilst in Italy teachers can adjust the activity a bit if there is not enough time.

Third, attention is paid by Italian teachers to long-term processes because of the autonomy given to schools to plan the whole year of activity, whilst in China, control is strictly exerted by the Ministry of Education. In general, an Italian teacher may have more freedom than a Chinese teacher to introduce innovative activities in order to fulfil the given goals, as time control is not exerted from the outside.

The above observations hint to institutional differences. We must acknowledge that in all cases institutional differences draw on cultural differences, as Italy is in the West, whilst China is under the influence of the so-called Confucian heritage culture (Leung, Graf, & Lopez-Real, 2006, 2015). This dialogue between Italian and Chinese culture is certainly partial and limited, though it is part of a more general dialogue fostered by ICMI in the last decade, as mentioned above (see Sect. 3.3). We hope such dialogue can continue and be deepened, allowing cultures to enrich each other.

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