Chapter 3
Didactics of Mathematics in the Netherlands

Marja Van den Heuvel-Panhuizen

Abstract This chapter highlights key aspects of the didactics of mathematics in the Netherlands. It is based on the Dutch contribution to the Thematic Afternoon session on European didactic traditions in mathematics, organised at ICME13 in Hamburg 2016. The chapter starts with a section in which mathematics education in the Netherlands is viewed from four perspectives in which subsequently attention is paid to the role of mathematics and mathematicians, the role of theory, the role of design, and the role of empirical research. In all these themes Hans Freudenthal has played a key role. Hereafter, the focus is on two Dutch mathematics educators (Adri Treffers for primary school and Jan de Lange for secondary school) who each left an important mark on how the didactics of mathematics has developed in the last half century and became known as Realistic Mathematics Education (RME). To illustrate the principles of this domain-specific instruction theory a concrete task is worked out in the section "Travelling to Hamburg". The chapter concludes with five sections featuring voices from abroad in which mathematics educators from other countries give a short reflection on their experiences with RME.

Keywords Realistic mathematics education · IOWO · Freudenthal · Mathematisation · Mathematics as a human activity · Didactical phenomenology · Empirical didactical research · Treffers · Students' own productions and constructions · De Lange · Contexts for introducing and developing concepts · Design-based research · Task design · Parametric curve · Findings from field tests

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3.1 Mathematics Education in the Netherlands Viewed from Four Perspectives

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3.1.1 The Role of Mathematics and Mathematicians in Mathematics Education in the Netherlands

When mathematics became a compulsory subject in primary and secondary school in the Netherlands at the start of the 19th century, professional mathematicians were not much involved. Furthermore, the Dutch government took a somewhat restrained position on what mathematics was taught and particularly on how it was taught. Decisions regarding the curriculum were considered an internal school affair. In 1917 this policy was formalized in the Dutch Constitution as Freedom of Education. In practice, this meant that changes in the school mathematics curriculum were usually discussed by teacher unions, or special committees of teachers, then approved by the school inspectors and only after that ratified by the government. Professional mathematicians played hardly any formal part in this process. They sometimes participated in secondary education final examinations, by acting as assessors in the oral examinations and checking the grading of students' work on the written examinations, but they were not responsible for these examinations, which were devised by a selected group of teachers and approved by school inspectors. Furthermore, professional mathematicians were scarcely involved in the production of textbooks for secondary and primary mathematics education, which were mainly written by teachers. The government left the production of textbooks to the market. Schools were free to choose those books they liked most. Although there is currently more government involvement in the 'what' of teaching through formulating standards and a series of compulsory tests, the freedom regarding textbooks still exists.

Professional mathematicians' interest in school mathematics began to grow in the first half of the 20th century. For example, in 1924 a mathematics journal was extended by an addendum in which didactical questions could be discussed; this addendum later became the still existing journal *Euclides*. The first issue of this journal paid attention to the argument between the mathematician Dijksterhuis and the physicist Ehrenfest-Afanassjewa. The latter had also studied mathematics in Göttingen with Klein and Hilbert. While Dijksterhuis insisted on keeping to the traditional approach to teaching school geometry, which was based on the formal characteristics of the discipline, Ehrenfest argued for making use of students' intuitive knowledge and starting with concrete activities in three-dimensional space. However, Ehrenfest's ideas and those of her discussion group on teaching mathematics which was to become the Mathematics Working Group in 1936, did not find much response from mathematicians. Even the famous Dutch mathematician Brouwer did not have any affinity with teaching mathematics in school.

Yet, from 1950 onwards, more attention was paid to mathematics education within the community of mathematicians. This was reflected, for example, in the establishment in 1954 of the Dutch Education Commission for Mathematics, an ICMI subgroup, in which mathematics teachers and mathematicians cooperated. Freudenthal became chair of the group shortly after its foundation. Although in those days more mathematicians felt that mathematics teaching needed modernisation, it was Freudenthal in particular who had a genuine and deep interest in the didactics of mathematics. So, it is no wonder that, after World War II, he had also joined Ehrenfest's group.

In 1961, the Dutch government appointed the Commission Modernisation of the Mathematics Curriculum (CMLW), which was a new phenomenon in the long history of government that was rather aloof with respect to curriculum issues. The founding of this new commission, consisting of professional mathematicians and teachers, was a direct consequence of the Royaumont conference. The Dutch government became convinced of the urgency of the modernization of mathematics education. Freudenthal was the most outstanding mathematician in this commission and he was also the only one who was heavily involved in the didactics of mathematics. He convinced the other commission members to focus on the teaching of mathematics instead of on the content of the curriculum and he also moved the attention of the commission to the lower grades of schooling. Later on, in 1971, IOWO (Institute for Development of Mathematics Education) was established with Freudenthal as its first director. The opening of the institute was the beginning of a long period of over forty years in which mathematics teachers and educators worked on the design and research of mathematics education in primary and secondary school and teacher education. The instructional designs and the underlying theory of Realistic Mathematics Education (RME) developed at this institute have changed the Dutch mathematics curriculum and the approach to teaching mathematics, and this happened without any government interference. Characteristic of this approach is that it starts with offering students problems in meaningful situations, from which contexts can gradually evolve into models that can be used to solve a broader scope of problems; through the process of progressive schematisation, students eventually end up understanding mathematics at a more formal level.

Until the late 1990s, RME was generally accepted for primary and secondary education. Also university mathematicians were involved in secondary education reform projects such as HEWET and PROFI. However, after 2000 some university mathematicians started to blame RME for the lack of basic mathematical skills of their first-year students. They wanted to return to the way of teaching mathematics that (in their view) was common some forty years ago. For primary school mathematics education, the Ministry of Education and the Netherlands Royal Academy of Sciences (KNAW) appointed a commission to arbitrate this Dutch Math War. The commission's conclusion was that there was no evidence that students' achievements would be better with either RME or the mechanistic back-to-basics approach. This conclusion resulted in RME being less in the firing line, and it became possible again to have a professional discussion among all stakeholders about primary school mathematics education. In the area of secondary education, the Ministry of Education appointed the Commission Future Mathematics Education (cTWO), consisting of mathematicians, mathematics education researchers and mathematics teachers, to revise the mathematics curriculum for upper secondary mathematics education. In addition, Regional Support Centres (Bètasteunpunten) were established to facilitate connections between secondary schools and universities. Also, the Mathematical Society and the mathematics teachers' association (NVvW), set up Platform Mathematics Netherlands, a new organisation for collaboration, which included, among other things, a commission for mathematics education. This commission only covers secondary mathematics education and not the teaching of mathematics in primary school.

Background information about the role of mathematics and mathematicians in mathematics education in the Netherlands can be found in Goffree, Van Hoorn, and Zwaneveld (2000), La Bastide-van Gemert (2015) and Smid (2018), for example.

3.1.2 The Role of Theory in Mathematics Education in the Netherlands

Mathematics education is not just about the process of teaching mathematics, but also encompasses ideas and knowledge about how students learn mathematics, about how mathematics can best be taught and what mathematical content should be taught and why. Finding answers to these questions is the main goal of the scientific discipline that in the Netherlands—in line with the European tradition—is called the didactics of mathematics.

At the beginning of the 19th century when the first textbooks were published in the Netherlands, the prefaces of these textbooks showed the initial efforts towards a theory of mathematics education that contributed to the development of the didactics of mathematics as a scientific discipline. A next step forward came in 1874 when the Dutch schoolteacher Versluys published his book on methods for teaching mathematics and for the scientific treatment of the subject. However, a decisive move towards a theoretical basis of mathematics education in the Netherlands was Freudenthal's unfinished manuscript *Rekendidactiek* (Arithmetic Didactics), written in 1944, but never published. Freudenthal's interest in mathematics education in primary school was triggered during World War II when he was teaching arithmetic to his sons, and observed their learning processes. He also carried out an extensive literature review of the didactics of arithmetic. A further advancement in Freudenthal's thinking about mathematics education occurred at the end of the 1950s when he worked with the Van Hieles and became familiar with the theory of levels. Inspired by this, he developed the very important didactic principle of guided reinvention in which decisions about guidance should be informed by analysing learning processes. For Freudenthal, the 're' in reinvention points to students' learning processes, and the adjective 'guided' to the instructional environment. Viewing learning as guided reinvention means striking a subtle balance between the students' freedom of invention and the power of the teachers' guidance.

Freudenthal's intention of giving mathematics education a scientific basis resulted in the publication of *Weeding and Sowing* in 1978, which he called a preface to a science of mathematics education. In this book, he introduced the didactical phenomenological analysis of mathematics, an approach which was further elaborated in *Didactical Phenomenology of Mathematical Structures*, published in 1983. According to Freudenthal, thorough analysis of mathematical topics is needed in order to show where the student might step into the learning process of mankind. In other words, a didactical phenomenology, rather than a pure epistemology of what constitutes mathematics, is considered to inform us on how to teach mathematics. This phenomenology includes how mathematical 'thought objects' can help organising and structuring phenomena in reality, which phenomena may contribute to the development of particular mathematical concepts, how students can come into contact with these phenomena, how these phenomena beg to be organised by the mathematics intended to be taught, and how students can be brought to higher levels of understanding.

Although mathematics plays a central role in these analyses, Freudenthal rejected the idea of taking the structure of mathematics or the expert knowledge of mathematicians as his point of departure. The goal was making mathematics accessible and understandable for students by taking their learning processes seriously. Freudenthal viewed working on the design of education and experience with educational practice as necessary requirements for making theory development possible. His work at IOWO, which was founded in 1971, and particularly his collaboration with the Wiskobas group around Treffers, and later with the Wiskivon group for secondary education, which both did a great deal of work with students and teachers in schools, was therefore crucial for Freudenthal's thinking.

At the same time, however, Freudenthal's involvement was important for IOWO as well. In addition to promoting mathematisation and mathematics as a human activity that is connected to daily life or an imagined reality, and emphasizing that students and even young children can generate a large amount of mathematical thinking—yet in an informal context-connected way—Freudenthal's participation was essential for another reason too. Being an authority in the field of mathematics as a discipline, Freudenthal legitimised the work done at IOWO from the perspective of mathematics.

Although the activities at IOWO, due to its focus on designing education, could be characterised as engineering work rather than as research, IOWO produced 'valuable splinters' which could be counted as research output. Freudenthal saw this approach as paradigmatic for how theory development must take place: from designing educational practice to theory. The theory that evolved from this work at IOWO was later called Realistic Mathematics Education (RME) and was initially described by Treffers in 1978, and published in 1987 in his book *Three Dimensions*. The principles of this domain-specific instruction theory have been reformulated over the years, including by Treffers himself, but are presently still seen as leading for RME.

The first principle, the activity principle, follows from the interpretation of mathematics as a human activity, and implies that students are treated as active participants in the learning process. The reality principle arises from considering mathematics as based in reality and developing from it through horizontal mathematisation, and entails that mathematics teaching provides students with meaningful problems that can be mathematised. The level principle highlights the idea that students pass through several stages of understanding, from informal, context-connected to formal mathematics. In this process, didactical models serve a bridging function and vertical mathematisation is stimulated. This level principle is also reflected in the procedure of progressive schematization. The intertwinement principle states that the mathematics curriculum is not split into isolated strands, but that, following the mathematisation of reality, the focus is on the connection and coherence of mathematical structures and concepts. The interactivity principle signifies the social-cognitive aspect of learning mathematics, and entails that students are offered opportunities to share their thinking with others in order to develop ideas for improving their strategies and to deepen their understanding through reflection. The guidance principle refers to organising education in such a way that guided reinvention is possible, through a proactive role of the teacher and educational programs based on coherent long-term teaching-learning trajectories that contain scenarios which have the potential to work as a lever to effect shifts in students' understanding. Several local instruction theories focusing on specific mathematical topics have been developed which align with these general principles of RME.

RME is not a fixed and finished theory of mathematics education and is still in development. Over the years the successors of IOWO—OW&OC (Research of Mathematics Education & Education Computer Centre), the Freudenthal Institute and the lately established Freudenthal Group[1]—have made different emphases. As a result of collaboration with researchers in other countries, RME has also been influenced by theories from abroad such as social constructivist approaches which contributed to the interaction principle of RME and provided RME with a lens for investigating classroom discourse. More recently, elicited by the use of new technology in mathematics education, approaches inspired by instrumentation theory have connected with RME to achieve a better understanding of how tool use and concept development are related. Finally, a further new avenue is the revitalisation of the activity principle of RME through the incorporation of embodied cognition and perception-action theories in which the focus is also on how students' concept development and deep learning can be understood and fostered.

However, when working on the further development of RME through integrating it with other theories, it is still important that mathematics should maintain its central place. Developing mathematics education and investigating learning and teaching processes should always be grounded in mathe-didactical analyses which unpack mathematics in didactic terms and take into account phenomenological, genetic-epistemological, and historical-cultural perspectives.

[1] In 2012 the Freudenthal Institute was split into two. The research and design work in early childhood, primary education, special education, and intermediate vocational education were moved to the Faculty of Social and Behavioural Sciences (FSW) and carried out by the *Freudenthal Group*. The research and design work in secondary education have remained part of the *Freudenthal Institute* in the Faculty of Science.

Background information about the role of theory in mathematics education in the Netherlands can, for example, be found in De Lange (1987), Freudenthal (1991), Treffers (1987a), and Van den Heuvel-Panhuizen and Drijvers (2014).

3.1.3 The Role of Design in Mathematics Education in the Netherlands

Making things work, looking for pragmatic solutions and being creative and innovative are typical features of Dutch culture and they occupy an important place in Dutch society. This emphasis on design can also be recognized in mathematics education and can be considered the most significant characteristic of the Dutch didactic tradition in the past half century.

The reform movement in mathematics education that started in the Netherlands at the end of the 1960s was all about designing 'new' education, which in those days meant working on an alternative to the mechanistic approach to teaching mathematics that was prevalent in the Netherlands at the time. This approach, which still has some followers today, is characterised by teaching mathematics at a formal level from the outset, an atomized and step-by-step way of teaching in which the teacher demonstrates how problems must to be solved, and the scant attention paid to the application of mathematics. At the same time that the need arose for an alternative for this mechanistic approach, two new approaches from abroad appeared: the empiricist trend in which students were set free to discover a great deal by themselves and were stimulated to carry out investigations, and the structuralistic trend propagated by the New Math movement in which the mathematics to be taught was directly derived from mathematics as a discipline. However, neither of these new approaches was well received in the Netherlands.

Therefore, at the end of the 1960s the Wiskobas group started to think about another way to improve teaching mathematics in primary school. From 1971 on this took place at the newly-established IOWO, which some time later was extended to include the Wiskivon group that had been formed to design a new approach to teaching mathematics in secondary education. All staff members of these two groups, except Freudenthal, had experience in school practice either as a mathematics teacher or as a mathematics teacher educator. This meant that their work was very practice-oriented. The theory development which resulted in RME was considered a derivative of this practical work and would later serve as a guide for further design activities. Because of the focus on the practice of teaching, it is not surprising that Freudenthal often stated that IOWO was not a research institute, and IOWO staff members did not regard themselves as researchers, but as producers of instruction, as engineers in the educational field. As implied by the latter term, this work was not done in isolation, but carried out with teachers and students in classrooms. Moreover, there was a strong collaboration with mathematics teacher educators, counsellors at teacher advisory centres, and textbook authors with whom the materials were discussed and who also contributed to their development. In this way, the implementation of the reform happened more or less naturally without specific government interference. By having these strong networks of people and institutions involved in mathematics education, new ideas for teaching mathematics could immediately be used in pre-service teacher education, in-service courses, and above all in textbooks. Of all the possible change agents, textbooks have played a key role in the reform of mathematics education in the Netherlands. For primary school mathematics, the same is also true for the mathematics education infrastructure that evolved from these networks. For secondary mathematics education a teachers' association (NVvW) had already been founded in 1925 and for primary school mathematics the infrastructure came into being later. In 1981 Panama was set up, which has come to involve a collaboration of institutions for pre-service and in-service mathematics teacher education, and in 1982 NVORWO was established as an association for primary school mathematics. The main purpose of the infrastructure as a whole was, and still is, to inform the mathematics education community in the Netherlands through national mathematics education conferences, professional journals, in-service courses and websites and to support national mathematical events for students.

The educational designs that have been produced over the years by IOWO and its respective successors, are multifaceted, ranging from tasks containing opportunities for mathematisation and paradigmatic contexts that evolve into level-shifting didactical models, to tasks for mathematics days and competitions for students, to elaborate teaching sequences for particular mathematical domains. Among other things, the design work in primary school mathematics led to helpful contexts such as the pizza context in which students could produce fractions by themselves through fair sharing activities, and the bus context in which students were encouraged to reason about passengers entering and exiting and so invent their own symbolic notations of what happens at a bus stop. The design work for primary school also resulted in some very powerful didactical models that can be found in most current textbooks in the Netherlands, such as the empty number line, the arithmetic rack, the percentage bar and the ratio table. With respect to upper secondary education, new programs were developed in the 1980s and 1990s for Mathematics A (preparing students for studies in the social sciences) and Mathematics B (preparing students for studies in the natural sciences). Additionally, at the turn of the century new RME-based modules on calculus and geometry were developed for Mathematics B in the upper grades of pre-university secondary education. A prominent design project that was carried out with the University of Wisconsin involved the development of a complete textbook series *Mathematics in Context* for Grade 5–8 of the U.S. middle school. This project began in the mid-1990s and ran for some ten years.

Another long-term design project was the TAL project that started in 1997. Its aim was the development of longitudinal conceptual teaching-learning trajectories that describe the pathway that students largely follow in mathematics from Kindergarten to Grade 6. The decision to work on such trajectories was innovative at that time. The basis for this TAL project was the so-called Proeve, a first version of a national curriculum for primary school mathematics that led to the official enactment of the first description of the core goals for mathematics at the end of primary school at the beginning of the 1990s. The teaching-learning trajectories were designed to describe how these core goals could be reached, thus providing teachers, textbook authors and test developers with an insight into the continuous learning line of learning mathematics, so contributing to making the curriculum more coherent.

The advent of computer-based technology in schools again brought new demands and challenges for design. In addition to exploring opportunities for computer-assisted instruction, much effort was also put into rethinking the subject of mathematics within the context of the virtual world and exploring how students could benefit from the dynamic and interactive qualities of the new technology. This led not only to the development of the so-called Digital Mathematics Environment in which teachers can adapt and design instructional material for their students including the use of mathematical tools and feedback, but also resulted in a seemingly inexhaustible flow of applets and mini-games for primary and secondary education that are freely available online.

Background information about the role of design in mathematics education in the Netherlands can, for example, be found in Bakker (2004), Doorman (2005), Drijvers (2003), Gravemeijer (1994), National Center for Research in Mathematical Sciences Education & Freudenthal Institute (1997–1998), Streefland (1993), and Van den Heuvel-Panhuizen (1996, 2003).

3.1.4 The Role of Empirical Research in Mathematics Education in the Netherlands

Research in the Netherlands into the learning and teaching of mathematics since the first half of the 20th century has always been empirical in one way or another. Initially this research was undertaken mostly by psychologists and pedagogues with an interest in mathematics, but later on it was also done by mathematics teachers. A prominent example of such research was the didactical experiment carried out by Van Hiele-Geldof in the 1950s on teaching geometry in the first year of secondary school. Her thesis about this experiment contained a very careful description of how she developed the teaching sequence that brought students from visually supported thinking to abstract thinking. She also recorded precise protocols of what happened in the classroom, which were then thoroughly analysed. Starting with what she called a psychological-didactical analysis of the mathematical content was part of her research method. In fact, Van Hiele-Geldof's work, greatly admired by Freudenthal, contained many important ingredients of the research into the learning and teaching of mathematics that was done in the Netherlands from then on.

Freudenthal's empirical didactical research began with observing his own children as he was teaching them arithmetic during World War II. It is noteworthy that he warned at first against overestimating the value of these observations, stating that he would like to do observations with a more diverse sample and on a larger scale. Later, he apparently changed his opinion. In the 1970s he emphasized the strengths of qualitative small case studies. He even called on research in the natural sciences ("One Foucault pendulum sufficed to prove the rotation of the earth") to prove the power of observing a student's learning process. For Freudenthal, one good observation was worth more than hundreds of tests or interviews. The reason for this preference was that observing learning processes led to the discovery of discontinuities in learning, which Freudenthal regarded as being of great significance for understanding how students learn mathematics. Merely comparing scores of a large sample of students collected at different measuring points would imply that only an average learning process, in which the discontinuities have been extinguished and all essential details have disappeared, can be analysed.

The design work that was carried out at the IOWO from 1971 onwards, with the aim of creating materials and teaching methods for the reform of Dutch mathematics education, was also highly informed by empirical research. In agreement with Freudenthal, the emphasis was on small-scale qualitative studies carried out in schools. Based on didactical-phenomenological analyses of the mathematical domains and making use of knowledge of students' learning processes and the classroom context, learning situations were initially designed using thought experiments. These were followed by actual experiments with students and teachers, and the reactions of both students and teachers were observed. In this way, IOWO staff members ran school experiments in which the mathematics to be taught and the teaching methods were continuously adapted based on experiences in classrooms and feedback and input from the teachers. In this development process, design, try-out, evaluation and adaptation followed each other in short, quick cycles. Reflections on what happened in the classrooms focused not only on whether or not a learning process had taken place, but also on what impeded or facilitated its occurrence. These reflections and the accompanying intensive deliberations among IOWO staff members about the designed learning situations provided important theoretical insights which evolved into the RME theory of mathematics education. In their turn, these theoretical insights led further designs. In other words, theory development and the development of education were strongly interwoven.

This type of research, initially called developmental research but later given the internationally more common name design research, was the backbone of RME-based research activities. Over the years, the method of design research was developed further. Data collection and analysis procedures which would contribute to the evidential value of the findings of design research were added. The theoretical grounding was also elaborated, through prior mathe-didactical analyses and through including findings and approaches from the education and learning sciences.

In addition to design research, depending on what specific research questions have to be answered, various other empirical research methods are used, such as quasi-experiments (including pretest-posttest intervention designs and micro-genetic designs), surveys (including questioning teachers about their classroom practice and beliefs about mathematics education and carrying out expert consultations), document studies (including textbook and software analyses and study on the history of mathematics education) and review studies and meta-analyses. Also, outside the circle of RME-affiliated didacticians, there is a large group of researchers in the Netherlands consisting of psychologists, orthopedagogues, and cognitive neuroscientists who focus particularly on investigating how specific student characteristics influence students' learning of mathematics. In this way, they complement the research done by didacticians. Similarly, research by educationalists who, among other things, investigate school organisation and classroom climate also provide relevant knowledge for all involved in mathematics education.

Relevant empirical data to direct the development of mathematics education were also acquired through the PPON studies that, from 1987 on, have been carried out every five years by Cito, the national institute for educational measurement. It is important that these studies gave an overview of changes over time in the mathematics achievements of Dutch primary school students and of the effect of the use of particular textbooks. Finally, PISA and TIMSS provide the international perspective on achievement data for both primary and secondary school students.

Background information about the role of empirical research in mathematics education in the Netherlands can, for example, be found in Freudenthal (1977), Goffree (2002), Van den Brink and Streefland (1979), and Van der Velden (2000).

3.2 Students' Own Productions and Own Constructions—Adri Treffers' Contributions to Realistic Mathematics Education

Marc van Zanten[2]

3.2.1 Introduction

The development of Realistic Mathematics Education (RME) started with the setup of the Wiskobas project in 1968. Wiskobas is an acronym for 'Wiskunde op de basisschool', meaning mathematics in primary school. Treffers was one of the leading persons within Wiskobas from the beginning onwards. He can be considered as one of the founding fathers of RME. In an interview held on the occasion of the ICME13 conference, Treffers (Fig. 3.1) stated that, in his view, the active input of students in the teaching and learning process is the basis for good mathematics education. This was one of the issues that came to the fore in the interview, together with other ideas Treffers published on over the years, which are mentioned also in the text. In the interview, Treffers highlighted the issues that are most important to him, and looked back on his earliest sources of inspiration for his work on mathematics education. He explained how certain people had had a major influence on his vision of mathematics education. Interestingly, they turned out to be very special people not belonging to the

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Fig. 3.1 Adri Treffers

community of mathematics education: he learned a lot about mathematics education from his five year older sister, his brother with whom he shared a bedroom, the father of his friend Beppie, and from Mr. Zwart, a very good teacher he had.

3.2.2 Treffers' Theoretical Framework for Realistic Mathematics Education

Treffers described in detail the principles of RME in his seminal publication *Three dimensions. A model of goal and theory description in mathematics instruction—The Wiskobas project* (1978, 1987a). The framework for an instruction theory for RME, formulated in the 1987 version, was built on the work of Wiskobas, established in collaboration with Freudenthal (1968, 1973), and his ideas about mathematics as a human activity, avoiding mathematics education as transmitting ready-made mathematics to students, and instead stimulating the process of mathematisation. This framework consisted of five instruction principles, derived from the didactical characteristics of Wiskobas. Over the decades these principles have been reformulated and further developed, both by Treffers himself and by other didacticians, and are still seen as the core teaching principles of RME, as described in the Sect. 3.1 of this chapter (see also Van den Heuvel-Panhuizen & Drijvers, 2014).

3.2.3 Students' Own Productions

According to Treffers (1987a, p. 249), teachers can help students to find their way to higher levels of understanding, but this "trip should be made by the pupil on his

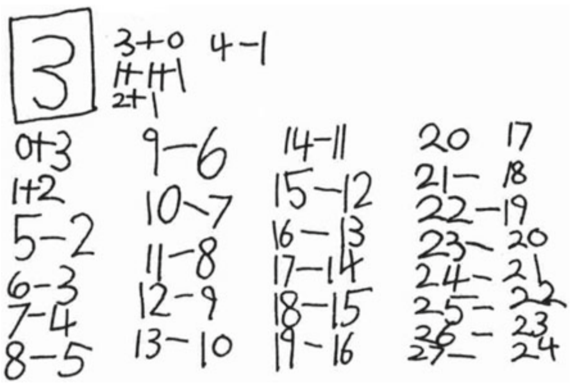

Fig. 3.2 An own production by a first grader (Grossman, 1975; see Treffers & De Moor, 1990, p. 163)

own two legs". Therefore, a decisive influence in the learning process comes from the students themselves in the form of their own productions and their own constructions. Referring to this contribution of the students to the learning process, Treffers (1987a, p. 250) speaks of an "essential factor", which explains his preference for students' own productions. In the interview Treffers revealed who formed the basis for this insight.

> Treffers: My first source of inspiration is the most important one. That is my five year older sister. You could say that she invented a new didactic principle. Playing school in the attic, she let me and my friend Beppie make up our own mathematical problems. Nowadays we would call them students' own productions. An example of a completely free production is: "Produce problems that have 7 as the answer." To be honest, I don't think that we did it like that back then, but now we do.

Students producing problems themselves is one of the ways in which they can actively contribute to their own learning process. This can take place from the first grade on, as shown in Fig. 3.2. Here the assignment was to make up problems that should have 3 as the answer. These students' own productions can serve as 'productive practice', which can be done alongside to regular practice. One benefit of productive practice is that it engages many students. Another is that students are not limited to a certain range of numbers. As a result, students can actually surprise their teachers with their productions. For example, a student in the first grade who was asked to make up problems with the answer 5 came up with the problems '100–95', '2000–1995' and '10,000–9995', which were problems far beyond the number range the student had been taught at that moment. What happens when students make these productions is that they make use of the structure of the number system, which is a form of mathematisation.

Students' own productions evoke reflection, which stimulates the learning process. In particular, asking students to produce simple, moderate and complex problems can cause students to reflect on their learning path. Figure 3.3 shows an example of how students' own productions can be elicited in order to make them aware of what they find easy and difficult in algorithmic subtraction.

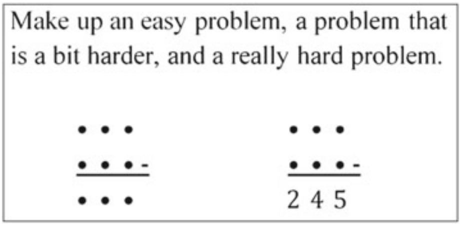

Fig. 3.3 Own productions in the context of algorithmic subtraction (Treffers, 2017, p. 85, p. 87)

While discussing students' own productions, all the difficulties of algorithmic subtraction can come to the fore, including borrowing once, borrowing more than once and borrowing from zero, Treffers points out. In line with this, Treffers comes up with another problem.

> Treffers: Cover up some digits of a subtraction. Can another student reconstruct the original subtraction? How many digits can be covered up at max?

These questions may lead to more productive practice, but they also connect students' own productions to problem solving. The latter is significant because of the importance of challenging students, which came up later in the interview.

3.2.4 Students' Own Constructions

As mentioned previously, Treffers does not see the learning of mathematics as a process of absorbing ready-made knowledge. Instead, he considers the understanding of mathematics as a process that is constructed by students themselves.

> Treffers: My second source of inspiration is my brother, with whom I shared a bedroom. He set me sometimes, teasingly, a few problems, like "you can't do those yet." But that taught me how to move along the imagined number line and flexible arithmetic, for example that you can calculate the multiplication tables in a smart way.

However, Treffers' view that students construct their own knowledge does not mean that mathematics education should rely on students' self-reliant discovery. Instead, instruction should make use of students' own contributions and should help them through 'guided reinvention', as Freudenthal called it. Students' own informal solution methods function as a starting point for such guided instruction. Figure 3.4 shows, as an example, informal constructions for 8×23 by students beginning to learn multi-digit multiplication.

In the various additive and multiplicative methods used by the students, several steps of the upcoming learning path are already recognizable. The teacher makes use of this in the interactive discussion of the students' solutions. He points out handy ways, such as in this case the use of products of the multiplication tables. In general, students are encouraged to think critically about their own solution methods

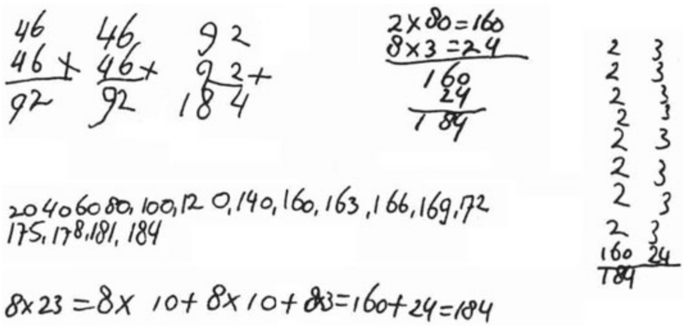

Fig. 3.4 Beginning third graders' constructions of solutions for 8 × 23 (Treffers, 1987b, p. 128)

Hans made a trip of 75 kilometer.
After 48 kilometer he took a brief rest.

1. How many kilometer must he do after the rest?
2. How do you think he travelled, by car, by bike, on foot, ...?
3. How long do you think the whole trip of 75 kilometer took him?

Fig. 3.5 The problem of Hans (Treffers, 1991, p. 338)

and compare them with their classmates' solutions and the solutions the teacher emphasizes. These latter are purposely chosen to guide the students to a gradual process of schematising, shortening and generalising.

The problem in Fig. 3.5 shows another type of problem that requires students' own constructions. Treffers (1991) is a strong proponent of these 'daily life' problems in which students have to make use of all kinds of measurement knowledge and have to figure out this knowledge based on their experiences with the situation involved. Moreover, this type of problems elicits reasoning and further questioning: How far is 75 km? Roughly how many km does a car cover in one hour? How fast does a bike go? What is the speed of a pedestrian? How long would a cyclist take to do 48 km? Taking a rest after half an hour, after two hours—what is sensible? What does a 'brief' rest mean? A quarter of an hour, half an hour, or a few hours? This kind of reasoning, involving arguing, proposing solutions and calculations in which knowledge of number and numerical data is both used and increased, is important for the development of students' numeracy. Treffers introduced this term and its importance in Dutch primary school mathematics education, leading to its inclusion in the officially established objectives for primary school.

3.2.5 Challenging Students with Classical Puzzles

Another feature of Treffers' work is the stimulation of students' thinking by offering them challenging problems, such as the subtraction problems mentioned earlier, but also more complicated, classical mathematical problems.

> Treffers: The father of my friend Beppie, a cobbler, gave us classical puzzles, including the 'Achilles and the Tortoise' paradox, and the famous 'Wheat and the Chessboard' problem. Later, a very good teacher, Mr. Zwart, elaborated these further, touching on problems we were still struggling with a bit. For example, whether 0,999… with the decimal 9 repeating infinitely is or is not equal to 1. Beppie's father and Mr. Zwart were my third and fourth sources of inspiration.

In his work, Treffers elaborated on the ways in which classical mathematical puzzles like the 'Wheat and the Chessboard' can be set in primary school so that there is, again, room for students' own constructions. Starting with students' informal approaches, discussing these in an interactive setting, leading the students to a shortened and structured procedure, all these features mentioned earlier are present in his descriptions. Due to the influential work of Treffers and his colleagues, these ideas made it into Dutch textbooks.

3.2.6 Students' Input Is the Basis of Everything

To conclude, Treffers comes with advice for mathematics education today.

> Treffers: I feel that of the things we spoke about, the issue of students' own productions is the most important one. Take for example, magic squares (Fig. 3.6). Students need to think about how to solve them, but they can also produce them for themselves. For the teachers, it means that they can see what students find easy or hard and how they can sometimes fly off far above the familiar range of number and the difficulty level of the problem they could have thought of themselves. Here you are talking about students who have an input in the teaching and learning process and that is the basis of everything.

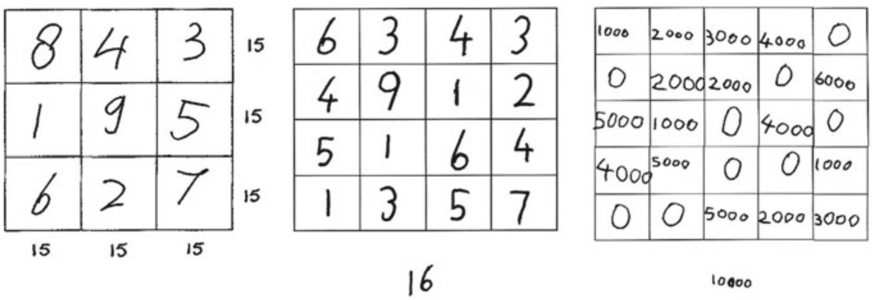

Fig. 3.6 Second-grade students' own productions of magic squares (De Goeij & Treffers, 2004)

Treffers sees students' own productions and constructions not only as a didactical tool, but also as a goal of mathematics education. Naturally this is true, since producing and constructing mathematics yourself is in essence mathematising. This ability is even more important in modern society than it already was in the early days of Wiskobas.

> Treffers: My general recommendation for the future of mathematics education is: enlarge the role of students' own productions and own constructions, in practice, in problem solving, and in the combination of the two.

3.3 Contexts to Make Mathematics Accessible and Relevant for Students—Jan de Lange's Contributions to Realistic Mathematics Education

Michiel Doorman[3]

3.3.1 Introduction

De Lange has been working at Utrecht University for 40 years. He led the Freudenthal Institute from 1981 and as Professor/Director from 1989 until 2005. He started as a mathematician and initially was more interested in upper secondary education. His most recent interests lie in the study of talents and competencies such as the scientific reasoning of very young children. De Lange worked on the theoretical basis of assessment design, carrying it through to practical impact in the Netherlands and internationally as chair of the PISA Mathematics Expert Group (Fig. 3.7).

De Lange's contributions to the ideas underpinning Realistic Mathematics Education (RME) are strongly connected to the role of contexts in mathematical problems. In traditional mathematics education, contexts are included in textbooks as word problems or as applications at the end of a chapter. These contexts play hardly any role in students' learning processes. Word problems are mostly short storylines presenting a mathematical problem that has a straightforward solution. Applications at the end of a chapter help students to experience how the acquired mathematical procedures can be applied in a context outside mathematics.

In RME, context problems have a more central role in students' learning process from the very start onwards. These problems are presented in a situation that can be experienced as realistic by the students and do not have a straightforward solution procedure. On the contrary, students are invited to mathematise the situation and to invent and create a solution. Ideally, students' intuitive and informal solutions anticipate the topics of the chapter and provide opportunities for the teacher to connect these topics to the students' current reasoning. In RME, such context problems are

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Fig. 3.7 Jan de Lange during the interview

expected to have a central role in the guided reinvention of mathematics by the students themselves.

This understanding of the potential of context problems was developed during the 1970s and 1980s and was largely inspired by the work of De Lange. His contributions to the Dutch didactic tradition consisted of developing a large collection of teaching units used mainly in innovation-oriented curriculum projects in the Netherlands and in the USA. In preparation for the Dutch contribution to ICME13 Thematic Afternoon session on European Didactic Traditions, De Lange was interviewed to reflect on his work and specifically on the importance of contexts in mathematics education.

3.3.2 Using a Central Context for Designing Education

After De Lange graduated in the 1970s he started his career as a mathematics teacher. Soon he discovered that students reacted quite differently to the topics that he tried to address in his lessons. Most surprising for him was that they did not recognize mathematics in the world around them. After De Lange moved to the Freudenthal Institute, one of his ambitions was to find contexts that could be used to make mathematics accessible. A teacher in lower secondary school asked him if he could do something for her students who had problems with trigonometric ratios. De Lange designed a unit intended for a couple of weeks of teaching.

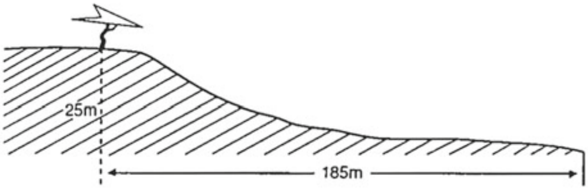
> De Lange: I started to work on one of my hobbies, planes, and I wrote a little booklet. It is called *Flying Through Maths*. It is about all kinds of different mathematics, all in one context. It is about glide ratios, vectors and sine and cosine.

All the mathematics in this teaching unit is presented in the context of planes and flying, which approach is later referred to by De Lange as an example of 'central context design' (De Lange, 2015). This means that the same context is used to

GLIDE RATIO

The glide ratio is used to compare the performances of gliders or hang gliders. The first hang glider had a glide ratio of 1:4 (one to four). The improved version had a glide ratio of 1:7.

4. Define what is meant by a glide ratio.



Otto Lilienthal made about 2500 flights. On one of his flights from the Rhinower Mountains near Berlin, he started from a height of 25 meters and flew a distance of 185 meters. On his next flight he changed his glider a bit, and then started at a height of 20 meters to reach 155 meters.

5. What were the glide ratios of Otto Lilienthal's two gliders? Was the second glider better than the first one?

Fig. 3.8 *Glider* problem from the booklet *Flying Through Maths* (De Lange, 1991, p. 7)

introduce students into various mathematical concepts. One of the concepts that was presented in this flying context was the glide ratio (Fig. 3.8).

The context of flying is used to encourage students to reason about covering distances when gliding from a certain height. By comparing different flights, students are expected to come up with some thinking about the glide ratio, i.e., the ratio between the distance covered and the starting height. This glide ratio plays a role in the context problems in the beginning of a chapter on slopes. In this way, the glide ratio is meaningful for students. They can use it for solving problems that they can experience as real problems. Later in the chapter this glide ratio is generalized to triangles and as a measure that can be used to calculate or compare slopes.

3.3.3 Contexts for Introducing and Developing Concepts

RME brought about a new perspective on the use of contexts in mathematics education. Contexts are not only considered as an area for application learned mathematics, but also have an important role in the introduction and development of mathematical concepts.

> De Lange: Applications is one thing. In the traditional textbooks, it was the end of the book. You started first with learning mathematics, and then you got the applications of mathematics. Through our theory developed at the Freudenthal Institute in the 70s, we changed that to developing concepts through context. So, you had to be very careful, because if the context is not very suited for the concept development, you are riding the wrong train on the wrong

> track. But I think in general we can say for a lot of concepts we found very nice contexts to start with.

An example is the context of exponential growth that can support students in developing the logarithm-concept (De Lange, 1987). In this context, the growth of water plants, students first calculate the exponential increase of the area covered by these water plants with the growth factor and the number of growing weeks. At a certain moment, the question in the context is reflected. The question is no longer what the area is after a certain number of weeks, but how many weeks are needed to get an area that is 10 times as much? Students will experience that this is independent of the starting situation and can be estimated by repeating the growth factor (e.g., with a growth factor of 2 this is a bit more than 3 weeks). After these introductory tasks, ${}^{2}\log 10$ is defined as the time needed to get 10 times the area of water plants when the growth factor per week is 2. With this context and the concrete contextual language in mind, students can develop basic characteristics of logarithmic relations such as ${}^{2}\log 3 + 1 = {}^{2}\log 6$ as follows: with this 1 extra week, you get 2 times more than 3, which equals 6. Similarly, ${}^{2}\log 6 + {}^{2}\log 2$ has to equal ${}^{2}\log 12$, as ${}^{2}\log 6$ is the number of weeks to get 6 times as much, and ${}^{2}\log 2$ is the number of weeks to get 2 times as much. The time needed to first get 6 times as much followed by the time needed to get 2 times as much has to be equal to the time needed to get 12 times as much.

With such a context problem, a concept is not only explored, but also more or less formalized. Such a concrete foundation is important because it offers opportunities for students in the future to reconstruct the procedure and meaning of the abstract calculation procedures by themselves.

3.3.4 Relevant Mathematics Education

The aforementioned examples show the potential of contexts for learning mathematics and for making that learning process meaningful and relevant for students. This approach to mathematics education connects to the RME instructional theory in which the learning of mathematics is interpreted as extending your common sense reasoning about the world around you. Hence, De Lange emphasises in his reflection on educational design that designers need to find contexts by meeting the real world outside mathematics and experience the potential of contexts by going to real classrooms (De Lange, 2015). Observing authentic classroom activities is crucial. He stresses the importance of direct observations without using video in order to observe much more. In such a direct observation, one is able to look at the students' notes, one has the possibility to participate with their work, and one can ask questions in order to understand why students do what they do.

Exploiting the real world guided De lange towards a wide variety of original and surprising contexts. One example is the art of ballooning. Flying a balloon depends completely on the strength and direction of the wind and the change of the wind

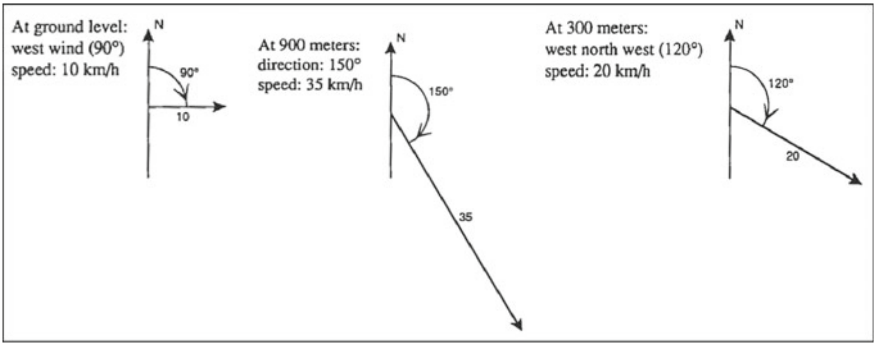


Fig. 3.9 Ballooning information from the booklet *Flying Through Maths* (De Lange, 1991, p. 34)

> In the Netherlands, one out of four high school students has used drugs (at least once). Design a study to find out how many students at your school have used drugs. Do you get a representative sample by using your school when you want to say something about the use of drugs at all schools in your city?

Fig. 3.10 *Drug use* problem (De Lange & Verhage, 1992, p. 12)

speed along different altitudes. The following example (Fig. 3.9) is also taken from the booklet *Flying Through Math* and is typical for a situation in which you are supposed to travel with a balloon from one spot to a target. The task for the students was to determine what happens when a balloon starts from Albuquerque and flies the first half hour at 300 m, then an hour at 900 m and finally, a half hour at 300 m.

In this context, not all information is available. Remaining questions are: How much time is needed to land? What happens when you go from one altitude to another and how much time does that take? The task becomes a real problem solving task for the students.

Through being in real classrooms De Lange could also look for contexts that trigger interest in students. In choosing these contexts, he was not afraid of using controversial situations. This can be recognized in a task for 15-year olds about drug use (see Fig. 3.10), for example.

In the interview, De Lange emphasised again that contexts serve many important roles in the teaching and learning of mathematics. They support conceptual development, can be motivating and raise interest, and also teach students how to apply mathematics.

> De Lange: We should be aware that contexts have to be mathematised. This means that we should be aware of what is the relevant mathematics in the contexts, which concepts plays an important role, and can the contexts serve as the starting point of modelling cycles. So, what you actually see is, that in the first phase of learning from context to concept, you use things, you do things, which are exactly the same as using the concept in a problem-solving activity.

> In a certain country, the national defence budget is $30 million for 1980. The total budget for that year is $500 million. The following year the defence budget is $35 million, while the total budget is $605 million. Inflation during the period covered by the two budgets was 10 percent.
>
> a. You are invited to give a lecture for a pacifist society. You intend to explain that the defence budget decreased over this period. Explain how you could do this.
>
> b. You are invited to lecture to a military academy. You intend to explain that the defence budget increased over this period. Explain how you would do this.

Fig. 3.11 *Military-budget* problem (De Lange, 1987, p. 87)

The examples of problems discussed above all illustrate how contexts can be used for designing relevant and meaningful mathematics education. What can also be recognized is that in the 1980s designers like De Lange already anticipated what we now call 21st century skills. A nice example of this is the *Military-budget* problem (Fig. 3.11), which some thirty years ago was designed by De Lange to stimulate students to become mathematically creative and critical.

3.3.5 Conclusion

Creative designers like De Lange, people who are able to convince others of the limitations of many textbooks and who are able to translate general educational ideas into original and attractive resources for students, are of crucial importance for realising meaningful and relevant mathematics education. In 2011, he was awarded the *ISDDE Prize for Excellence in Design for Education*. Malcolm Swan wrote on behalf of the prize committee: “He has a flair for finding fresh, beautiful, original, contexts for students and shows humour in communicating them.” Without De Lange’s contributions many ideas in the Dutch didactic tradition would have been less well articulated, less well illustrated, and less influential in the world outside the Dutch context.

3.4 Travelling to Hamburg

Paul Drijvers[4]

3.4.1 Introduction

This section describes an example of a task that was designed and field-tested for the ICME13 conference. Its aim is to illustrate how the principles of Realistic Mathematics Education (RME) (see, e.g., Van den Heuvel-Panhuizen & Drijvers, 2014). can guide the design of a new task. Indeed, task design is a core element in setting up mathematics education according to an RME approach. This is one of the reasons why design-based research is an important research methodology in many studies on RME (Bakker & Van Eerde, 2015).

The task presented here involves setting up a graph that many students are not familiar with, because it displays one distance plotted against another. For several reasons, it makes sense to have students work on such less common graphs. First, graphs in mathematics education in almost all cases involve an independent variable, often called x, on the horizontal axis, and a dependent variable, for example y or $f(x)$, on the vertical axis. However, there are also other types of graphs than these common x-y graphs. In economics, the independent variable—not always x but also t for time—can also be plotted on the vertical axis rather than on the horizontal. In physics—think about phase diagrams—the independent variable may be a parameter that is not plotted on one of the axes. The latter case reflects the mathematical notion of parametric curve, in which the independent variable remains implicit. In short, students should be prepared for other types of graphs as well.

A second, more general reason to address non-typical types of graphs is the worldwide call for mathematical thinking and problem solving as overarching goals in mathematics education (Devlin, 2012; Doorman et al., 2007; Schoenfeld, 1992). If students are to be educated to become literate citizens and versatile professionals, they should be trained to deal with uncommon problem situations that invite flexibility. As such, mathematical thinking has become a core aim in the recent curriculum reform in the Netherlands (Drijvers, 2015; Drijvers, De Haan, & Doorman, submitted).

In this section, first the task will be presented together with some design considerations that led to its present form. Next, a brief sketch is provided of the results of the field test in school and of what the task brought to the fore at the ICME13 conference. As a task may need adaptation to the specific context in which it is used, its elaboration is described next for the purpose of in-service teacher training, including a three-dimensional perspective. Finally, the main points will be revisited in the conclusion section.

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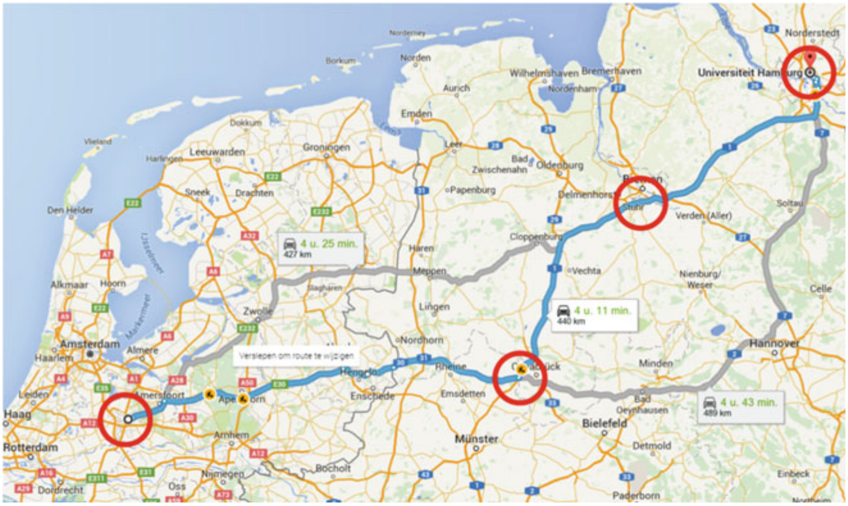


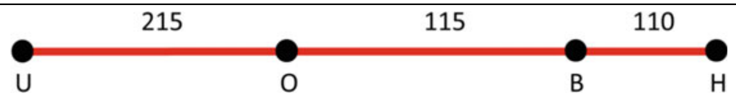
To attend the ICME13 conference, we travelled to Hamburg by car. To prepare for this, I consulted Google Maps. Here you see the result: we leave Utrecht, in the Netherlands. The distance to Hamburg is 440 km. After 215 km, we pass by Osnabrück. Another 115 and we pass by Bremen. Next, we need to drive the final 110 km to reach Hamburg.

Fig. 3.12 Setting the scene for the task

3.4.2 Task Design

Figure 3.12 shows the presentation of the task in the form of a picture, displayed through a data projector, and a suggested text that might be spoken by the teacher. As the problem situation is a somewhat personal story from 'real life', it is preferable to deliver the text orally rather than in written form. It is expected that the task becomes 'experientially real' through this form of presentation. In the task, the perspective taken is that of a participant in the ICME13 conference in Hamburg, Germany. Of course, this perspective could easily be adapted to other situations that are more relevant to the audience.

Figure 3.13 shows how the task presentation might continue. It shows a schematisation of the problem situation, in which the 'noise' of the real map has been removed. In the text that might be spoken, this schematisation and the underlying mental step of representing the highway as a line segment are explicitly addressed. Depending on the audience and the intended goal of the task, of course, one might consider leaving this step up to the student and to reduce guidance at the benefit of opportunities for guided reinvention. For the field tests addressed in the next section, it was decided not to do so, due to the expected level of the students and the time constraints. The text in Fig. 3.13 ends with the problem statement. Students are invited to use their worksheet, which contained two coordinate systems like the one shown in Fig. 3.14.



If I depict the highways as straight-line segments, I can globally represent the situation in this way. U stands for Utrecht, 215 to O (Osnabrück), 115 to B (Bremen), and 110 to H (Hamburg). Now I am wondering the following. During the trip, while driving, my distances to both Osnabrück and Bremen constantly change. What would a graph look like, if I put the distance to Bremen vertically, and my distance to Osnabrück horizontally? Could you please take a few minutes to sketch the graph in which the distance to B during the trip is vertically plotted against the distance to O on the horizontal axis? Please use your worksheet!

Fig. 3.13 Schematising the situation and posing the problem

Fig. 3.14 Coordinate system that is shown on the worksheet

One might wonder if this is a realistic task. Who would be so silly as to raise this question? Well, we were. We were three mathematicians who were bored during the long trip to Hamburg. How to "sell" this to students? A possible approach could be to "play the card of the strange mathematician", but this should not be exaggerated. Our experience is that students may be intrigued by such problem situations, even if the question itself is not solving a 'real' problem. Also, even if one might ask "Why do we need to know this?", the problem situation is sound in the mathematical sense, and has possibilities for applications in mathematics and science, as explained in Sect. 3.4.1.

After the task is presented, students can start to work in pairs, in small groups, or individually. As it is expected that there will be quite a bit of differentiation in the class—some students might solve this task in a minute, whereas others may not have a clue where to start—the students who finish quickly can be provided orally with an additional task:

> If you feel you are doing well, please think of a question that you might use to help a peer who doesn't know how to start, a question that might serve as a scaffold.

Also, some scaffolding questions are prepared that may serve as a hint to react to students who have difficulties with the task and raise their hands for help. For example: How can you make a start? Do you know a similar but easier problem?

Does this resemble a problem that you have seen in the past? Where are you in the *O*-*B* plane when you are leaving Utrecht? And when you arrive in Hamburg? And when you pass by Osnabrück?

To be effective, such a class activity needs a whole-class wrap-up. It might start with the question of how to help peers to make a start, or how you started yourself. For example, one can consider finding the position in the plane for the special moments of leaving Utrecht and arriving in Hamburg. This leads to the points $(O, B) = (215, 330)$ and $(O, B) = (225, 110)$. How about, when passing Osnabrück and Bremen? Another option is to imagine what happens in between O and B: the distance to O increases as much as the distance to B decreases. How does this affect the graph? A natural question that emerges, is whether the driving speed should be constant, and if it matters at all. Would the graph look different if you walk from Utrecht to Hamburg rather than driving (not recommended, of course)? In an advanced class, with many students coming up with a sensible graph, it might be interesting to show an animation in a dynamic geometry environment, which in its turn may invite setting up parametric equations. Of course, how far one can go in such a wrap-up largely depends on the students' progress. If needed, postponing the presentation of the results to the next lesson may be an appropriate 'cliff-hanger teaching strategy'. The expertise and the experience of the teacher in leading the whole-class wrap-up are decisive in making the task work in class.

In retrospective, the following considerations guided the design of this task:

– To make the problem situation come alive for the audience at the ICME conference, the trip to Hamburg was chosen as a point of departure. The 'experientially real' criterion was decisive.
– In the beginning, there was some hesitation on whether to travel by train or by car. The advantage of the train would have been that, contrary to cars driving on highways, trains do pass through the city centres. However, it was estimated that the car version would be more recognizable to the audience, particularly in combination with the Google Maps image and driving directions. As an aside, the designers of the tasks did not travel by car to Hamburg themselves; the story is based on another car trip. The point in designing this task is not the truth of the story behind it, but its experiential reality and mathematical soundness.
– How openly to phrase the problem? When designing the problem different versions came up, with different levels of support. Indeed, the version we had in mind might be quite a surprising challenge to students, but it was expected that through the scaffolding hints mentioned above, it would be possible to have the students start.
– How to present the problem? It was decided to present the task orally to the class as a whole, supported by slides displayed through a data projector. The idea here was that this would enhance the personal character of the problem situation. Also, such an oral whole-class introduction is expected to provide a collaborative setting, while working on a shared problem. Finally, an oral presentation can be a welcome change after many textbook-driven activities.

– It was decided to provide the students with the crucial linear representation (Fig. 3.13) and, in this way, give away the first schematisation step. Other choices can make perfect sense here. All depends on the level of the students, their preliminary knowledge, the time available, and the learning goals.
– To deal with student differences in this task, a second layer was built in, namely, that of thinking of hints for peers. In this way, students who finished the task quickly were invited to put themselves in the place of their slower peers, and, as a consequence, reflect on the thinking process needed to solve the task.

3.4.3 Field Tests

To prepare the activity for the ICME13 conference, the task was field-tested in a bilingual class in a rural school in the Netherlands. The students, 13- to 14-year olds, took part in the pre-university stream within secondary education. The pilot took one 50-minute lesson. After the oral introduction, students went to work. The question needed to be repeated once or twice. Also, we had a short whole-class discussion after the first tentative graphs, and invited the students to sketch a second one afterwards.

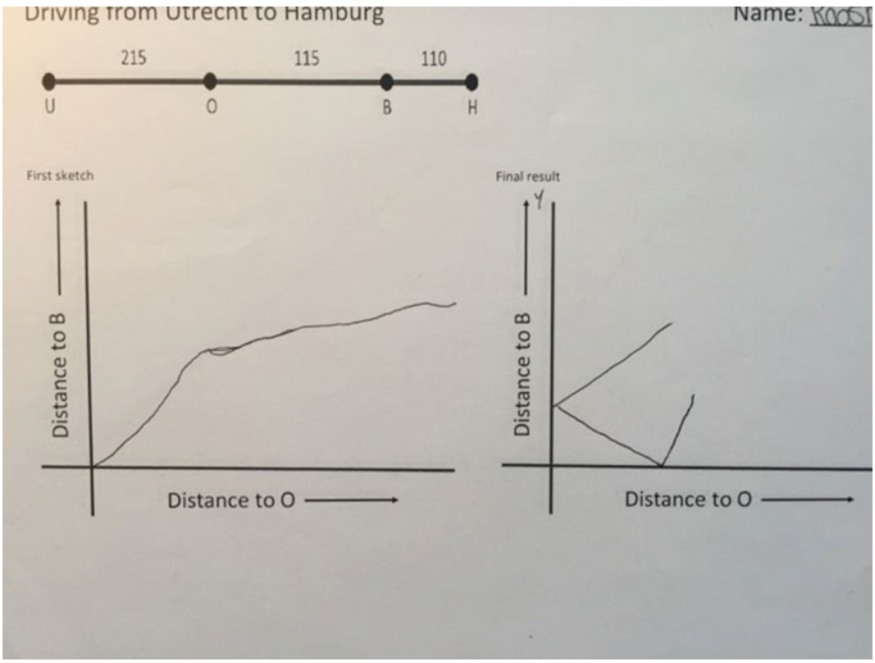


Fig. 3.15 Graphs by Student 1

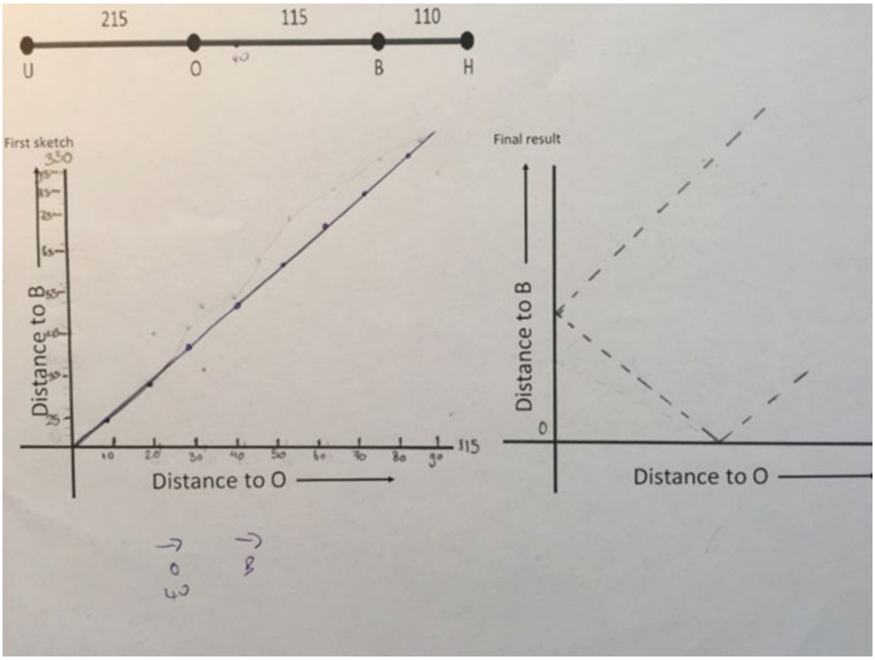


Fig. 3.16 Graphs by Student 2

Figure 3.15 shows the work of Student 1 who initially seemed to identify the graph with the map, not an uncommon phenomenon while introducing graphs.

The second sketch made by Student 2, shown in Fig. 3.16, is much better, even if the first and the last part of the graph are not parallel. Student 2's first graph was linear, suggesting a proportional increase of both distances. Clearly, this student did not have a correct mental image of the problem situation at the start. After the whole-class interruption and maybe some discussion with peers, the second graph was close to perfect.

In the whole-class wrap-up, Student 2 explained his initial reasoning, but was interrupted by Student 3, who introduced the notion of linearity.

Student 2: First, I had like this, but I thought, you can't be in the origin at the same time, you can't be in *B* and in *O* at the same time ….

Teacher: Yeah, you cannot.

Student 2: So, I thought like, maybe they, yeah, I don't know, I can't really explain it.

Student 3: It's a linear formula.

Teacher: Wow, how come? Why is it…. Please, explain.

Student 3: Well, ehm, since there isn't, eh yes, since the amount added always is the same, the first step, it's a linear formula.

This short one-lesson intervention confirmed the initial expectations, that the problem situation was rich and could give rise to interesting discussions.

During the ICME13 conference in Hamburg, the task was piloted again in a similar way with an audience of about 250 attendants. Of course, individual help was hard to deliver in this large-scale setting. Still, in comparison to the field test in class, similar patterns could be observed. Also, the need for level differentiation was bigger than in the secondary class, due to the heterogeneity of the ICME audience. It was surprising how mathematics teachers, researchers and educators have their schemes for graphing, and can get quite confused once these schemes are challenged by new situations.

3.4.4 Possible Task Extensions

As already mentioned, guided reinvention, meaning and experiential reality are subtle matters. To be able to deal with this subtlety appropriately in a setting with students of different levels, a good task should provide teachers with opportunities to simplify the task, to provide variations, and to deepen and extend the task. A straightforward way to simplify the task is to leave out one of the two cities between Utrecht and Hamburg, or even to leave out both and ask for the graph of the distance to Hamburg against the distance to Utrecht. These might be appropriate first steps towards solving the original problem. As variations, one may consider similar situations, such as the already mentioned trip by train, or a bike ride from home to school.

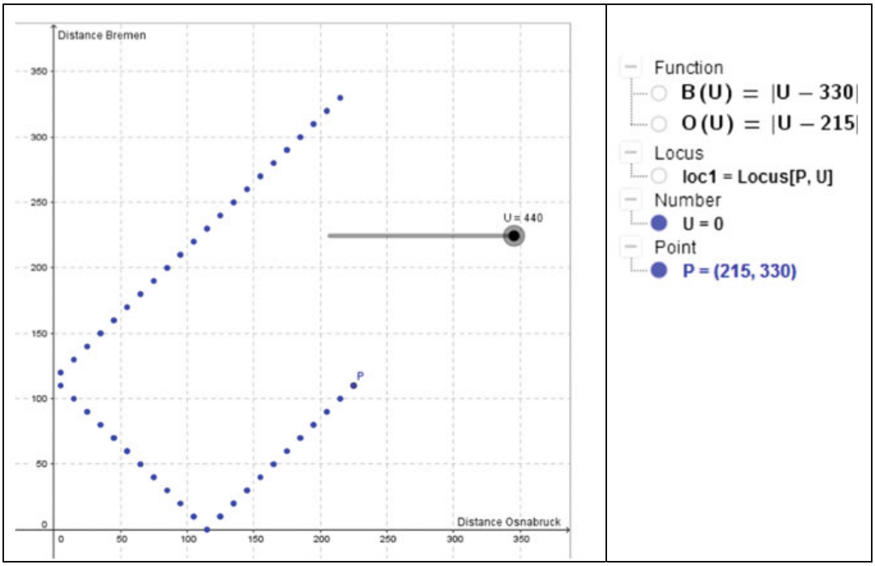


Fig. 3.17 Animation in Geogebra (left) and the underlying function definitions (right)

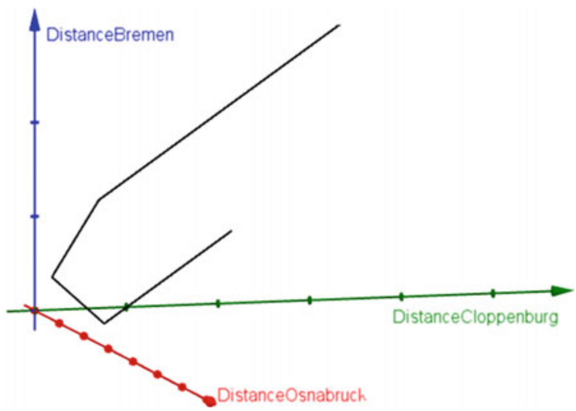
As students may come up with different graphs and will make all kinds of gestures while explaining their reasoning, it might be convenient to show the graph through an animation in a dynamic geometry system. The left screen of Fig. 3.17 shows such an example in Geogebra, using a slider bar to move the point. This may help to illustrate the resulting graph. In the meantime, however, this raises a deeper question: How can you make this animation, which equations and definitions are needed? The right screen in Fig. 3.17 provides the answer. The following definitions were used:

• Distance to Utrecht: U	(Independent variable)
• Distance to Osnabrück: $O(U) = \lvert U - 215\rvert$	(Dependent variable)
• Distance to Bremen $B(U) = \lvert U - 330\rvert$	(Dependent variable)
• Point in the plane: $P(U) = (O(U), B(U))$	

In this way, we take a mathematical perspective and the problem forms a gateway to the fascinating world of parametric curves.

As a final extension, also a third city between Utrecht and Hamburg can be considered. For example, Cloppenburg is about in the middle of Osnabrück and Bremen: Osnabrück–Cloppenburg is 60 km, and Cloppenburg–Bremen is 55 km. Can you plot a graph, indicating how the distances to Osnabruck, Cloppenburg and Bremen co-vary during the trip? Note that this task, in line with its higher level, is phrased in a somewhat more abstract way. Of course, the graph in this case will be in three dimensions rather than in two. Again, an animation can be built in Geogebra (Fig. 3.18). Rotating the graph shows a familiar form (Fig. 3.19) and in a natural way raises new, interesting questions, such as on the angle between the trajectory and the planes. This latter extension to the third dimension was used in a teacher professional development course, in which the participants found the two-dimensional case relatively easy, but were intrigued by the problem situation.

Fig. 3.18 3D graph in Geogebra



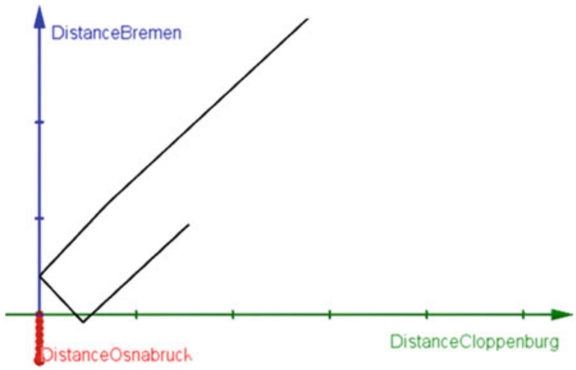


Fig. 3.19 3D graph in Geogebra seen from one of the axes

3.4.5 Conclusion

This example on task design according to RME principles revealed that for both students in Dutch secondary school and for participants of the ICME13 conference, it was hard to have the flexibility to refrain from the conventional time-distance graph paradigm and to open the horizon towards distance-distance graphs. This type of mathematical flexibility, needed in this unconventional and non-routine task, is core in problem solving, and at the heart of what RME sees as an essential value in mathematics education. The point of departure is 'realistic' in the sense that both target groups could imagine the situation and seemed to perceive it as realistic. What makes the task suitable from an RME perspective is that it can be used in different variations, appropriate for different levels of students and for different mathematical learning goals. Also, there are different, more and less mathematical, approaches and solution strategies, as well as follow-up questions. Finally, the somewhat surprising character of the task, may lead to the kind of lively and mathematically interesting interactions among students and between students and their teacher that are so important in the co-construction of mathematical meaning.

These task characteristics are central in RME and reflect the approach to mathematics education in the Dutch didactic tradition. To design tasks that elicit genuine mathematical activity in students is a challenge, not only in the Netherlands but in the mathematics education community world-wide!

3.5 Voices from Abroad

The chapter concludes with five sections which give a flavour of the international life of RME. From the beginning of the development of RME, mathematics educators all over the world were interested in it. This led to cooperation with a large number of countries where RME ideas and materials were tried out, discussed and adapted.

The countries that are represented here in these short notes about their experiences with RME all had a prominent place in a certain phase of the development of RME, and cover all corners of the globe, including the United States, Indonesia, England and the Cayman Islands, South Africa, and Belgium.

3.5.1 Realistic Mathematics Education in the United States

David Webb,[5] **Frederick Peck**[6]

The origins of Realistic Mathematics Education (RME) in the United States can be traced back to a proof-of-concept study (De Lange, Burrill, Romberg, & van Reeuwijk, 1993) at a high school in Milwaukee organized by Romberg (University of Wisconsin) and De Lange (Freudenthal Institute). The success of this pilot study illustrated how RME design principles could be applied in U.S. classrooms. More recently, RME continues to be articulated largely through professional development opportunities offered at innovation centres. As we trace the spread and scale of RME in the United States, the instantiation of RME is best characterised as a teacher-centred approach that involves principled reconsideration of how students learn mathematics. Reconsideration of beliefs and conceptions is often motivated when teachers re-experience mathematics through the lens of progressive formalisation and related didactic approaches (Webb, Van der Kooij, & Geist, 2011). Teachers' participation in the interpretation and application of RME in U.S. classrooms has led to systemic innovation that has been sustained, inspired and supported by professional development and curricula, and by fellow teachers who provide their colleagues with a proof-of-concept in their local context.

3.5.2 Two Decades of Realistic Mathematics Education in Indonesia

Zulkardi,[7] **Ratu Ilma Indra Putri**,[8], **Aryadi Wijaya**[9]

In Indonesia, some two decades ago, the process of adapting Realistic Mathematics Education (RME) began (Sembiring, Hadi, & Dolk, 2008). The Indonesian approach is called 'Pendidikan Matematika Realistik Indonesia' (PMRI). This process began in 1994 when Sembiring from the Institut Teknologi Bandung met De Lange, the director of the Freudenthal Institute of Utrecht University, who was presenting a keynote

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at the ICMI conference in Shanghai. The next step was the decision of the Indonesian government to send six doctoral candidates to the Netherlands to learn about RME. Afterwards the development and implementation of RME was continued through a Dutch-Indonesian project called 'Do-PMRI' (Dissemination of PMRI). Moreover, implementation strategies were carried out such as developing a master's program on RME, designing learning materials using RME theory and the development of a national contest of mathematics literacy using context-based mathematics tasks similar to those employed in the PISA test (Stacey et al., 2015). Recently, there were two new initiatives at Sriwijaya University in Palembang, namely the development of a Centre of Excellence of PMRI and the establishment of a doctoral programme on PMRI.

3.5.3 Implementing Realistic Mathematics Education in England and the Cayman Islands

Paul Dickinson,[10] **Frank Eade**,[11] **Steve Gough**,[12] **Sue Hough**,[13] **Yvette Solomon**[14]

Realistic Mathematics Education (RME) has been implemented in various projects over the past ten years in the secondary and post-16 sectors of the English education system. All of these projects can be characterised as needing to deal with clashing educational ideologies. In particular, pressure towards early formalisation and the heavy use of summative assessment has influenced how far it is possible to change teachers' practices and classroom cultures. Nevertheless, intervention studies based on the use of RME materials and RME-inspired pedagogic design showed that, in post-tests, students were willing to 'have a go' at problems, indicating confidence in their ability to make sense of a problem and to apply their mathematics in different contexts. Also, students were able to use a range of strategies to answer questions, including a use of models which reflected higher levels of sense-making in mathematics than before (Dickinson, Eade, Gough, & Hough, 2010). Even in post-16 national examination resit classes with students who had experienced long-term failure in mathematics, and where teaching is normally focused on examination training, small but significant achievement gains were found in number skills following a short intervention (Hough, Solomon, Dickinson, & Gough, 2017). While questionnaire data did not show any improvement in students' attitudes to mathematics, analysis of interview data suggested that this finding reflected the long-term legacy of their previous experience of learning mathematics by learning rules without meaning, but nevertheless, many students reported enjoying mathematics more in RME classes. In

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the Cayman Islands, with an education system that is influenced by British tradition, but which is distant from many of its politically driven accountability pressures and measures, the RME approach with primary students who had poor number sense led to a substantial gain in achievement. Also in secondary school, students were very positive about RME materials and made improvements in achievement as well. Teachers agreed to continue using contexts, interactive approaches and models to support problem solving rather than focus on formal algorithms. So, despite the problems encountered in these projects, there are reasons to remain optimistic about the potential of an RME approach in the English system.

3.5.4 Reflections on Realistic Mathematics Education in South Africa

Cyril Julie,[15] **Faaiz Gierdien**[16]

The project Realistic Mathematics Education in South Africa (REMESA) was introduced in South Africa during a period when curriculum changes were introduced to fit the educational ideals of the 'new' South Africa. In the project, a team comprising staff from the Freudenthal Institute and the Mathematics Education sector of the University of the Western Cape develop several RME-based modules, which were implemented in classrooms. One of the modules was *Vision Geometry* (Lewis, 1994). It was deemed that this topic was a sound way to manifest the RME approach. Content of vision geometry such as lines of sight, angle of sight, perspective, was encapsulated in activities for students. Although the students found the activities enjoyable and not above their abilities, the teachers had concerns about the time needed for the activities, the curriculum coverage and the examinability of the module's content. This scepticism remained after the module was somewhat adapted. Another module that was developed was *Global Graphs* (Julie et al., 1999). This module was also adapted by the South African staff. For example, another introduction was chosen than in the original RME version, namely instead of having the students construct graphs, asking them to match graphs and situations. A difference with the other module was that it was developed with a larger group of practising teachers. Moreover, the teaching experiments occurred when there was a more stable, albeit contested, operative curriculum in the country. Overall, teachers expressed satisfaction about the usefulness of the module *Global Graphs*. In contrast with the module *Vision Geometry*, the module *Global Graphs* was more readily accepted, which also can be ascribed to the prominence of graphical representations in South African school mathematics curricula. The lesson learned from the REMESA project is that the proximity of innovative approaches to the operative curriculum plays an important role in teachers' adoption of resources for their practice. In addition, the REMESA

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project has contributed positively to current research and development endeavours to address the issue of high-quality teaching of mathematics in secondary schools in low socio-economic environments in a region in South Africa.

3.5.5 Influences of Realistic Mathematics Education on Mathematics Education in Belgium

Dirk De Bock,[17] **Wim Van Dooren**,[18] **Lieven Verschaffel**,[19] **Johan Deprez**[20], **Dirk Janssens**[21]

The second half of the last century was a turbulent time for mathematics education in Belgium. In the 1960s and 1970s, mathematics education—as in many other countries—was drastically changed by the New Math movement that broke through (Noël, 1993). This revolution first took place in secondary school and entered primary school a few years later. Then, for about twenty years, the official curricula in Belgium followed this New Math approach faithfully. When from the 1980s on, New Math was increasingly criticised in Flanders, it was opted for a reform along the lines of Realistic Mathematics Education (RME) (De Bock, Janssens, & Verschaffel, 2004). In the end, this led to a reformed Flemish primary school curriculum that indeed is strongly inspired by the Dutch RME model, but that certainly is not simply a copy of that model. Although, for example, more attention was paid to linking numbers to quantities and to solution methods based on heuristic strategies in addition to the standard computational algorithms, the Flemish curriculum maintained the valuable elements of the strong Belgian tradition in developing students' mental and written calculation skills. Also in secondary school the critique of New Math led to using elements of the Dutch RME model to enrich the Belgian mathematics curriculum. Among other things, this resulted in giving solid geometry a more prominent place, making geometry more connected to measurement, and having a less formal, intuitive-graphical way of introducing calculus. In addition to other content changes, this reform brought also a number of didactic innovations, such as the role given to modelling and applications and to authentic mathematical exploration, discovery and simulation.

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