How Variance and Invariance Can Inform Teachers' Enactment of Mathematics Lessons

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Abstract The use of systematic variance and invariance has been identified as a critical aspect for the design of mathematics lessons in many countries where different forms of lesson study and learning study are common. However, a focus on specific teaching strategies is less frequent in the literature. In particular, the use of systematic variation to inform teachers' continuous decision-making during class is uncommon. In this chapter, we report on the use of variation theory in the Math Minds Initiative, a project focused on improving mathematics learning at the elementary level. We describe how variation theory is embedded in a teaching approach consisting of four components developed empirically through the longitudinal analysis of more than 5 years of observations of mathematics lessons and students' performance in mathematics. We also discuss the pivotal role of the particular teaching resource used in the initiative. To illustrate, we offer an analysis of our work with a Grade 1 lesson on understanding tens and ones and a Grade 5 lesson on distinguishing partitive and quotitive division.

Keywords Variation pedagogy · Math Minds · Elementary · Formative assessment · Division · Place value

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1 Introduction

Systematic variation has been identified as an important element of lesson study and learning study (e.g., Huang and Li 2017), which have become popular models for teacher professional development around the world (Hart et al. 2011; Huang and Shimizu 2016). Generally, lesson study involves the design of a lesson by a team of teachers and teacher educators; the implementation of the lesson in the classroom, observed by the team; the refinement of the lesson; a second implementation; and a report of the results (sometimes with further iterations). Learning study is similar to lesson study but includes testing students before and after the lesson as well as the use of variation theory as a theoretical framework (Huang and Li 2017; Marton et al. 2004). Pang et al. (2017) explained that two complementary perspectives on “variation pedagogy” (Watson 2017) have developed, first in parallel and later through extensive interaction between researchers from each tradition: “Bianshi Jiaoxue” was developed in China, and pedagogical approaches based on the variation theory of learning proposed by Marton (2015) and colleagues were first developed in Sweden. Both perspectives are prominent in the literature regarding learning study (Huang and Li 2017; Marton and Häggström 2017; Marton et al. 2004) and have been implemented in lesson study (cf. Bruce et al. 2016; Han et al. 2017). This chapter addresses the model of teacher professional development developed by the Math Minds Initiative, which includes elements of lesson study enacted in a particular way and builds on both Marton’s approach to variation and Chinese perspectives on variation.

While different forms of lesson study have been implemented in Canada (Bruce and Ladky 2011; Bruce et al. 2016; Preciado-Babb and Liljedahl 2012; Tepylo and Moss 2011), there have been other forms of collaboration in the design of teaching artifacts as a means of professional development. For instance, Preciado and Liljedahl elaborated on different modes of “teachers’ collaborative design” (p. 23), which involve elements of lesson study such as codesign, enactment of a lesson, debriefing of results, and refinement of the lesson. Local constraints such as timetables and lack of supports for substitute teachers have made it difficult to implement lesson study in Canada. In our work with the Math Minds Initiative, we also found that having few or no participating teachers teaching the same grade level made it very difficult for teachers to teach, redesign, and reteach a lesson during the same year. Nonetheless, we have included key elements of lesson study in our work, as we describe in this chapter.

A significant feature of our approach to lesson study has been the use of systematic variation for both lesson adaptation and for directly informing in-the-moment teaching decisions. In this chapter, we describe a lesson observation protocol that we use as a tool for analyzing and refining lessons with teachers. It was developed through the analysis of classroom observations and longitudinal data on students’ performance in mathematics. Systematic variation is key to each aspect of the protocol.

This protocol represents a theoretical contribution that has the potential to inform different modes of teachers' collaborative design, including lesson and learning study. While a focus on systematic variation has been widely reported in the literature on learning study as critical for task design and research analysis of mathematics lessons, we assert that such focus can also have a significant impact on how the tasks are enacted in the classroom. Like Kullberg et al. (2014), who reported how the same task can be enacted in different ways and offer different possibilities to learn, we have observed (Preciado-Babb et al. 2016a) how the use of the same lesson plans by different teachers yielded contrasting results in terms of students' learning (as measured by standardized tests and as observed in terms of student engagement in lessons). In this chapter, we stress the importance of both continuous assessment during the enactment of a lesson and teachers' appropriate responses to student feedback. We posit that systematic variation can inform both. This approach emphasizes not only lesson design (which can be anticipated) but also emergent situations that require teachers to improvise during class. The chapter elaborates on the methodology and key findings of the Math Minds Initiative, describes how variation has informed the development of the observation protocol, and presents two examples of how the protocol has been used in the project.

2 The Math Minds Initiative

The Math Minds Initiative started in 2012 with the intent of improving mathematics instruction at the elementary level through design-based research (Cobb et al. 2003) that involved the collaboration of several organizations, including the University of Calgary, Calgary Catholic School District, Golden Hills School Division, JUMP Math (2018), and Suncor Energy Foundation (the latter as sponsor). In Phase 1 (2012–2017), the initiative focused on an elementary school with a highly diverse demography and a long history of low performance in mathematics. Although the initiative started in 2012, the project was not fully implemented until September 2013. A second elementary school was added in 2014 and a third school from another school district in 2016. In total, the initiative has included professional development and weekly mathematics lesson observations for 31 participant teachers and the video recording of about 300 lessons. The study also included a longitudinal analysis of student performance in mathematics, as measured by the Canadian Test for Basic Skills (CTBS; Nelson 2018), and 44 teacher interviews and 228 student interviews. We are currently involved in a second phase of the project that aims to design a model for teacher professional development that is based on the results from the first phase.

In Phase 1 of the initiative, we observed consistent improvement in student performance in mathematics. Scores from the mathematics component of the CTBS (Grades 2 to 6) were collected each year. We used a linear mixed model (LMM) to accommodate this *unbalanced* study design. In this way, "not all individuals need to have the same number of observations and not all individuals need to

Table 1 LMM estimates of average t-scores for CTBS

	n	2013	2014	2015	2016	2017
School 1	147	43.582	45.691	47.511	49.654	–
School 2	131	–	47.113	49.373	53.202	52.09
School 3	85	–	–	–	46.881	49.398
All schools	363	44.054	45.943	47.725	50.052	50.93

All estimates are significant at $p < 0.001$

be measured at the exact same time points" (West 2009, p. 212). CTBS scores were converted to t-scores and normalized before conducting the analysis. Table 1 summarizes the results of the longitudinal analysis per school and for the whole intervention. The analysis showed a significant improvement in student performance in mathematics from Year 2 to Year 5 of the project (the study was fully implemented in its second year), with national percentile rankings rising from 27 to 55, which is equivalent to a rise in t-scores from 44.054 to 50.93 (see Table 1).

The results were not the same for all groups (Preciado-Babb et al. 2016a). By contrasting results and observations from different classrooms, we were able to identify teaching approaches associated with higher rates of improvement in CTBS scores, particularly conceptual understanding, and higher levels of student engagement in mathematical activities, as observed during classroom visits. These teaching approaches informed the observation protocol described in this chapter.

Lesson analysis has been an essential component of our work with teachers in the Math Minds Initiative. Because we worked with small schools, there were limited opportunities for regular cycles of lesson study in the same year. Often, there was only one participant teacher teaching a particular grade level. The initiative's approach to lesson analysis, debriefing, redesign, and re-implementation was therefore based mostly on modification from 1 year to the next and between closely related lessons within and across grade levels. We also considered longitudinal results of students' performance to inform lesson redesign.

The interaction of teachers and researchers and the use of a shared resource provided by JUMP Math (2018) have been pivotal to efforts to revise lessons and compare results of different implementations. JUMP Math resources include students' *assessment-and-practice* books, teacher guides, and predesigned SMART® Notebook slides for use with interactive whiteboards. In turn, this work informed professional development sessions for teachers. Thus, we have been able to test not only the improvement of lesson plans but also the teaching approaches used during the lessons. Thus, the Math Minds approach emerged iteratively with a close focus on individual lessons and themes that became sites of attention that encompassed teachers from different grades, over multiple years, and over multiple research sites. Over the course of the project, our work has grown in ways that distinguish it from common approaches often situated as opposite in North America; it does not neatly align with either traditional (commonly associated with direct teaching and rote memorization) or progressive (often associated with inquiry-based learning) teaching approaches (cf. Metz et al. 2016).

It is important to stress that many classrooms in Canada have considerable diversity regarding achievement levels in mathematics. The classrooms we observed all had such gaps, spanning up to three grade levels. The strategies we describe were developed to support all students in such diverse classrooms.

3 Approach to Variation

Marton's (2015) theory is based on the conjecture that "novel meanings are acquired through contrast and not through induction" (Marton and Häggström 2017, p. 391). Close attention to what varies and what remains the same in the examples and tasks that students are exposed to is essential. Contrast supports learners in discerning critical features of the intended object of learning. The term *critical discernment* is used here to indicate a critical feature that students need to discern in order to learn something.

We emphasize that a mathematical concept involves multiple discernments *woven* together into a coherent, powerful idea. There is a cumulative aspect to these discernments: As concepts are developed, they become elements of contrast needed for discerning new concepts. In other words, learning a concept involves (a) becoming familiar with the discernments that comprise it and (b) appropriately integrating those discernments with one another. For instance, when learning to represent numbers from 9 to 20 in the decimal system, it is important for students to integrate two critical discernments: (a) grouping by tens and (b) assigning value with respect to place (e.g., there are one ten and three units in 13).

Teaching, thus, involves formatting a concept in a manner that supports learning, starting with the identification of necessary discernments. Taken together, these discernments and the relationships between them comprise the *intended object of learning* (Marton 2015). We note that the manner in which discernments are woven together must take into account the limits of learners' abilities to hold multiple discernments in mind. The variation theory of learning can inform a systematic sequencing of activities and examples offered to students; such sequencing becomes the *enacted object of learning*. It is important that the teacher assesses all students' understanding every time a new discernment is presented and as students respond to prompts intended to support their integration into broader mathematical structures; when done well, doing so can offer insight into what Marton described as the *lived object of learning*. Further, this assessment can inform how the teacher decides to proceed with the lesson.

3.1 The Teaching Approach

The teaching approach developed in Math Minds consists of four elements enacted recursively during class. These now form the basis of our framework for lesson design and classroom observation. The elements are (a) raveling, (b) prompting, (c) interpreting, and (d) deciding, which we now elaborate.

Raveling refers to the identification of a critical discernment within a coherent sequence of discernments. One part of *raveling* happens prior to teaching and involves long-range planning on the part of teacher and resource. The teacher and/or the resource thoroughly decompose a concept (to identify fine-grained discernments that are embedded within) and recompose that concept, drawing attention to how discernments may be knitted back together and to how the concept fits within a broader network of mathematical understanding; we refer to this as *macro*-raveling. *Micro*-raveling occurs at the level of an individual lesson. Ideally, students are continuously prompted to critical discernments woven together into a meaningfully connected big idea that stays focused on the heart of lesson. Each new idea is anchored to previously developed ideas that have been summarized to carry forward (e.g., a quick summary, a key word, an image); there is attention to transforming the new into something familiar and to anticipating next steps. We see strong connections between micro-raveling and Gu, Huang, and Marton's (Gu et al. 2004) descriptions of conceptual and procedural variation in the Bianshi tradition.

Prompting is about engaging students in ways that (a) channel their attention to each critical discernment, through the use effective patterns of variation to draw attention to each discernment, and (b) require students to *make those distinctions*. For instance, the teacher might offer relevant contrasts that are clearly juxtaposed, highlighted, and appropriately sequenced followed by a task in which students are asked to make the intended distinctions.

Interpreting involves getting a read from each learner on the sense being made of the critical discernment. Essential to effective interpreting are student responses that quickly and clearly inform the teacher as to whether all have made the intended discernment or noticed the intended connection.

Deciding is the way the teacher chooses between stepping back, lingering, or pressing on, based on learners' demonstrated understandings. The teacher's decisions might involve clarification of prompts, modification of tasks, and/or adjustment of the pace of the lesson in ways that benefit all learners. An important feature of deciding is the way students are offered opportunities to extend significant ideas during class.

In addition to the variation theory of learning, we have also attended to the extensive research showing the limitations of working memory (Engle et al. 1992): Humans have limited capacity to attend simultaneously to multiple pieces of information. Working memory has been widely studied in the context of mathematics learning (e.g., Berg 2008). A detailed review of the literature on this subject is beyond the scope of the chapter; rather, we stress the relevance of considering its limitations when designing for learning. For instance, visually juxtaposing two well-crafted examples can support students' discernment of key distinctions.

We note both tensions and complementarities imposed by the manner in which human perception is drawn to *difference* while simultaneously having strict limits on how much can be attended to simultaneously. Contrast requires at least two elements. We have seen many cases where attempts to reduce demands on working memory have eliminated the very contrast needed to draw attention to a particular discernment. We differentiate our approach to working memory from those that focus primarily on explicit guidance: Clark et al. (2012) claimed that "Teachers are more effective when they provide explicit guidance accompanied by practice and feedback, not when they require students to discover many aspects of what they must learn" (p. 6). While our approach may be seen as offering explicit guidance, such guidance involves creating conditions for students to make critical discernments of the targeted object of learning.

We also contrast our teaching practices with problem-based and project-based approaches to teaching mathematics that offer limited guidance. Lessons in these approaches often open with a *rich* question that offers investigative spaces where students collaborate to develop deep and connected understanding of big mathematical ideas. However, some students may lack both the skills to effectively structure their own inquiries and sufficient background knowledge to attend to multiple ideas at once, thereby overwhelming their working memories. We have found that the teaching strategies identified in Math Minds can support students in developing the conceptual understanding and the skills necessary to address more open explorations. The focus is on supporting students to engage in activities that help them notice and integrate critical features of the targeted learning outcomes.

Continuous assessment that informs teachers' immediate decisions during class is critical to the teaching approach and the observation protocol that we describe in this chapter. This is consistent with recent work in formative assessment (Chappuis 2015; Wiliam 2011; Wiliam and Leahy 2015). Wylie and Wiliam (2007) identified the need for *hinge* questions to support teachers who struggled with continuously assessing all students during class and with providing appropriate immediate responses. Hinge questions can be answered in less than a minute, allowing the teacher to assess students' understanding and make informed decisions about how the lesson might proceed. Although there has been important work in the last decade on the development of such questions, particularly in mathematics (Wiliam 2011; Wiliam and Leahy 2015; Wylie and Wiliam 2007), most of the advice to teachers has been limited to re-explaining or reteaching, with little emphasis on *how* teaching might be effectively modified.

Ongoing decisions about teaching that are based on continuous assessment support both learners who struggle and those who quickly meet targeted learning goals. Preciado-Babb et al. (2017) reported how a focus on variation has resulted in a better way to support students in making critical discernments during class based on "on-the-fly" teacher's decisions. Clear juxtaposition and systematic variation can support students in both making discernments and proceeding from a familiar question or task to further challenges as they meet the learning goals during class.

3.2 Teaching Resources

We have found it important to consider how responsibility for the four elements of the Math Minds model might be effectively shared between a teacher and a teaching resource. While a resource can identify and sequence discernments in a manner that supports long-term coherence, a teacher can be responsible for attending to the needs and development of particular learners as they engage with key ideas.

At Math Minds we have found that, in contrast to other commonly available classroom resources designed to support Alberta's Program of Studies for Mathematics, JUMP Math (2018) offers a carefully measured presentation of information and emphasizes assessment of each key idea. We hypothesize that this approach respects limitations on students' working memories (Engle et al. 1992) and makes key ideas explicit. However, we have found that modifications to the JUMP Math materials based on systematic variation have often resulted in more effective ways of *drawing students' attention to critical features* or *connecting* key mathematical ideas (Metz et al. 2017).

The JUMP Math (2018) resource is consistent with the idea of constantly assessing students and responding accordingly during class. The teacher guides suggest "stepping back" to a place where students can reengage in the lesson when they have trouble in class (Mighton et al. 2010a). The guides also recommend offering bonus questions that challenge students who move quickly through assigned tasks. We have observed that both suggestions are often difficult for teachers (Preciado-Babb et al. 2016b): Step back to where? How should they create effective bonus questions? We have found that systematic variation can support both "stepping back" and "bonusing." Effective adaptations, however, often have more to do with effective variation than with smaller steps or bigger numbers. While the JUMP Math resource has carefully identified critical discernments to be noticed by learners and has sequenced topics coherently, we have observed that adapting the resource using more clearly structured variation (Marton 2015; Pang et al. 2016; Watson 2017) and remaining mindful of broader learning targets can provide better opportunities for students' learning. Doing so has opened pathways that both supported the weakest students and challenged even the most capable students (Metz et al. 2017; Preciado-Babb et al. 2017).

4 Examples from the Classroom

We offer two examples to show how the four elements of teaching described here have been used for both lesson analysis and redesign. The first example also shows the elements of lesson study enacted in the initiative in which lessons have been analyzed, redesigned, and implemented in subsequent years; excerpts from one classroom are included to show instances of interpreting and deciding. The second example illustrates how a focus on raveling and prompting was used to analyze a lesson plan and support teacher professional learning; we also include suggestions for further adaptations to the lesson.

4.1 *Example 1: Representing Numbers Up to 20*

Early in the Math Minds Initiative, we observed a Grade 1 lesson in which the teacher made numerous attempts to support students in representing numbers to 20 using base ten blocks. In addition to providing several opportunities for each child to respond to questions, the teacher paid regular attention to one student, Heidi, who struggled far more than her classmates. The teacher's careful use of assessment, persistence in seeking an effective response, and effective adaptation of the given variation allowed everyone to reach understanding by the end of the lesson. The episode also included examples of extensions for students who met the expected outcomes during class. Insights gleaned during this lesson informed other lessons as well as new implementations in subsequent years of the project.

Representing numbers in the decimal system requires understanding and combining (raveling) the notions of (a) grouping by powers of ten and (b) identifying the value associated with the position of each digit in a numeral. Base ten blocks are commonly used for representing numbers and were used to help prompt students' attention to these critical discernments.

4.1.1 Analysis of the Lesson

The teacher started the lesson with a series of questions and took answers either in chorus or from individual students. Some students gave wrong answers. In particular, there seemed to be confusion about the number of ten-blocks and one- blocks used to represent a particular number. The instructional sequence involved using blocks to represent 11, 12, 13, 14, 15, and 18; students were asked to identify the number of ten-blocks and one-blocks and then write the number. Note that in this sequence, the number of units varied, while the number of tens remained constant (always 1 ten).

After offering the first item in the sequence, the teacher checked on each student individually. She then offered explanations for 10, 11, 12, and 13. At this point, she asked the class to represent 14 and checked each student's response. Heidi was struggling, so the teacher gave her 4 one-blocks and 1 ten-block. Then she requested the attention of all students and provided an explanation on the board. The teacher then asked students to show blocks for 18. She walked through the class checking on every student, offering assistance and posing further challenges as needed (19, 17). These actions illustrate the deciding step in the observation protocol. The actions gave her some time to work individually with some students. Only two or three students, including Heidi, still had trouble representing the numbers.

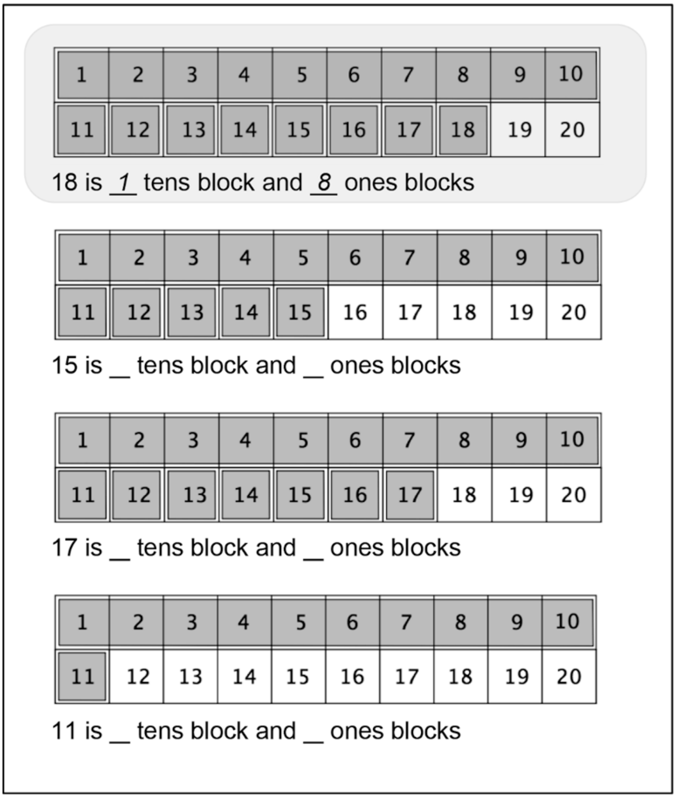


Fig. 1 Student's task from *Assessment and Practice* book. (Image based on Mighton et al. 2009a, p. 48)

The class moved to the next part of the lesson, in which the students worked in their *Assessment and Practice* books (see Fig. 1). The teacher introduced this work by using a document camera to show the example solved in the students' book for the case of 18. She then asked individual students to solve the rest of the exercises in front of the class.

When the teacher asked how many ten-blocks were in 15, some students answered in chorus: "10." The teacher then explained that there was only 1 ten-block. She gave every student 1 ten-block, asking "How many tens blocks did I give you?" until every student answered "one."

Subsequent questions required students to identify tens and ones without the support of colored 20 charts: They were now required to place blocks on a given 20 charts to represent the given number and then complete a sentence similar to those in Fig. 1. For instance, the first task was:

14 is ___ tens block and ___ ones blocks.

One student showed the answer on the board, and the teacher re-explained the use of one-blocks and ten-blocks. Subsequent tasks in the practice book asked students to consider 19, 11, 13, 12, and 20. Note that so far, all but one of the examples focused on a single ten paired with varying numbers of units. The exception comes at the very end, where it is unlikely to serve as a useful contrast.

As students worked individually, the teacher walked through the class checking on each of them. At least three students, including Heidi, still had trouble differentiating the ten-block from the one-block. At this point, the teacher started giving these students more ten-blocks. She started with Heidi, showing 3 ten-blocks and counting, "One, two three." For the first time, the pattern of variation now included a contrast in the number of ten-blocks.

The teacher went to another student, who had apparently written a 5 in the space for ten-blocks; she indicated that in the case of 15, there were not 5 ten- blocks, but only 1 ten-block. This student seemed to understand and corrected his work.

Another student had written 15 in the space for ten-blocks. The teacher showed her 15 ten-blocks while saying "Here are 15 ten-blocks" and asking "How many ten-blocks are there [in the given example]?" The teacher gave 1 ten-block to the student, asking: "How many blocks did I give you?" The following dialogue ensued:

Student: One.
[Teacher gave another]
Teacher: Now how many blocks?
Student: Two.
[Teacher took away one block]
Teacher: OK? How many ten-blocks did I give you right now?
Student: One
Teacher: Then write one.

The class finished, and the teacher asked Heidi to stay; she spent 4 min working with her. The teacher held up 10 ten-blocks.

Teacher: This is 10 ten-blocks.
[Teacher gave the blocks to Heidi to hold in her hand]
Teacher: See ten blocks?

Heidi started giving the blocks back, one by one to the teacher. The teacher counted, while the student gave the blocks back.

Teacher: One, two, three tens. Can you show me 5 ten-blocks?
Heidi: One, two, three, four, five [moving the tens-blocks one by one]
Teacher: So that's 5 tens blocks. Now show me 1 ten-block. Just one.
[Heidi moved one block.]
Teacher: Now, how many ten-blocks do you have here?
Heidi: One.
Teacher: So, write the number one. How many one-blocks? How many ones?

In this episode, we stress that the teacher led group discussions in which one student or students in chorus responded to her questions, instead of using an all-student response system. Nonetheless, it was clear to her that there was confusion about the distinction between 1 ten-block and 10 one-blocks. The teacher prompted attention to this difference several times during the lesson, which included giving 1 ten-block to each student and asking, "How many blocks did I give you?" Still, some students were confused. By the end of the lesson, the teacher started to vary the number of ten-blocks in the examples and questions posed to students. This was more effective for the students who were still confused.

4.1.2 Adaptations to Other Lessons

We discussed this issue with other teachers in the project, suggesting to vary the tens digit to draw attention to the meaning of the tens place in similar lessons: They did so in different ways. One teacher prompted attention to the meaning of the tens digit by including numbers beyond 19. She simply kept a handful of ten-blocks at her side as she worked with her students. When the predicted difficulty with 1 ten-block- vs. 10 one-blocks came up, she was prepared with several ten-blocks to show the difference.

Another adaptation based on this suggestion included an explicit prompt to numbers with and without ten-blocks. The teacher started the lesson by offering a series of slides in which students were supposed to identify the number of dots, preferably by counting on from a full ten frame (Fig. 2).

Many students struggled to identify the number of dots in the slides. After a few more tries, the teacher recognized that continuing with the planned lesson was unlikely to be productive. She decided to switch to a task in which students had to identify a number based on the number of dots she held up. She presented a yellow card with ten dots and a blue card with anywhere from zero to nine dots (see Fig. 3).

For each question (prompt), she held up a blue card *with or without the yellow ten-card*, and the students' job was to name the number. Note that although multiple tens were not involved, as in the previous case, the presence of the *zero-or-one-ten* option helped to draw students' attention to role of the ten's digit. The students quickly became adept at recognizing the numbers she held up.

In another case, a second-grade lesson that already included variation in the number of tens was adapted to more clearly highlight this variation. Initially, the four examples shown on the left in Fig. 4 were presented separately. By clearly juxtaposing them (as on the right), students found it easier to attend to key contrasts in 28 as (a) 28 ones, (b) 1 ten and 18 ones, and (c) 2 tens and 8 ones (something many struggled with during their first attempt with this lesson).

These adaptations resulted from the teacher's awareness that students were struggling. She requested a meeting with a lead teacher and a researcher after class

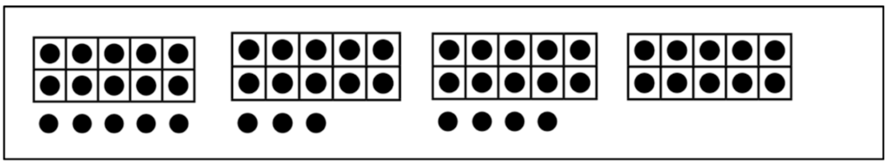


Fig. 2 Modification of the JUMP Math slides to juxtapose examples. (Mighton et al. 2009a)

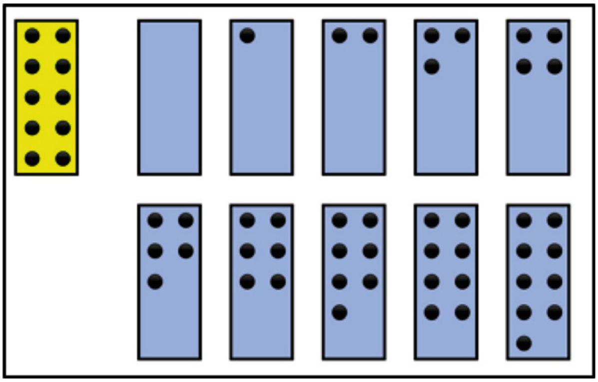


Fig. 3 Slide showing variations in the use of ten or no ten for representing numbers

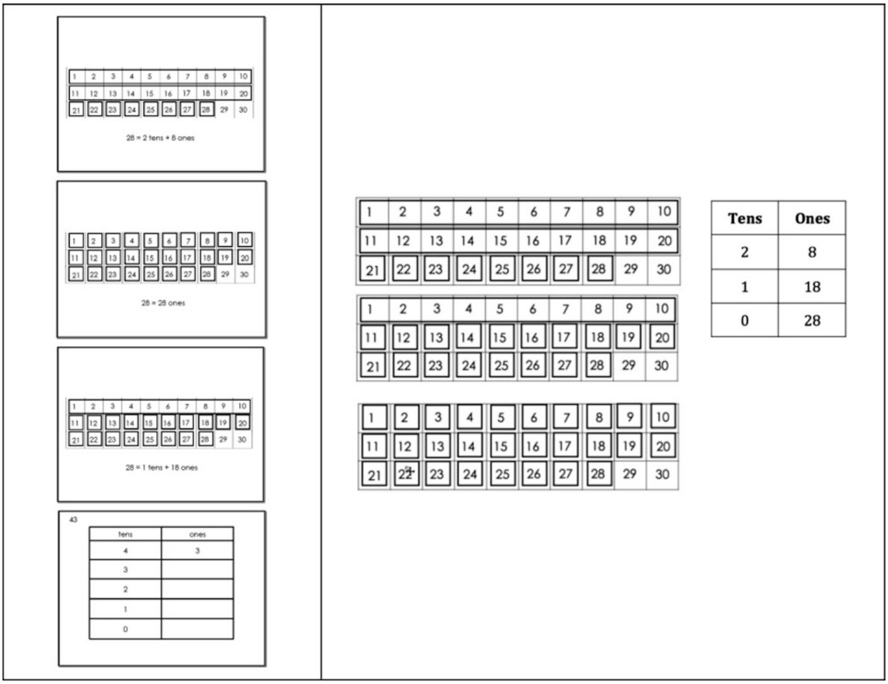


Fig. 4 On the left, original sequence of slides shown to students; on the right, modified version with a single slide. (Mighton et al. 2010a, p. B-58)

to adapt the lesson, with the intent to reteach it the following day. It was then that they decided to juxtapose the first three images and the T-Table on a single slide. Further, students were to identify what was the same and different in the three representations of 28. Following this, students were asked to identify possible numbers of tens and ones in other numbers, supported by an erasable hundred chart taped to a whiteboard (see Fig. 5); the use of the hundred chart was also an

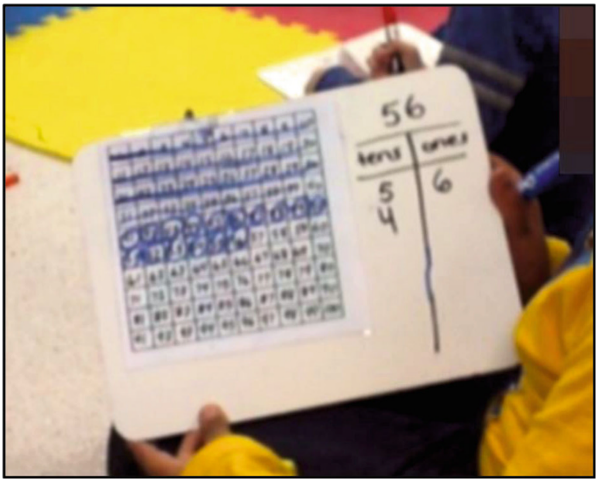


Fig. 5 Individual whiteboard and hundred chart used to represent numbers

adaptation from the original lesson, during which students used base ten blocks that they found hard to keep track of. Here, the teacher indicated that a ten-block must be shown by a continuous line through an entire row, while a one-block must be shown with a slash mark through a single number, thus further highlighting the contrast between a ten-block and a one-block. The total numbers of each were recorded in a T-Table positioned directly beside the hundred chart on the whiteboard. Students experienced greater success with this approach.

The decision to use the whiteboards and the hundred charts also facilitated the process of interpreting students' understanding: It allowed the teacher to quickly check individual understanding of all students during this part class.

Note that in contrast to previous lessons where the number being represented varied, here, the number remained the same but was represented by varying numbers of ten and one-blocks. Including the T-Table on the same slide allowed further contrast in terms of representation before moving to other numbers (as in the fourth slide on the left in Fig. 4).

4.2 Example 2: Partitive vs. Quotitive Division

The following example focuses on the distinction between partitive and quotitive division. Learners' and teachers' difficulties distinguishing partitive and quotitive division have been widely reported in the literature. These include limitations in correctly solving word problems related to division (e.g., Fischbein et al. 1985; Graeber et al. 1989; Tirosh 2000). The JUMP Math resource (Mighton et al. 2010b) includes a Grade 5 lesson, "Two Ways of Sharing" (p. 81) intended to draw attention to manners of grouping that are consistent with partitive and quotitive division. This lesson precedes other lessons on knowing whether to multiply or divide in particular contexts and understanding the long division algorithm. In Math Minds, we have

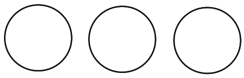
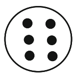
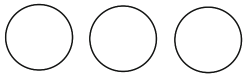
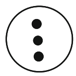
Original Contrast (Mighton et al. 2009b, p. 80)	**Modified Contrast**
Tory has 18 cookies. There are two ways she can share or divide her cookies equally: 1) She can decide how many sets (or groups) of cookies she wants to make: Tory wants to make 3 sets of cookies. She draws 3 circles. She then puts one cookie at a time into the circles until has placed 18 cookies.  2) She can decide how many cookies she wants to put in each set: Tory wants to put 6 cookies in each set. She counts out 6 cookies. She counts out sets of 6 cookies until she has placed 18 cookies in sets. 	Tory has 18 cookies. There are two ways she can share or divide her cookies equally: 1) She can decide how many sets (or groups) of cookies she wants to make: Tory wants to make 3 sets of cookies. She draws 3 circles. She then puts one cookie at a time into the circles until has placed 18 cookies. We can describe her action as 18 divided into 3 sets, or 18 ÷ 3.  2) She can decide how many cookies she wants to put in each set: Tory wants to put 3 cookies in each set. She counts out 3 cookies. She counts out sets of 3 cookies until she has placed 18 cookies We can describe her action as 18 divided into sets of 3, or 18 ÷ 3. 

Fig. 6 Two tasks intended to prompt attention to the difference between partitive and quotitive division

observed that neither students nor teachers who engaged with this lesson made the desired distinction in subsequent lessons. For instance, teachers made no reference to the two types of sharing (division) and explained the division algorithm in terms of quotitive language (e.g., "how many times does 3 goes into . . .") even though the resource refers explicitly to images of partitive division (e.g., splitting 600 blocks into 3 equal groups). We decided to make this lesson the focus of a group discussion with participant teachers, including one who was teaching Grade 5 at that time.

During the session with teachers, we analyzed ways to support learners' understanding of the difference between the two types of division. Then, we asked teachers to compare the tasks in Fig. 6 and to discuss which version would better draw attention to the difference between partitive and quotitive division. The first task is included JUMP Math (Mighton et al. 2009b) for the selected lesson; the second was a modification of this task.

Note that in the original (Fig. 6, left) version of the task, the circle/dot diagrams both show 3 sets of 6 cookies, with no reference to division notation; had this been included, one would have had a divisor of 6 and the other a divisor of 3. Here, the actions performed in forming the groups (based on what information is given in the problem) change, while the resulting groups are identical. In the modified version (Fig. 6, right), the circle/dot diagrams correspond to two different actions, both of which can be represented by 18 ÷ 3; in one case, the resulting diagram shows 3 sets of 6 cookies; the other shows 6 sets of 3 cookies. In this case, the only thing that varies is the type of division. This distinction prompted several "a-ha's" among the teachers; it was here that the point of the partitive/quotitive distinction became more clearly apparent to them. After this discussion, we proceeded to analyze the rest of the lesson plan.

4.2.1 Analysis of the Original Lesson Plan: Two Ways of Sharing

The "Two Ways of Sharing" lesson offered in JUMP opens with a number of contrasts. The first is between two situations in which 12 learners are "numbered off" to form 2 teams:

(a) 1, 1, 2, 1, 1, 3, 1, 1, 4, 1, 1, 1
(b) 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4

Here, what varies is whether or not the groups are equal; the point of emphasis is equal-sized groups. Following a's uneven distribution, students are invited to make the teams fair by reassigning numbers to some students.

Next, students are offered containers labeled 1, 2, 3, and 4 and invited to distribute 12 blocks into the 4 containers. Compared with the previous task, this offers a variation in the manner of representing the 12 students being divided.

Then, the resource invites learners to consider sharing 12 cookies among 4 people. This offers a contrast in context; there are still 12 objects being split into 4 groups, but the particular objects change. Suggested questions in the teacher's guide draw attention to the number of containers and counters needed and to what these represent (cookies and people). Learners are then invited to consider circles for containers and dots for cookies, and the teacher models placing 1 cookie in the first circle, then 1 in the second, etc. until the 12 cookies have been distributed. Here, we see another contrast between this example and the previous one: representation with containers and counters vs. representation with circles and dots. Students have an opportunity to practice this by dividing 12 students into 4 cars. Again, the total remains constant, as does the number of groups into which the total must be divided; the only thing that changes is the context.

Note that so far, the variations in the lesson have drawn attention to the need for equal-sized groups, to how objects and groups might be represented (by people standing together, by counters in containers, or by dots in circles), and to the idea that even if contexts change, the numbers remain constant (though this is not explicitly mentioned).

Following this, a set of practice examples varies the number of cookies and people (12 cookies with 3 people, 15 cookies with 5 people, 10 cookies with 2 people), each of which produces a different number in each group.

At this point, it is important to note that, based on our previous observations to the enactment of this lesson, all of the examples used so far could easily be visualized by most students as four groups of three; they did not need to divide (or partition their counters) to find the answer. Students responded to questions of this nature with reference to a multiplication sentence; for example, when asked how they knew there were 3 students on a team, students responded: "Because $4 \times 3 = 12$."

The resource then announces, to the teacher, a shift in intention: "So far, questions have told students how many piles (or sets) to make. Now questions will tell students how many to put in each pile (or in each set) and they will need to determine the number of sets" (Mighton et al. 2010b, p. 82). However, there is no suggestion that the teacher should flag this shift for students, nor are there direct contrasts between the quotitive contexts that follow and the partitive ones with which they have already worked.

The first question students are invited to consider in this portion of the lesson is a person with 30 apples who wants to give five apples to each friend. Returning to counters and containers, the teacher is supposed to ask students what the containers and counters represent and to ask students if they know how many containers are needed and if they know how many counters should go in each container. If students can remember back to their earlier work with containers, this offers a potential contrast between known and unknown number of containers, but this contrast is not highlighted, and the two are not directly juxtaposed. Nonetheless, this is first time that partitive vs. quotitive grouping has been opened as a dimension of variation.

As the lesson continues, the direct contrast between partitive and quotitive fades from view: Examples that contrast different quotitive contexts become the focus of the lesson. The next contrast involves modeling the apple problem a second time, this time with circles and dots (another contrast in representation with no contrast regarding what is known/unknown). Once the 30 apples have been drawn in groups of five, students are asked to consider 30 apples in groups of three; this might draw attention to the fact that when you use smaller groups you get more groups. Practice examples then invite students to divide varying numbers of dots into sets of known size: 9, 6, and 12 dots into sets of 3; 6 and 12 dots into sets of 2; 15 dots into sets of 5; 12 dots into sets of 4; and 16 dots into sets of 2. Further practice involves varying numbers of apples being placed into boxes of varying sizes (10 apples, 2 per box; 12 apples, 3 per box; 18 apples, 6 per box). Finally, practice shifts to situations in which students are required to highlight "important information," which includes total and group size and to find the number in each group. In each of these cases, set sizes and totals vary, but the manner of grouping does not.

At this point, the distinction between the two "methods of sharing" is made explicit to students, and they are invited to make this distinction.

a) 15 children with 5 in each canoe, b) 15 children with 5 canoes, and c) a girl with 40 stickers who gives 8 stickers to each of her friends.

Fig. 7 Slide 19 in the "Two Ways of Sharing" lesson

a) There are 24 strings on 4 guitars. b) There are 3 hands on each clock. There are 15 hands altogether. c) There are 18 holes in 6 sheets of paper. d) There are 15 rings on 5 binders. e) 15 people sit on 5 couches.

Fig. 8 Slide 21 in "Two Ways of Sharing" lesson

g) There are 15 people. 5 people fit on each couch. h) There are 4 cans of tennis balls. There are 3 tennis balls in each can. i) There are 20 chairs in 4 rows. j) There are 3 rows of chairs. There are 9 chairs altogether

Fig. 9 Slide 22 in the "Two Ways of Sharing" lesson

> Sometimes a problem provides the number of sets, and sometimes it provides the number of objects in each set. Have your students tell you if they are provided with the number of sets or the number of objects in each set for the following problems (Mighton et al. 2010a, b, p. 83).

The situations offered include a slide showing three cases simultaneously (see Fig. 7). The only thing that changes between the two canoe cases is how the 15 is divided into 5: Is it divided into 5 groups or groups of 5? This is the first time that students are invited to make this distinction. Note, however, that students are not asked to represent either situation with a division sentence. We also observe that the third example on the slide could in fact distract from the clear contrast in the first two, as it might seem to suggest three independent examples.

In the slides that follow, various contextual situations are offered that require students to identify (a) what is being shared, (b) how many sets, and (c) how many are in each set (using a question mark to denote the unknown element). The distinction, then, is phrased in terms of what is known vs. what is not known. These distinctions must be discerned against a background of changing context, numbers, and manner of grouping. Nevertheless, none of these examples are presented as pairs in which only the manner of grouping changes (as in the canoe example). Here, total objects, number of sets, and number in each set all change from example to example, with a couple of exceptions. There is only one pair that forms a clear partitive/quotitive contrast (clear in that only the manner of grouping changes), but it is split between two slides, as shown in Figs. 8 and 9. Note the contrast between Slide 21 (e) and Slide 22 (g).

	What has been shared or divided into sets?	How many sets?	How many in each set?
a) Kathy has 30 stickers. She put 6 stickers in each box.	30 stickers	?	6
b) 24 children are in 6 vans.	24 children	6	?
c) Andy has 14 apples. He gives them to 7 friends.			
d) Manju has 24 comic books. She puts 3 in each bin.			
e) 35 children sit at 7 tables.			
f) 24 people are in 2 boats.			
g) 12 books are shared among 4 children.			
h) 10 flowers are shared in 2 rows.			
i) 8 hamsters are in 4 cages.			

Fig. 10 Task of the "Two Ways of Sharing" lesson corresponding to *Assessment and Practice* student book. (Mighton et al. 2009a, b, p. 81)

We also observed that the distinction between known and unknown can be difficult to discern when the numbers are such that both are immediately obvious due to students' knowledge of basic multiplication facts; in other words, if both are easily known, the known/unknown distinction becomes hazy, and this is critical to distinguishing partitive and quotitive division.

The practice examples in the *Assessment and Practice* book (Mighton et al. 2009b) follow a pattern similar to the lesson; there is a set of practice involving partitive contexts followed by a set of practice involving quotitive contexts. Finally, there is a chart describing numerous groupings in which students are to identify total, number of sets (if possible), and number in each set (if possible); the distinction between known and unknown is key (see Fig. 10). The first two cases are solved for students as examples. Notice that among the nine statements that students are offered, only two are quotitive: (a) and (d).

In summary, we have highlighted three features that may prevent students from noticing key critical features of the partitive and quotitive division. First, there were few opportunities to contrast the two ways of sharing in which only the meaning of sharing was changed (e.g., the canoe situation in Fig. 7). Second, these two ways of sharing were not connected explicitly to division. This connection is not explicit in subsequent lessons, either. Finally, the reference to known and unknown may be ambiguous for students who know multiplication facts. For instance, if a student knows the multiplication fact $30 = 5 \times 6$ and connects it with $30 \div 6$, then the result of the division, 5, is known by the student.

4.2.2 Adaptations and Suggestions for Implementation

We considered that the contrast between quotitive and partitive division should be made more explicit in the lesson. For a revised version of the lesson, we encouraged the teacher to juxtapose partitive and quotitive examples from the beginning of the lesson. We also emphasized the importance of prompting students' attention to what changes and what stays the same from example to example. For instance, the teacher lesson used a modified version of Slide 19 (Fig. 7) with only the first two examples. In a further implementation of this part of the lesson, we observed that all students were able to discern the difference between the two types of division when the modified slide was used.

A relevant discernment for this lesson is the distinction between number of sets and number of objects in each set. This is an example of *raveling*, as a critical discernment was identified and connected to other discernments in the lesson. The chart in the *Assessment and Practice* book (Fig. 10) was modified to include (a) direct contrasts between partitive and quotitive contexts (in which only the

	What has been shared or grouped into sets?	How many sets?	How many in each set?	Draw the sets.	Division Sentence	P or Q?
Kathy has 30 stickers. She put 6 stickers in each box.	30 stickers	?	6		30 ÷ 6 = 5	
Kathy has 30 stickers. She has 6 boxes.					30 ÷ 6 = 5	
24 children are in 6 vans.						
There are 24 children with 6 in each van.						
Andy has 14 apples. He gives them to 7 friends.						
Andy has 14 apples. He gives 7 apples to each friend.						
Manju has 24 comics. She puts 3 in each bin.						
Manju has 24 comics. She puts them in 3 bins.						

Fig. 11 Modified chart

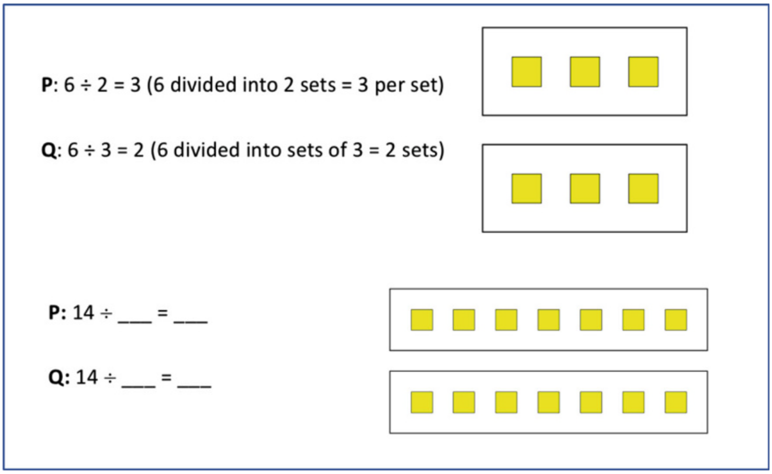


Fig. 12 Examples of two different ways of division corresponding to the same image

manner of grouping changed), (b) space to sketch the groupings, and (c) space for a division sentence (see Fig. 11). This adaptation may also be considered part of the *raveling* approach to teaching, as key discernments are both separated and connected. The adapted chart includes an explicit reference to division, which supports connections to future lessons (*macro-raveling*). The chart is also an example of *prompting*, as the examples are presented in contrasting pairs that support students distinguishing between the two types of division.

We also suggest to further emphasize the significance of *known quantity/unknown quantities*: Is there a *known number of sets* or a *known number in each set*? Although the unknown factor is flagged with a question mark, we observed in the classroom that the relation between the position of the question mark and the type of division remained a point of confusion for some students. Nonetheless, most students successfully identified the known and unknown elements and found an appropriate solution, and several invented their own pairs of examples as a bonus (an example of *deciding* that supported advanced students).

Finally, students who complete the modified chart might be offered images showing numbers split into sets and asked to identify the mathematical expressions that indicate partitive and quotitive groupings (Fig. 12). This would allow a broader contrast involving the same groupings with different number sentences.

5 Conclusion

The four teaching strategies outlined in this chapter – raveling, prompting, interpreting, and deciding – are informed by Marton's (2015) variation theory of learning. While *raveling* refers to the identification of critical discernments, and their connections to other mathematical ideas and concepts, *prompting* refers to the sequence of examples and tasks offered during class that require students to make distinctions relating to each critical discernment. Earlier, we noted that the selection of what should vary and what should remain invariant is not obvious; consistent with Marton, we have observed that varying the critical feature often works better to prompt learners' attention. While raveling and prompting can be supported by teaching resources such as textbooks, lesson plans, and teaching guides, teachers must be aware of the need for potential modifications, which may include clearer patterns of variation and more direct juxtaposition of tasks and examples that contrast key features of the intended object of learning.

The role of the teacher is more significant in interpreting students' understanding during class, as well as in deciding how to proceed. Systematic variation can inform the questions or tasks posed to students as assessment, as well as teachers' decisions regarding how best to support students who are struggling or those who require additional challenges to extend their learning.

Although the Math Minds Initiative has involved only selected aspects of lesson study, the focus on variation theory makes it relevant for the literature on lesson study and learning study. In particular, we suggest paying attention not only to lesson plans but also to the decision-making process during lesson implementation. Systematic variation can inform teachers' decisions that can immediately address the needs of all students in class rather than waiting for another iteration of the lesson to be developed (though this is also important). On another level of iteration, the four elements of the Math Minds protocol are also impacting the redesign of the JUMP Math (2018) materials, as suggestions for adapting lessons are informing future editions of the teaching materials.

The approach to teaching described here has been developed from empirical evidence in the Math Minds Initiative. In Phase 2 of the project, we continue to refine lessons plans and to more clearly articulate the four strategies presented in this paper so as to better support teachers in understanding and enacting them in class.

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