## Chapter 8

## Geometric Hardy Inequalities on Stratified Groups

Given a domain in the space, the 'geometric' version of Hardy inequalities usually refers to the Hardy type inequalities where the weight is given in terms of the distance to the boundary of the domain. In this chapter we discuss $L^{2}$ and $L^{p}$ versions of the geometric Hardy inequality on the stratified group $\mathbb{G}$. For the clarity of the exposition, we first deal with the half-space domains, and then with more general convex domains.

The results presented in this chapter have been obtained in [RSS18b], and our exposition here follows this paper. In particular, we discuss $L^{2}$ and $L^{p}$ versions of the (subelliptic) geometric Hardy inequalities in half-spaces and convex domains on general stratified groups. As usual, these imply the geometric versions of the uncertainty principles. A certain current drawback of the methods in the case of convex domains is that the convexity is understood in the Euclidean sense.

## 8.1 $\quad L^{2}$-Hardy inequality on the half-space

In this section, we discuss an $L^{2}$-version of the geometric Hardy inequality on the half-space of the stratified group $\mathbb{G}$. We start by recalling a few known results and by putting the further analysis in perspective.

## Remark 8.1.1.

1. If $\Omega$ is a convex open set of the Euclidean space, then the geometric version of the Hardy inequality is well understood and given by

$$
\int_{\Omega}|\nabla u|^{2} d x \geq \frac{1}{4} \int_{\Omega} \frac{|u|^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}} d x
$$

for $u \in C_{0}^{\infty}(\Omega)$, with the sharp constant $1 / 4$. Nowadays, there are many studies related to this subject, here we can mention, for example, [Anc86], [D'A04b], [AL10], [AW07], [Dav99] and [OK90].
2. In the setting of the Heisenberg group $\mathbb{H}$, the geometric Hardy inequality on the half-space

$$
\mathbb{H}^{+}:=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{H} \mid x_{3}>0\right\}
$$

takes the form

$$
\int_{\mathbb{H}^{+}}\left|\nabla_{\mathbb{H}^{+}} u\right|^{2} d x \geq \int_{\mathbb{H}^{+}} \frac{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}{x_{3}^{2}}|u|^{2} d x,
$$

for all $u \in C_{0}^{\infty}\left(\mathbb{H}^{+}\right)$. This inequality was obtained in [LY08], and we can also recapture it as a consequence in Corollary 8.1.5. There are further extensions to geometric $L^{p}$-Hardy inequalities as well as to the convex domains of the Heisenberg group obtained in [Lar16].

The following construction can be traced back to Garofalo [Gar08].
Definition 8.1.2 (Half-space and angle function). Let $\mathbb{G}$ be a stratified group. In this section the half-space of $\mathbb{G}$ will be defined by

$$
\mathbb{G}^{+}:=\{x \in \mathbb{G}:\langle x, \nu\rangle>d\}
$$

where $d \in \mathbb{R}$, and $\nu:=\left(\nu_{1}, \ldots, \nu_{r}\right)$ with $\nu_{j} \in \mathbb{R}^{N_{j}}, j=1, \ldots, r$, is the Riemannian outer unit normal to $\partial \mathbb{G}^{+}$. The Euclidean distance to the boundary $\partial \mathbb{G}^{+}$will be denoted by $\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)$and given by the formula

$$
\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)=\langle x, \nu\rangle-d
$$

The angle function on $\partial \mathbb{G}^{+}$is defined by

$$
\begin{equation*}
\mathcal{W}(x):=\sqrt{\sum_{i=1}^{N}\left\langle X_{i}(x), \nu\right\rangle^{2}} \tag{8.1}
\end{equation*}
$$

In what follows we will be working in the setting of Definition 8.1.2.
Theorem 8.1.3 (Geometric $L^{2}$-Hardy inequality on half-space). Let $\mathbb{G}^{+}$be a halfspace of a stratified group $\mathbb{G}$.
(1) Let $\beta \in \mathbb{R}$ and set $C_{1}(\beta):=-\left(\beta^{2}+\beta\right)$. Then we have

$$
\begin{align*}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq & C_{1}(\beta) \int_{\mathbb{G}^{+}} \frac{\mathcal{W}(x)^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}|u|^{2} d x \\
& +\beta \int_{\mathbb{G}^{+}} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}|u|^{2} d x \tag{8.2}
\end{align*}
$$

for all $u \in C_{0}^{\infty}\left(\mathbb{G}^{+}\right)$.
(2) We have

$$
\begin{equation*}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq \frac{1}{4} \int_{\mathbb{G}^{+}} \frac{|u|^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}} d x \tag{8.3}
\end{equation*}
$$

for all $u \in C_{0}^{\infty}\left(\mathbb{G}^{+}\right)$.
Remark 8.1.4 (Uncertainty principle and step 2 case). In the step 2 case we have the following simplification of Part (1) of Theorem 8.1.3.

1. Note that for the stratified groups of step 2 it follows from Proposition 1.2.19 that one can use the following basis of the left invariant vector fields

$$
\begin{equation*}
X_{i}=\frac{\partial}{\partial x_{i}^{\prime}}+\sum_{s=1}^{N_{2}} \sum_{m=1}^{N} a_{m, i}^{s} x_{m}^{\prime} \frac{\partial}{\partial x_{s}^{\prime \prime}}, \tag{8.4}
\end{equation*}
$$

where $i=1, \ldots, N$ and $a_{m, i}^{s}$ are the constants depending on the group. In addition, we can also write $x=\left(x^{\prime}, x^{\prime \prime}\right)$ with

$$
x^{\prime}=\left(x_{1}^{\prime}, \ldots, x_{N}^{\prime}\right), x^{\prime \prime}=\left(x_{1}^{\prime \prime}, \ldots, x_{N_{2}}^{\prime \prime}\right),
$$

and also $\nu=\left(\nu^{\prime}, \nu^{\prime \prime}\right)$ with

$$
\nu^{\prime}=\left(\nu_{1}^{\prime}, \ldots, \nu_{N}^{\prime}\right), \nu^{\prime \prime}=\left(\nu_{1}^{\prime \prime}, \ldots, \nu_{N_{2}}^{\prime \prime}\right) .
$$

Then the statement of Theorem 8.1.3, Part (1), can be simplified as follows: for all $u \in C_{0}^{\infty}\left(\mathbb{G}^{+}\right)$and $\beta \in \mathbb{R}$ we have

$$
\begin{align*}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq & C_{1}(\beta) \int_{\mathbb{G}^{+}} \frac{\mathcal{W}(x)^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}|u|^{2} d x  \tag{8.5}\\
& +K(a, \nu, \beta) \int_{\mathbb{G}^{+}} \frac{|u|^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)} d x
\end{align*}
$$

where $C_{1}(\beta):=-\left(\beta^{2}+\beta\right)$ and $K(a, \nu, \beta)=\beta \sum_{s=1}^{N_{2}} \sum_{i=1}^{N} a_{i, i}^{s} \nu_{s}^{\prime \prime}$.
2. In the standard way Theorem 8.1.3, Part (2), implies the geometric uncertainty principle on the half-space $\mathbb{G}^{+}$for general stratified groups $\mathbb{G}$. Indeed, (8.3) and the Cauchy-Schwarz inequality imply

$$
\begin{aligned}
\int_{\mathbb{G}^{+}} & \left|\nabla_{H} u\right|^{2} d x \int_{\mathbb{G}^{+}} \operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}|u|^{2} d x \\
& \geq \frac{1}{4} \int_{\mathbb{G}^{+}} \frac{1}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}|u|^{2} d x \int_{\mathbb{G}^{+}} \operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}|u|^{2} d x \\
& \geq \frac{1}{4}\left(\int_{\mathbb{G}^{+}}|u|^{2} d x\right)^{2} .
\end{aligned}
$$

That is, we have

$$
\left(\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x\right)^{\frac{1}{2}}\left(\int_{\mathbb{G}^{+}} \operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}|u|^{2} d x\right)^{\frac{1}{2}} \geq \frac{1}{2} \int_{\mathbb{G}^{+}}|u|^{2} d x
$$

for all $u \in C_{0}^{\infty}\left(\mathbb{G}^{+}\right)$.

Proof of Theorem 8.1.3. Proof of Part (1). For the proof we apply the method of factorisation. So, for any real-valued $W:=\left(W_{1}, \ldots, W_{N}\right), W_{i} \in C^{1}\left(\mathbb{G}^{+}\right)$, which will be chosen later, a direct calculation gives

$$
\begin{aligned}
0 \leq \int_{\mathbb{G}^{+}}\left|\nabla_{H} u+\beta W u\right|^{2} d x & =\int_{\mathbb{G}^{+}}\left|\left(X_{1} u, \ldots, X_{N} u\right)+\beta\left(W_{1}, \ldots, W_{N}\right) u\right|^{2} d x \\
& =\int_{\mathbb{G}^{+}}\left|\left(X_{1} u+\beta W_{1} u, \ldots, X_{N} u+\beta W_{N} u\right)\right|^{2} d x \\
& =\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left|X_{i} u+\beta W_{i} u\right|^{2} d x \\
& =\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}+2 \operatorname{Re} \beta W_{i} u X_{i} u+\beta^{2} W_{i}^{2}|u|^{2}\right] d x \\
& =\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}+\beta W_{i} X_{i}|u|^{2}+\beta^{2} W_{i}^{2}|u|^{2}\right] d x \\
& =\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}-\beta\left(X_{i} W_{i}\right)|u|^{2}+\beta^{2} W_{i}^{2}|u|^{2}\right] d x
\end{aligned}
$$

From the above expression we get the inequality

$$
\begin{equation*}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq \int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left[\left(\beta\left(X_{i} W_{i}\right)-\beta^{2} W_{i}^{2}\right)|u|^{2}\right] d x . \tag{8.6}
\end{equation*}
$$

Let us now take $W_{i}$ in the form

$$
\begin{equation*}
W_{i}(x)=\frac{\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}=\frac{\left\langle X_{i}(x), \nu\right\rangle}{\langle x, \nu\rangle-d} \tag{8.7}
\end{equation*}
$$

where

$$
X_{i}(x)=(\overbrace{0, \ldots, 1}^{i}, \ldots, 0, a_{i, 1}^{(2)}\left(x^{\prime}\right), \ldots, a_{i, N_{r}}^{(r)}\left(x^{\prime}, x^{(2)}, \ldots, x^{(r-1)}\right)),
$$

and

$$
\nu=\left(\nu_{1}, \nu_{2}, \ldots, \nu_{r}\right), \nu_{j} \in \mathbb{R}^{N_{j}}
$$

Now $W_{i}(x)$ can be written as

$$
W_{i}(x)=\frac{\nu_{1, i}+\sum_{l=2}^{r} \sum_{m=1}^{N_{l}} a_{i, m}^{(l)}\left(x^{\prime}, \ldots, x^{(l-1)}\right) \nu_{l, m}}{\sum_{l=1}^{r} x^{(l)} \cdot \nu_{l}-d}
$$

By a direct computation we have

$$
\begin{align*}
X_{i} W_{i}(x) & =\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle \operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)-\left\langle X_{i}(x), \nu\right\rangle X_{i}\left(\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)\right)}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}} \\
& =\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}-\frac{\left\langle X_{i}(x), \nu\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}, \tag{8.8}
\end{align*}
$$

where

$$
\begin{aligned}
X_{i}\left(\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)\right) & =X_{i}\left(\sum_{k=1}^{N} x_{k}^{\prime} \nu_{1, k}+\sum_{l=2}^{r} \sum_{m=1}^{N_{l}} x_{m}^{(l)} \nu_{l, m}-d\right) \\
& =\nu_{1, i}+\sum_{l=2}^{r} \sum_{m=1}^{N_{l}} a_{i, m}^{(l)}\left(x^{\prime}, \ldots, x^{(l-1)}\right) \nu_{l, m} \\
& =\left\langle X_{i}(x), \nu\right\rangle .
\end{aligned}
$$

Now combining (8.8) with (8.6) we arrive at the inequality

$$
\begin{aligned}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq & -\left(\beta^{2}+\beta\right) \int_{\mathbb{G}^{+}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}|u|^{2} d x \\
& +\beta \int_{\mathbb{G}^{+}} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}|u|^{2} d x,
\end{aligned}
$$

which completes the proof of Part (1).
Proof of Part (2). Let $x:=\left(x^{\prime}, x^{(2)}, \ldots, x^{(r)}\right) \in \mathbb{G}$ with $x^{\prime}=\left(x_{1}^{\prime}, \ldots, x_{N}^{\prime}\right)$ and $x^{(j)} \in \mathbb{R}^{N_{j}}, j=2, \ldots, r$. By taking $\nu:=\left(\nu^{\prime}, 0, \ldots, 0\right)$ with $\nu^{\prime}=\left(\nu_{1}^{\prime}, \ldots, \nu_{N}^{\prime}\right)$, we have that

$$
X_{i}(x)=(\overbrace{0, \ldots, 1}^{i}, \ldots, 0, a_{i, 1}^{(2)}\left(x^{\prime}\right), \ldots, a_{i, N_{r}}^{(r)}\left(x^{\prime}, x^{(2)}, \ldots, x^{(r-1)}\right))
$$

so that

$$
\sum_{i=1}^{N}\left\langle X_{i}(x), \nu\right\rangle^{2}=\sum_{i=1}^{N}\left(\nu_{i}^{\prime}\right)^{2}=\left|\nu^{\prime}\right|^{2}=1
$$

and

$$
X_{i}\left\langle X_{i}(x), \nu\right\rangle=X_{i} \nu_{i}^{\prime}=0
$$

Substituting this in (8.2) we get

$$
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{2} d x \geq-\left(\beta^{2}+\beta\right) \int_{\mathbb{G}^{+}} \frac{|u|^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}} d x
$$

To optimize we differentiate the right-hand side expression with respect to $\beta$, that is, we put $-2 \beta-1=0$, or $\beta=-\frac{1}{2}$ in this inequality, implying (8.3).

### 8.1.1 Examples of Heisenberg and Engel groups

Let us give examples of the geometric $L^{2}$-Hardy inequality on half-spaces from Theorem 8.1.3 in the cases of groups of steps 2 and 3. The example of general
stratified groups of step 2 was considered in Remark 8.1.4, Part 1, and now we look at the special case of the Heisenberg group. In particular, it yields the estimate that was given in Remark 8.1.1, Part 2.

Corollary 8.1.5 (Geometric $L^{2}$-Hardy inequality on half-space of the Heisenberg group). Let $\mathbb{H}^{+}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{H} \mid x_{3}>0\right\}$ be a half-space of the Heisenberg group $\mathbb{H}$. Then for all $u \in C_{0}^{\infty}\left(\mathbb{H}^{+}\right)$we have

$$
\int_{\mathbb{H}^{+}}\left|\nabla_{\mathbb{H}} u\right|^{2} d x \geq \int_{\mathbb{H}^{+}} \frac{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}{x_{3}^{2}}|u|^{2} d x,
$$

where $\nabla_{\mathbb{H}}=\left(X_{1}, X_{2}\right)$.
Proof of Corollary 8.1.5. Since the left invariant vector fields on the Heisenberg group can be given by

$$
X_{1}=\frac{\partial}{\partial x_{1}}+2 x_{2} \frac{\partial}{\partial x_{3}}, \quad X_{2}=\frac{\partial}{\partial x_{2}}-2 x_{1} \frac{\partial}{\partial x_{3}}
$$

with the commutator

$$
\left[X_{1}, X_{2}\right]=-4 \frac{\partial}{\partial x_{3}}
$$

choosing $\nu=(0,0,1)$ as the unit vector in the direction of $x_{3}$ and taking $d=0$ in inequality (8.2), we get

$$
X_{1}(x)=\left(1,0,2 x_{2}\right) \quad \text { and } \quad X_{2}(x)=\left(0,1,-2 x_{1}\right),
$$

and

$$
\begin{aligned}
& \left\langle X_{1}(x), \nu\right\rangle=2 x_{2}, \quad \text { and } \quad\left\langle X_{2}(x), \nu\right\rangle=-2 x_{1}, \\
& X_{1}\left\langle X_{1}(x), \nu\right\rangle=0, \quad \text { and } \quad X_{2}\left\langle X_{2}(x), \nu\right\rangle=0,
\end{aligned}
$$

where $x=\left(x_{1}, x_{2}, x_{3}\right)$. Thus, with $\mathcal{W}(x)$ as in (8.1), we get

$$
\frac{\mathcal{W}(x)^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}=4 \frac{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}{x_{3}^{2}}
$$

Inserting these to (8.2) with $\beta=-\frac{1}{2}$ we obtain

$$
\int_{\mathbb{H}^{+}}\left|\nabla_{\mathbb{H}} u\right|^{2} d x \geq \int_{\mathbb{H}^{+}} \frac{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}{x_{3}^{2}}|u|^{2} d x,
$$

completing the proof.
Next, let us give an example for a class of stratified groups of step $r=3$, namely, the case of the Engel group.

Definition 8.1.6 (Engel group). The Engel group $\mathcal{E}$ is the space $\mathbb{R}^{4}$ with the group law given by

$$
x \circ y=\left(x_{1}+y_{1}, x_{2}+y_{2}, x_{3}+y_{3}+P_{1}, x_{4}+y_{4}+P_{2}\right)
$$

where

$$
\begin{aligned}
P_{1} & =\frac{1}{2}\left(x_{1} y_{2}-x_{2} y_{1}\right) \\
P_{2} & =\frac{1}{2}\left(x_{1} y_{3}-x_{3} y_{1}\right)+\frac{1}{12}\left(x_{1}^{2} y_{2}-x_{1} y_{1}\left(x_{2}+y_{2}\right)+x_{2} y_{1}^{2}\right)
\end{aligned}
$$

The left invariant vector fields of $\mathcal{E}$ are generated by (the basis)

$$
\begin{array}{ll}
X_{1}=\frac{\partial}{\partial x_{1}}-\frac{x_{2}}{2} \frac{\partial}{\partial x_{3}}-\left(\frac{x_{3}}{2}-\frac{x_{1} x_{2}}{12}\right) \frac{\partial}{\partial x_{4}}, & X_{3}=\frac{\partial}{\partial x_{3}}+\frac{x_{1}}{2} \frac{\partial}{\partial x_{4}} \\
X_{2}=\frac{\partial}{\partial x_{2}}+\frac{x_{1}}{2} \frac{\partial}{\partial x_{3}}+\frac{x_{1}^{2}}{12} \frac{\partial}{\partial x_{4}}, & X_{4}=\frac{\partial}{\partial x_{4}}
\end{array}
$$

The group $\mathcal{E}$ is stratified of step 3, with the nonzero commutation relations given by

$$
\left[X_{1}, X_{2}\right]=X_{3},\left[X_{1}, X_{3}\right]=X_{4}
$$

So we have
Corollary 8.1.7 (Geometric $L^{2}$-Hardy inequality on half-space of the Engel group). Let $\mathcal{E}^{+}=\left\{x:=\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathcal{E} \mid\langle x, \nu\rangle>0\right\}$ be a half-space of the Engel group $\mathcal{E}$. Then for all $\beta \in \mathbb{R}$ and $u \in C_{0}^{\infty}\left(\mathcal{E}^{+}\right)$we have

$$
\begin{align*}
\int_{\mathcal{E}^{+}}\left|\nabla_{\mathcal{E}} u\right|^{2} d x \geq & C_{1}(\beta) \int_{\mathcal{E}^{+}} \frac{\left\langle X_{1}(x), \nu\right\rangle^{2}+\left\langle X_{2}(x), \nu\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \mathcal{E}^{+}\right)^{2}}|u|^{2} d x  \tag{8.9}\\
& +\frac{\beta}{3} \int_{\mathcal{E}^{+}} \frac{x_{2} \nu_{4}}{\operatorname{dist}\left(x, \partial \mathcal{E}^{+}\right)}|u|^{2} d x
\end{align*}
$$

where $\nabla_{\mathcal{E}}=\left(X_{1}, X_{2}\right)$, $\nu:=\left(\nu_{1}, \nu_{2}, \nu_{3}, \nu_{4}\right)$, and $C_{1}(\beta)=-\left(\beta^{2}+\beta\right)$.
In particular, if we take $\nu_{4}=0$ in (8.9), then by taking $\beta=-\frac{1}{2}$, we get the following inequality on such $\mathcal{E}^{+}$:

$$
\int_{\mathcal{E}^{+}}\left|\nabla_{\mathcal{E}} u\right|^{2} d x \geq \frac{1}{4} \int_{\mathcal{E}^{+}} \frac{\left\langle X_{1}(x), \nu\right\rangle^{2}+\left\langle X_{2}(x), \nu\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \mathcal{E}^{+}\right)^{2}}|u|^{2} d x
$$

Proof of Corollary 8.1.7. Using the above basis of the left invariant vector fields, we have

$$
\begin{aligned}
& X_{1}(x)=\left(1,0,-\frac{x_{2}}{2},-\left(\frac{x_{3}}{2}-\frac{x_{1} x_{2}}{12}\right)\right) \\
& X_{2}(x)=\left(0,1, \frac{x_{1}}{2}, \frac{x_{1}^{2}}{12}\right)
\end{aligned}
$$

It is then straightforward to see that

$$
\begin{aligned}
\left\langle X_{1}(x), \nu\right\rangle & =\nu_{1}-\frac{x_{2}}{2} \nu_{3}-\left(\frac{x_{3}}{2}-\frac{x_{1} x_{2}}{12}\right) \nu_{4} \\
\left\langle X_{2}(x), \nu\right\rangle & =\nu_{2}+\frac{x_{1}}{2} \nu_{3}+\frac{x_{1}^{2}}{12} \nu_{4} \\
X_{1}\left\langle X_{1}(x), \nu\right\rangle & =\frac{x_{2}}{12} \nu_{4}+\frac{x_{2}}{4} \nu_{4}=\frac{x_{2} \nu_{4}}{3} \\
X_{2}\left\langle X_{2}(x), \nu\right\rangle & =0
\end{aligned}
$$

Now plugging these in inequality (8.2) we get the desired inequality (8.9).

## 8.2 $\quad L^{p}$-Hardy inequality on the half-space

Now we discuss an $L^{p}$ version of the geometric Hardy inequality on the half-space of $\mathbb{G}$ as an extension of the previous $L^{2}$ arguments. We recall that the $p$-version of Garofalo's angle function from Definition 8.1.2 can be defined by the formula

$$
\begin{equation*}
\mathcal{W}_{p}(x)=\left(\sum_{i=1}^{N}\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}\right)^{1 / p} \tag{8.10}
\end{equation*}
$$

with $\mathcal{W}(x):=\mathcal{W}_{2}(x)$, and where $N$ denotes the dimension of the first stratum of $\mathbb{G}$. As before let $\mathbb{G}^{+}$be a half-space of a stratified group $\mathbb{G}$. The $L^{p}$ version of the geometric Hardy inequality from Theorem 8.1.3 can be written in the following form.

Theorem 8.2.1 (Geometric $L^{p}$-Hardy inequality on half-space). Let $\mathbb{G}^{+}$be a halfspace of a stratified group $\mathbb{G}$ and let $1<p<\infty$. Then for all $u \in C_{0}^{\infty}\left(\mathbb{G}^{+}\right)$and all $\beta \in \mathbb{R}$ we have

$$
\begin{align*}
\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left|X_{i} u\right|^{p} d x \geq & C_{2}(\beta, p) \int_{\mathbb{G}^{+}} \frac{\mathcal{W}_{p}(x)^{p}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p}}|u|^{p} d x  \tag{8.11}\\
& +\beta(p-1) \int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}|u|^{p} d x
\end{align*}
$$

where $C_{2}(\beta, p):=-(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right)$.
Remark 8.2.2. Note that for $p \geq 2$, since

$$
\left|\nabla_{H} u\right|^{p}=\left(\sum_{i=1}^{N}\left|X_{i} u\right|^{2}\right)^{p / 2} \geq \sum_{i=1}^{N}\left(\left|X_{i} u\right|^{2}\right)^{p / 2}
$$

the proof will also yield the inequality

$$
\begin{align*}
\int_{\mathbb{G}^{+}}\left|\nabla_{H} u\right|^{p} d x \geq & C_{2}(\beta, p) \int_{\mathbb{G}^{+}} \frac{\mathcal{W}_{p}(x)^{p}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p}}|u|^{p} d x  \tag{8.12}\\
& +\beta(p-1) \int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}|u|^{p} d x
\end{align*}
$$

Proof of Theorem 8.2.1. For $W \in C^{\infty}\left(\mathbb{G}^{+}\right)$and $f \in C^{1}\left(\mathbb{G}^{+}\right)$, a direct computation with Hölder's inequality gives

$$
\begin{align*}
& \int_{\mathbb{G}^{+}} \operatorname{div}_{H}(f W)|u|^{p} d x=-\int_{\mathbb{G}^{+}} f W \cdot \nabla_{H}|u|^{p} d x=-p \int_{\mathbb{G}^{+}} f\left\langle W, \nabla_{H} u\right\rangle|u|^{p-1} d x \\
& \quad \leq p\left(\int_{\mathbb{G}^{+}}\left|\left\langle W, \nabla_{H} u\right\rangle\right|^{p} d x\right)^{1 / p}\left(\int_{\mathbb{G}^{+}}|f|^{\frac{p}{p-1}}|u|^{p} d x\right)^{\frac{p-1}{p}} \tag{8.13}
\end{align*}
$$

For $p>1$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we will use Young's inequality

$$
a b \leq \frac{a^{p}}{p}+\frac{b^{q}}{q}, \text { for } a \geq 0, b \geq 0
$$

with

$$
a:=\left(\int_{\mathbb{G}^{+}}\left|\left\langle W, \nabla_{H} u\right\rangle\right|^{p} d x\right)^{1 / p} \quad \text { and } \quad b:=\left(\int_{\mathbb{G}^{+}}|f|^{\frac{p}{p-1}}|u|^{p} d x\right)^{\frac{p-1}{p}} .
$$

Using this Young inequality in (8.13) and rearranging the terms, we get

$$
\begin{equation*}
\int_{\mathbb{G}^{+}}\left|\left\langle W, \nabla_{H} u\right\rangle\right|^{p} d x \geq \int_{\mathbb{G}^{+}}\left(\operatorname{div}_{H}(f W)-(p-1)|f|^{\frac{p}{p-1}}\right)|u|^{p} d x . \tag{8.14}
\end{equation*}
$$

Now choosing $W:=I_{i}$, which has the following form $I_{i}=(\overbrace{0, \ldots, 1}^{i}, \ldots, 0)$ and setting

$$
f=\beta \frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p-1}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p-1}},
$$

we calculate

$$
\begin{aligned}
\operatorname{div}_{\mathrm{H}}(W f) & =\left(\nabla_{H} \cdot I_{i}\right) f=X_{i} f=\beta X_{i}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-1} \\
& =\beta(p-1)\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2} X_{i}\left(\frac{\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right) \\
& =\beta(p-1)\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}-\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{2}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{2}}\right) \\
& =\beta(p-1)\left[\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)-\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p}}\right],
\end{aligned}
$$

and

$$
|f|^{\frac{p}{p-1}}=|\beta|^{\frac{p}{p-1}} \frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p}}
$$

Moreover, we have

$$
\left\langle W, \nabla_{H} u\right\rangle=\overbrace{(0, \ldots, 1}^{i}, \ldots, 0) \cdot\left(X_{1} u, \ldots, X_{i} u, \ldots, X_{N} u\right)^{T}=X_{i} u
$$

Substituting these in (8.14) and summing over $i=1, \ldots, N$, we obtain

$$
\begin{align*}
\int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left|X_{i} u\right|^{p} d x \geq & -(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right) \int_{\mathbb{G}^{+}} \sum_{i=1}^{N} \frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)^{p}}|u|^{p} d x  \tag{8.15}\\
& +\beta(p-1) \int_{\mathbb{G}^{+}} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}\right)^{p-2} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}\left(x, \partial \mathbb{G}^{+}\right)}|u|^{p} d x .
\end{align*}
$$

This completes the proof.

## $8.3 \quad L^{2}$-Hardy inequality on convex domains

In this and the following sections we extend the proceeding arguments from halfspaces to convex domains in the stratified groups. Here, however, the convex domain is understood in the sense of the Euclidean space. Thus, let $\Omega$ be a convex domain of a stratified group $\mathbb{G}$ and let $\partial \Omega$ be its boundary. Here for $x \in \Omega$ we denote by $\nu(x)$ the unit normal for $\partial \Omega$ at a point $\hat{x} \in \partial \Omega$, determined by the condition

$$
\operatorname{dist}(x, \partial \Omega)=\operatorname{dist}(x, \hat{x})
$$

For the half-space, we have the distance from the boundary dist $(x, \partial \Omega)=\langle x, \nu\rangle-d$. As it was already defined in (8.10), we will use the $p$-version of the angle function

$$
\mathcal{W}_{p}(x)=\left(\sum_{i=1}^{N}\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}\right)^{1 / p}
$$

with $\mathcal{W}(x):=\mathcal{W}_{2}(x)$. We have the following extension of Theorem 8.1.3.
Theorem 8.3.1 (Geometric $L^{2}$-Hardy inequality on convex domains). Let $\Omega$ be a convex domain of a stratified group $\mathbb{G}$. Then for all $u \in C_{0}^{\infty}(\Omega)$ and all $\beta<0$ we have

$$
\begin{equation*}
\int_{\Omega}\left|\nabla_{H} u\right|^{2} d x \geq C_{1}(\beta) \int_{\Omega} \frac{\mathcal{W}(x)^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}}|u|^{2} d x+\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}|u|^{2} d x \tag{8.16}
\end{equation*}
$$

where $C_{1}(\beta):=-\left(\beta^{2}+\beta\right)$.
Proof of Theorem 8.3.1. As elsewhere in this chapter, we follow the proof for general stratified groups of [RSS18b], based on the convex polytope approach used by Larson [Lar16] in the case of the Heisenberg group.

We denote the facets of $\Omega$ by $\left\{\mathcal{F}_{j}\right\}_{j}$ and unit normals of these facets by $\left\{\nu_{j}\right\}_{j}$, which are directed inward. So, $\Omega$ can be viewed as the union of the disjoint sets

$$
\Omega_{j}:=\left\{x \in \Omega: \operatorname{dist}(x, \partial \Omega)=\operatorname{dist}\left(x, \mathcal{F}_{j}\right)\right\}
$$

Now we follow the method as in the case of the half-space $\mathbb{G}^{+}$for each element $\Omega_{j}$ with one exception that not all the boundary values are zero when we use the partial integration. As before we calculate

$$
\begin{aligned}
0 \leq \int_{\Omega_{j}}\left|\nabla_{H} u+\beta W u\right|^{2} d x= & \int_{\Omega_{j}} \sum_{i=1}^{N}\left|X_{i} u+\beta W_{i} u\right|^{2} d x \\
= & \int_{\Omega_{j}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}+2 \operatorname{Re} \beta W_{i} u X_{i} u+\beta^{2} W_{i}^{2}|u|^{2}\right] d x \\
= & \int_{\Omega_{j}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}+\beta W_{i} X_{i}|u|^{2}+\beta^{2} W_{i}^{2}|u|^{2}\right] d x \\
= & \int_{\Omega_{j}} \sum_{i=1}^{N}\left[\left|X_{i} u\right|^{2}-\beta\left(X_{i} W_{i}\right)|u|^{2}+\beta^{2} W_{i}^{2}|u|^{2}\right] d x \\
& +\beta \int_{\partial \Omega_{j}} \sum_{i=1}^{N} W_{i}\left\langle X_{i}(x), n_{j}(x)\right\rangle|u|^{2} d \Gamma_{\partial \Omega_{j}}(x)
\end{aligned}
$$

where $n_{j}$ is the unit normal of $\partial \Omega_{j}$ which is directed outward. Since $\mathcal{F}_{j} \subset \partial \Omega_{j}$ we have $n_{j}=-\nu_{j}$. That is, we have

$$
\begin{align*}
\int_{\Omega_{j}}\left|\nabla_{H} u\right|^{2} d x \geq & \int_{\Omega_{j}} \sum_{i=1}^{N}\left[\left(\beta\left(X_{i} W_{i}\right)-\beta^{2} W_{i}^{2}\right)|u|^{2}\right] d x  \tag{8.17}\\
& -\beta \int_{\partial \Omega_{j}} \sum_{i=1}^{N} W_{i}\left\langle X_{i}(x), n_{j}(x)\right\rangle|u|^{2} d \Gamma_{\partial \Omega_{j}}(x)
\end{align*}
$$

The boundary terms on $\partial \Omega$ disappears since $u$ is compactly supported in $\Omega$. Thus, we only need to deal with the parts of $\partial \Omega_{j}$ in $\Omega$. Note that for every facet of $\partial \Omega_{j}$ there exists some $\partial \Omega_{l}$ which shares this facet. Denote by $\Gamma_{j l}$ the common facet of $\partial \Omega_{j}$ and $\partial \Omega_{l}$, with $\left.n_{k}\right|_{\Gamma_{j l}}=-\left.n_{l}\right|_{\Gamma_{j l}}$.

Now we choose $W_{i}$ in the form

$$
W_{i}(x)=\frac{\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)}=\frac{\left\langle X_{i}(x), \nu_{j}\right\rangle}{\left\langle x, \nu_{j}\right\rangle-d}
$$

and a direct computation shows that

$$
\begin{equation*}
X_{i} W_{i}(x)=\frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)}-\frac{\left\langle X_{i}(x), \nu_{j}\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)^{2}} \tag{8.18}
\end{equation*}
$$

Substituting (8.18) into (8.17) we get

$$
\begin{align*}
& \int_{\Omega_{j}}\left|\nabla_{H} u\right|^{2} d x \geq-\left(\beta^{2}+\beta\right) \int_{\Omega_{j}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)^{2}}|u|^{2} d x  \tag{8.19}\\
& \quad+\beta \int_{\Omega_{j}} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)}|u|^{2} d x-\beta \int_{\Gamma_{j l}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle\left\langle X_{i}(x), n_{j l}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}|u|^{2} d \Gamma_{j l} .
\end{align*}
$$

Now we sum over all partition elements $\Omega_{j}$ and let $n_{j l}=\left.n_{k}\right|_{\Gamma_{j l}}$, i.e., the unit normal of $\Gamma_{j l}$ pointing from $\Omega_{j}$ into $\Omega_{l}$. Then we have

$$
\begin{aligned}
\int_{\Omega}\left|\nabla_{H} u\right|^{2} d x \geq & -\left(\beta^{2}+\beta\right) \int_{\Omega} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu\right\rangle^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}}|u|^{2} d x \\
& +\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}|u|^{2} d x \\
& -\beta \sum_{j \neq l} \int_{\Gamma_{j l}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle\left\langle X_{i}(x), n_{j l}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}|u|^{2} d \Gamma_{j l} \\
= & -\left(\beta^{2}+\beta\right) \int_{\Omega} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu\right\rangle^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}}|u|^{2} d x \\
& +\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}|u|^{2} d x \\
& -\beta \sum_{j<l} \int_{\Gamma_{j l}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}-\nu_{l}\right\rangle\left\langle X_{i}(x), n_{j l}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}|u|^{2} d \Gamma_{j l} .
\end{aligned}
$$

Here we have used the fact that (by the definition) $\Gamma_{j l}$ is a set with

$$
\operatorname{dist}\left(x, \mathcal{F}_{j}\right)=\operatorname{dist}\left(x, \mathcal{F}_{l}\right)
$$

From

$$
\Gamma_{j l}=\left\{x: x \cdot \nu_{j}-d_{j}=x \cdot \nu_{l}-d_{l}\right\}
$$

rearranging $x \cdot\left(\nu_{j}-\nu_{l}\right)-d_{j}+d_{l}=0$ we see that $\Gamma_{j l}$ is a hyperplane with a normal $\nu_{j}-\nu_{l}$. So, $\nu_{j}-\nu_{l}$ is parallel to $n_{j l}$ and one only needs to check that $\left(\nu_{j}-\nu_{l}\right) \cdot n_{j l}>0$. Since $n_{j l}$ points out and $\nu_{j}$ points into $j$ th partition element, $\nu_{j} \cdot n_{j l}$ is non-negative. Similarly, we see that $\nu_{l} \cdot n_{j l}$ is non-positive. That is, $\left(\nu_{j}-\nu_{l}\right) \cdot n_{j l}>0$. On the other hand, it is easy to see that

$$
\begin{aligned}
\left|\nu_{j}-\nu_{l}\right|^{2}=\left(\nu_{j}-\nu_{l}\right) \cdot\left(\nu_{j}-\nu_{l}\right) & =2-2 \nu_{j} \cdot \nu_{l} \\
& =2-2 \cos \left(\alpha_{j l}\right)
\end{aligned}
$$

which implies that

$$
\left(\nu_{j}-\nu_{l}\right) \cdot n_{j l}=\sqrt{2-2 \cos \left(\alpha_{j l}\right)}
$$

where $\alpha_{j l}$ is the angle between $\nu_{j}$ and $\nu_{l}$. So we obtain

$$
\begin{aligned}
\int_{\Omega}\left|\nabla_{H} u\right|^{2} d x \geq & -\left(\beta^{2}+\beta\right) \int_{\Omega} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu\right\rangle^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}}|u|^{2} d x \\
& +\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}|u|^{2} d x \\
& -\beta \sum_{j<l} \sum_{i=1}^{N} \int_{\Gamma_{j l}} \sqrt{1-\cos \left(\alpha_{j l}\right)} \frac{\left\langle X_{i}(x), n_{j l}\right\rangle^{2}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}|u|^{2} d \Gamma_{j l} .
\end{aligned}
$$

Here with $\beta<0$ and due to the boundary term signs we prove the desired inequality for the polytope convex domains.

Now we are ready to consider the general case, that is, when $\Omega$ is an arbitrary convex domain. For each $u \in C_{0}^{\infty}(\Omega)$ one can always choose an increasing sequence of convex polytopes $\left\{\Omega_{j}\right\}_{j=1}^{\infty}$ such that $u \in C_{0}^{\infty}\left(\Omega_{1}\right), \Omega_{j} \subset \Omega$ and $\Omega_{j} \rightarrow \Omega$ as $j \rightarrow \infty$. Assume that $\nu_{j}(x)$ is the above map $\nu$ (corresponding to $\Omega_{j}$ ), and then we can calculate

$$
\begin{aligned}
& \int_{\Omega}\left|\nabla_{H} u\right|^{2} d x=\int_{\Omega_{j}}\left|\nabla_{H} u\right|^{2} d x \\
& \quad \geq-\left(\beta^{2}+\beta\right) \int_{\Omega_{j}} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)^{2}}|u|^{2} d x+\beta \int_{\Omega_{j}} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)}|u|^{2} d x \\
& \quad=-\left(\beta^{2}+\beta\right) \int_{\Omega} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle^{2}}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)^{2}}|u|^{2} d x+\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \partial \Omega_{j}\right)}|u|^{2} d x \\
& \quad \geq-\left(\beta^{2}+\beta\right) \int_{\Omega} \sum_{i=1}^{N} \frac{\left\langle X_{i}(x), \nu_{j}\right\rangle^{2}}{\operatorname{dist}(x, \partial \Omega)^{2}}|u|^{2} d x+\beta \int_{\Omega} \sum_{i=1}^{N} \frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}(x, \partial \Omega)}|u|^{2} d x
\end{aligned}
$$

Now we obtain the desired result by letting $j \rightarrow \infty$.

## 8.4 $\quad L^{p}$-Hardy inequality on convex domains

The same arguments as in the previous section give the general $L^{p}$-version of Theorem 8.3.1.

Theorem 8.4.1 (Geometric $L^{p}$-Hardy inequality on convex domains). Let $\Omega$ be a convex domain of a stratified group $\mathbb{G}$. Then for all $u \in C_{0}^{\infty}(\Omega)$ and all $\beta<0$
we have

$$
\begin{align*}
\int_{\Omega} \sum_{i=1}^{N}\left|X_{i} u\right|^{p} d x \geq & C_{2}(\beta, p) \int_{\Omega} \frac{\mathcal{W}_{p}(x)^{p}}{\operatorname{dist}(x, \partial \Omega)^{p}}|u|^{p} d x  \tag{8.20}\\
& +\beta(p-1) \int_{\Omega} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}(x, \partial \Omega)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}\right)|u|^{p} d x
\end{align*}
$$

where $C_{2}(\beta, p):=-(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right)$.
Remark 8.4.2. Note that for $p \geq 2$, since

$$
\begin{equation*}
\left|\nabla_{H} u\right|^{p}=\left(\sum_{i=1}^{N}\left|X_{i} u\right|^{2}\right)^{p / 2} \geq \sum_{i=1}^{N}\left(\left|X_{i} u\right|^{2}\right)^{p / 2} \tag{8.21}
\end{equation*}
$$

instead of (8.20) we have the inequality

$$
\begin{align*}
\int_{\Omega}\left|\nabla_{H} u\right|^{p} d x \geq & C_{2}(\beta, p) \int_{\Omega} \frac{\mathcal{W}_{p}(x)^{p}}{\operatorname{dist}(x, \partial \Omega)^{p}}|u|^{p} d x  \tag{8.22}\\
& +\beta(p-1) \int_{\Omega} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}(x, \partial \Omega)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}\right)|u|^{p} d x
\end{align*}
$$

Proof of Theorem 8.4.1. As in the proof of Theorem 8.3.1, let us first assume that $\Omega$ is the convex polytope. Thus, for $f \in C^{1}\left(\Omega_{j}\right)$ and $W \in C^{\infty}\left(\Omega_{j}\right)$, we calculate

$$
\begin{align*}
& \int_{\Omega_{j}} \operatorname{div}_{\mathbb{G}}(f W)|u|^{p} d x=-p \int_{\Omega_{j}} f\left\langle W, \nabla_{H} u\right\rangle|u|^{p-1} d x+\int_{\partial \Omega_{j}} f\left\langle W, n_{j}(x)\right\rangle|u|^{p} d \Gamma_{\partial \Omega_{j}}(x) \\
& \leq p\left(\int_{\Omega}\left|\left\langle W, \nabla_{H} u\right\rangle\right|^{p} d x\right)^{\frac{1}{p}}\left(\int_{\Omega_{j}}|f|^{\frac{p}{p-1}}|u|^{p} d x\right)^{\frac{p-1}{p}}+\int_{\partial \Omega_{j}} f\left\langle W, n_{j}(x)\right\rangle|u|^{p} d \Gamma_{\partial \Omega_{j}}(x), \tag{8.23}
\end{align*}
$$

where $\Omega_{j}$ is the partition as in the proof of Theorem 8.3.1. In the last line we have used the Hölder inequality. By using Young's inequality in (8.23) and rearranging the terms, we get

$$
\begin{align*}
\int_{\Omega_{j}}\left|\left\langle W, \nabla_{H} u\right\rangle\right|^{p} d x \geq & \int_{\Omega}\left(\operatorname{div}_{\mathbb{G}}(f W)-(p-1)|f|^{\frac{p}{p-1}}\right)|u|^{p} d x  \tag{8.24}\\
& -\int_{\partial \Omega_{j}} f\left\langle W, n_{j}(x)\right\rangle|u|^{p} d \Gamma_{\partial \Omega_{j}}(x)
\end{align*}
$$

Choosing $W:=I_{i}$ as a unit vector of the $i$ th component and letting

$$
f=\beta \frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p-1}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p-1}}
$$

we calculate

$$
\operatorname{div}_{\mathbb{G}}(W f)=X_{i} f=\beta X_{i}\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}
$$

$$
\begin{aligned}
& =\beta(p-1)\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-2} X_{i}\left(\frac{\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right) \\
& =\beta(p-1)\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}-\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{2}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{2}}\right) \\
& =\beta(p-1)\left[\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)-\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p}}\right]
\end{aligned}
$$

and

$$
|f|^{\frac{p}{p-1}}=|\beta|^{\frac{p}{p-1}} \frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p}}
$$

Moreover, we have

$$
\left\langle W, \nabla_{H} u\right\rangle=(\overbrace{0, \ldots, 1}^{i}, \ldots, 0) \cdot\left(X_{1} u, \ldots, X_{i} u, \ldots, X_{N} u\right)^{T}=X_{i} u .
$$

Substituting these into (8.24) and summing over all $i=1, \ldots, N$, we obtain

$$
\begin{align*}
\int_{\Omega_{j}} \sum_{i=1}^{N}\left|X_{i} u\right|^{p} d x \geq & -(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right) \int_{\Omega_{j}} \sum_{i=1}^{N} \frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p}}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p}}|u|^{p} d x  \tag{8.25}\\
& +\beta(p-1) \int_{\Omega_{j}} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu_{j}\right\rangle}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)|u|^{p} d x \\
& -\beta \int_{\partial \Omega_{j}} \sum_{i=1}^{N}\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j}(x)\right\rangle|u|^{p} d \Gamma_{\partial \Omega_{j}}(x)
\end{align*}
$$

Now summing up over $\Omega_{j}$, and with the interior boundary terms we get

$$
\begin{aligned}
\int_{\Omega} \sum_{i=1}^{N}\left|X_{i} u\right|^{p} d x \geq & -(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right) \sum_{i=1}^{N} \int_{\Omega} \frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}}{\operatorname{dist}(x, \partial \Omega)^{p}}|u|^{p} d x \\
& +\beta(p-1) \sum_{i=1}^{N} \int_{\Omega}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}(x, \partial \Omega)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}\right)|u|^{p} d x \\
& -\beta \sum_{j \neq l} \sum_{i=1}^{N} \int_{\Gamma_{j l}}\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j l}(x)\right\rangle|u|^{p} d \Gamma_{j l} \\
= & -(p-1)\left(|\beta|^{\frac{p}{p-1}}+\beta\right) \sum_{i=1}^{N} \int_{\Omega} \frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|^{p}}{\operatorname{dist}(x, \partial \Omega)^{p}}|u|^{p} d x \\
& +\beta(p-1) \sum_{i=1}^{N} \int_{\Omega}\left(\frac{\left|\left\langle X_{i}(x), \nu\right\rangle\right|}{\operatorname{dist}(x, \partial \Omega)}\right)^{p-2}\left(\frac{X_{i}\left\langle X_{i}(x), \nu\right\rangle}{\operatorname{dist}(x, \partial \Omega)}\right)|u|^{p} d x \\
& -\beta \sum_{j<l} \sum_{i=1}^{N} \int_{\Gamma_{j l}}\left[\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j l}(x)\right\rangle\right.
\end{aligned}
$$

$$
\left.-\left(\frac{\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{l}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j l}(x)\right\rangle\right]|u|^{p} d \Gamma_{j l}
$$

As in the proof of Theorem 8.3.1, if the boundary term is positive we can discard it, so we need to show that

$$
\left[\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j l}(x)\right\rangle-\left(\frac{\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{l}\right)}\right)^{p-1}\left\langle X_{i}(x), n_{j l}(x)\right\rangle\right] \geq 0
$$

Since $n_{j l}=\frac{\nu_{j}-\nu_{l}}{\sqrt{2-2 \cos \left(\alpha_{j l}\right)}}$ and $\operatorname{dist}\left(x, \mathcal{F}_{j}\right)=\operatorname{dist}\left(x, \mathcal{F}_{l}\right)$ on $\Gamma_{j l}$, we have

$$
\begin{aligned}
& \frac{1}{2-2 \cos \left(\alpha_{j l}\right)}\left[\left(\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{j}\right)}\right)^{p-1}\left\langle X_{i}(x), \nu_{j}-\nu_{l}\right\rangle\right. \\
& \left.-\left(\frac{\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|}{\operatorname{dist}\left(x, \mathcal{F}_{l}\right)}\right)^{p-1}\left\langle X_{i}(x), \nu_{j}-\nu_{l}\right\rangle\right] \\
& =\frac{\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p}-\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p-1}\left\langle X_{i}(x), \nu_{l}\right\rangle}{\left(2-2 \cos \left(\alpha_{j l}\right)\right) \operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p-1}} \\
& \quad+\frac{-\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|^{p-1}\left\langle X_{i}(x), \nu_{j}\right\rangle+\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|^{p}}{\left(2-2 \cos \left(\alpha_{j l}\right)\right) \operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p-1}} \\
& =\frac{\left(\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|-\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|\right)\left(\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|^{p-1}-\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|^{p-1}\right)}{\left(2-2 \cos \left(\alpha_{j l}\right)\right) \operatorname{dist}\left(x, \mathcal{F}_{j}\right)^{p-1}} \geq 0 .
\end{aligned}
$$

Here we have used the equality

$$
(a-b)\left(a^{p-1}-b^{p-1}\right)=a^{p}-a^{p-1} b-b^{p-1} a+b^{p-1}
$$

with $a=\left|\left\langle X_{i}(x), \nu_{j}\right\rangle\right|$ and $b=\left|\left\langle X_{i}(x), \nu_{l}\right\rangle\right|$. Thus, for $\beta<0$ by discarding the above boundary term (integral) we complete the proof.

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