Chapter 2

Hardy Inequalities on Homogeneous Groups

This chapter is devoted to Hardy inequalities and the analysis of their remainders in different forms. Moreover, we discuss several related inequalities such as Rellich inequalities and uncertainty principles.

In this chapter we will use all the notations given in Chapter 1 concerning homogeneous groups and the operators defined on it. In particular, $\mathbb{G}$ *is always a homogeneous group of homogeneous dimension* $Q \geq 1$. Some statements will hold for $Q \geq 2$ or for $Q \geq 3$ but we will be specifying this explicitly in formulations when needed.

2.1 Hardy inequalities and sharp remainders

In this section we analyse the anisotropic version of the classical L^p-Hardy inequality

$$\left\| \frac{f}{|x|_E} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n, \tag{2.1}$$

where ∇ is the standard gradient in $\mathbb{R}^n$, $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm, $f \in C_0^\infty(\mathbb{R}^n)$, and the constant $\frac{p}{n-p}$ is known to be sharp. We also discuss in detail its critical cases and remainder estimates. As consequences, we derive Rellich type inequalities and the corresponding uncertainty principles.

2.1.1 Hardy inequality and uncertainty principle

First we establish the L^p-Hardy inequality and derive a formula for the remainder on a homogeneous group $\mathbb{G}$ of homogeneous dimension $Q \geq 2$. The radial operator $\mathcal{R}$ from (1.30) is entering the appearing expressions.

Theorem 2.1.1 (Hardy inequalities on homogeneous groups)**.** *Let* $|\cdot|$ *be any homogeneous quasi-norm on* $\mathbb{G}$*.*

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(i) *Let* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *be a complex-valued function. Then we have*

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R}f\|_{L^p(\mathbb{G})}, \quad 1 < p < Q, \tag{2.2}$$

where the constant $\frac{p}{Q-p}$ *is sharp. Moreover, the equality in* (2.2) *is attained if and only if* $f = 0$.

(ii) *For a real-valued function* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *and with the notations*

$$u := u(x) = -\frac{p}{Q-p}\mathcal{R}f(x),$$
$$v := v(x) = \frac{f(x)}{|x|},$$

we have

$$\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} = p\int_{\mathbb{G}} I_p(v,u)|v-u|^2 dx, \tag{2.3}$$

where

$$I_p(h,g) = (p-1)\int_0^1 |\xi h + (1-\xi)g|^{p-2}\xi d\xi. \tag{2.4}$$

(iii) *For* $Q \geq 3$, *for a complex-valued function* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have*

$$\|\mathcal{R}f\|^2_{L^2(\mathbb{G})} = \left(\frac{Q-2}{2}\right)^2 \left\|\frac{f}{|x|}\right\|^2_{L^2(\mathbb{G})} + \left\|\mathcal{R}f + \frac{Q-2}{2}\frac{f}{|x|}\right\|^2_{L^2(\mathbb{G})}, \tag{2.5}$$

that is, when $p = 2$, (2.3) *holds for complex-valued functions as well.*

Remark 2.1.2.

1. In the case of $\mathbb{G} = \mathbb{R}^n$ and $|x| = |x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$ the Euclidean norm, we have $Q = n$ and $\mathcal{R} = \partial_r$ is the usual radial derivative, and (2.2) implies the classical Hardy inequality (2.1). Indeed, in this case for $1 < p < n$ inequality (2.2) yields

$$\begin{aligned}\left\|\frac{f}{|x|_E}\right\|_{L^p(\mathbb{R}^n)} &\leq \frac{p}{n-p}\|\mathcal{R}f\|_{L^p(\mathbb{R}^n)} = \frac{p}{n-p}\|\partial_r f\|_{L^p(\mathbb{R}^n)} \\ &= \frac{p}{n-p}\left\|\frac{x}{|x|_E}\cdot\nabla f\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\|\nabla f\|_{L^p(\mathbb{R}^n)},\end{aligned} \tag{2.6}$$

   in view of the Cauchy–Schwarz inequality for the Euclidean norm.

   An interesting feature of the Hardy inequality in Part (i) is that the constant in (2.2) is sharp for any homogeneous quasi-norm $|\cdot|$.

2. In the setting of Part 1 above the remainder formula (2.3) for the Euclidean norm $|\cdot|_E$ in $\mathbb{R}^n$ was analysed by Ioku, Ishiwata and Ozawa [IIO17].

3. In Theorem 2.1.1, Part (ii) implies Part (i). To show it one can notice that the right-hand side of (2.3) is non-negative, which implies that

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \left\| \mathcal{R} f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < Q, \tag{2.7}$$

for any real-valued $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Moreover, by using the following identity we obtain the same inequality for all complex-valued functions: for all $z \in \mathbb{C}$ we have

$$|z|^p = \left(\int_{-\pi}^{\pi} |\cos\theta|^p d\theta \right)^{-1} \int_{-\pi}^{\pi} |\mathrm{Re}(z)\cos\theta + \mathrm{Im}(z)\sin\theta|^p \, d\theta, \tag{2.8}$$

which is a consequence of the decomposition of a complex number $z = r(\cos\phi + i\sin\phi)$.

That is, we obtain inequality (2.2), and also that the constant $\frac{p}{Q-p}$ is sharp, in view of the remainder formula. Now let us show that this constant is attained only for $f = 0$. Identity (2.8) says that it is sufficient to look only for real-valued functions f. If the right-hand side of (2.3) vanishes, then we must have $u = v$, that is,

$$-\frac{p}{Q-p} \mathcal{R} f(x) = \frac{f(x)}{|x|}.$$

This also means that $\mathbb{E} f = -\frac{Q-p}{p} f$. Lemma 1.3.1 implies that f is positively homogeneous of order $-\frac{Q-p}{p}$, i.e., there exists a function $h : \wp \to \mathbb{C}$ such that

$$f(x) = |x|^{-\frac{Q-p}{p}} h\left(\frac{x}{|x|}\right), \tag{2.9}$$

where $\wp$ is the unit sphere for the quasi-norm $|\cdot|$. It confirms that f cannot be compactly supported unless it is identically zero.

4. The identity (2.8) has been often used in similar estimates for passing from real-valued to complex-valued functions, see, e.g., Davies [Dav80, p. 176].

5. Let us denote by $H^1_{\mathcal{R}}(\mathbb{G})$ the functional space of the functions $f \in L^2(\mathbb{G})$ with $\mathcal{R} f \in L^2(\mathbb{G})$. Then Theorem 2.1.1 can be extended for functions in $H^1_{\mathcal{R}}(\mathbb{G})$, that is, the proof of (2.2) given above works in this case. As for the sharpness and the equality in (2.2), having (2.9) also implies that $\frac{f(x)}{|x|} = |x|^{-\frac{Q}{p}} h\left(\frac{x}{|x|}\right)$ is not in $L^p(\mathbb{G})$ unless $h = 0$ and $f = 0$.

Remark 2.1.2, Part 2, shows that (2.3) implies Part (i) of Theorem 2.1.1, that is, we only need to prove Parts (ii) and (iii). However, we now give an independent proof of (2.2) for complex-valued functions without relying on the formula (2.8). We see that this argument will be also useful in the proof of Part (ii).

Proof of Theorem 2.1.1. Proof of Part (i). Using the polar decomposition from Proposition 1.2.10, a direct calculation shows that

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^p}dx &= \int_0^\infty \int_{\wp} \frac{|f(ry)|^p}{r^p} r^{Q-1} d\sigma(y)dr \\ &= -\frac{p}{Q-p}\int_0^\infty r^{Q-p}\,\mathrm{Re}\int_{\wp} |f(ry)|^{p-2}f(ry)\overline{\frac{df(ry)}{dr}}d\sigma(y)dr \\ &= -\frac{p}{Q-p}\mathrm{Re}\int_{\mathbb{G}} \frac{|f(x)|^{p-2}f(x)}{|x|^{p-1}}\overline{\frac{df(x)}{d|x|}}dx. \end{aligned} \tag{2.10}$$

Now by the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$ we obtain

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^p}dx &= -\frac{p}{Q-p}\mathrm{Re}\int_{\mathbb{G}} \frac{|f(x)|^{p-2}f(x)}{|x|^{p-1}}\overline{\frac{df(x)}{d|x|}}dx \\ &\le \frac{p}{Q-p}\left(\int_{\mathbb{G}}\left|\frac{|f(x)|^{p-2}f(x)}{|x|^{p-1}}\right|^q dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}}\left|\frac{df(x)}{d|x|}\right|^p dx\right)^{\frac{1}{p}} \\ &= \frac{p}{Q-p}\left(\int_{\mathbb{G}}\frac{|f(x)|^p}{|x|^p}dx\right)^{1-\frac{1}{p}}\left\|\frac{df(x)}{d|x|}\right\|_{L^p(\mathbb{G})}. \end{aligned}$$

This proves inequality (2.2) in Part (i).

Proof of Part (ii). Since

$$u := u(x) = -\frac{p}{Q-p}\mathcal{R}f, \quad \text{and} \quad v := v(x) = \frac{f(x)}{|x|},$$

the formula (2.10) can be restated as

$$\|v\|^p_{L^p(\mathbb{G})} = \mathrm{Re}\int_{\mathbb{G}} |v|^{p-2}v\overline{u}dx. \tag{2.11}$$

For a real-valued f the formula (2.10) becomes

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^p}dx = -\frac{p}{Q-p}\int_{\mathbb{G}} \frac{|f(x)|^{p-2}f(x)}{|x|^{p-1}}\frac{df(x)}{d|x|}dx$$

and (2.11) becomes

$$\|v\|^p_{L^p(\mathbb{G})} = \int_{\mathbb{G}} |v|^{p-2}vudx. \tag{2.12}$$

Moreover, for any L^p-integrable real-valued functions u and v, we have

$$\begin{aligned}&\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} + p\int_{\mathbb{G}}(|v|^p - |v|^{p-2}vu)dx \\ &\quad = \int_{\mathbb{G}}(|u|^p + (p-1)|v|^p - p|v|^{p-2}vu)dx = p\int_{\mathbb{G}} I_p(v,u)|v-u|^2dx, \end{aligned} \tag{2.13}$$

where

$$I_p(v,u) = (p-1)\int_0^1 |\xi v + (1-\xi)u|^{p-2}\xi d\xi.$$

To show the last equality in (2.13), we observe the identity, for real numbers $u \neq v$,

$$\begin{aligned}
&\frac{1}{p}|u|^p + \left(1-\frac{1}{p}\right)|v|^p - |v|^{p-2}vu \\
&\quad = \left(1-\frac{1}{p}\right)(|v|^p - |u|^p) - u(|v|^{p-2}v - |u|^{p-2}u) \\
&\quad = (p-1)\int_0^1 |\xi v + (1-\xi)u|^{p-2}(\xi v + (1-\xi)u)d\xi\,(v-u) \\
&\qquad - (p-1)\int_0^1 |\xi v + (1-\xi)u|^{p-2}d\xi\, u(v-u) \\
&\quad = (p-1)\int_0^1 |\xi v + (1-\xi)u|^{p-2}\xi d\xi\,(v-u)^2,
\end{aligned}$$

using the integral expression for the remainder in the Taylor expansion formula.

Combining (2.13) with (2.12) we arrive at

$$\|u\|^p_{L^p(\mathbb{G})} - \|v\|^p_{L^p(\mathbb{G})} = p\int_{\mathbb{G}} I_p(v,u)|v-u|^2 dx.$$

It completes the proof of Part (ii).

Proof of Part (iii). When $p=2$, the equality (2.11) for complex-valued functions reduces to

$$\|v\|^2_{L^2(\mathbb{G})} = \operatorname{Re}\int_{\mathbb{G}} v\overline{u}dx.$$

Then we have

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{G})} - \|v\|^2_{L^2(\mathbb{G})} &= \|u\|^2_{L^2(\mathbb{G})} - \|v\|^2_{L^2(\mathbb{G})} + 2\int_{\mathbb{G}}(|v|^2 - \operatorname{Re} v\overline{u})dx \\
&= \int_{\mathbb{G}}(|u|^2 + |v|^2 - 2\operatorname{Re} v\overline{u})dx = \int_{\mathbb{G}} |u-v|^2 dx,
\end{aligned}$$

that is, (2.5) is proved. □

As a direct consequence of the inequality (2.2) we obtain the corresponding uncertainty principle:

Corollary 2.1.3 (Uncertainty principle on homogeneous groups)**.** *For every complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left(\int_{\mathbb{G}} \left|\frac{df(x)}{d|x|}\right|^p dx\right)^{1/p} \left(\int_{\mathbb{G}} |x|^q |f|^q dx\right)^{1/q} \geq \frac{Q-p}{p}\int_{\mathbb{G}} |f|^2 dx. \tag{2.14}$$

Here $Q \geq 2$, $|\cdot|$ is an arbitrary homogeneous quasi-norm on $\mathbb{G}$, $1 < p < Q$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The inequality (2.7) and the Hölder inequality imply that

$$\left(\int_{\mathbb{G}}\left|\frac{df(x)}{d|x|}\right|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{G}}|x|^q|f|^q dx\right)^{\frac{1}{q}}$$
$$\geq \frac{Q-p}{p}\left(\int_{\mathbb{G}}\frac{|f|^p}{|x|^p}dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{G}}|x|^q|f|^q dx\right)^{\frac{1}{q}} \geq \frac{Q-p}{p}\int_{\mathbb{G}}|f|^2 dx,$$

This shows (2.14). □

Remark 2.1.4. In the Abelian case $\mathbb{G}=(\mathbb{R}^n,+)$ with the standard Euclidean distance $|x|_E$, we have $Q=n$, so that (2.14) with $p=q=2$ and $n\geq 3$ implies the uncertainty principle

$$\int_{\mathbb{R}^n}\left|\frac{x}{|x|_E}\cdot\nabla u(x)\right|^2 dx\int_{\mathbb{R}^n}|x|_E^2|u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2\left(\int_{\mathbb{R}^n}|u(x)|^2 dx\right)^2, \quad (2.15)$$

which in turn implies the classical uncertainty principle for $\mathbb{G}\equiv\mathbb{R}^n$:

$$\int_{\mathbb{R}^n}|\nabla u(x)|^2 dx\int_{\mathbb{R}^n}|x|_E^2|u(x)|^2 dx \geq \left(\frac{n-2}{2}\right)^2\left(\int_{\mathbb{R}^n}|u(x)|^2 dx\right)^2, \quad n\geq 3.$$

2.1.2 Weighted Hardy inequalities

In this section $\mathbb{G}$ is a homogeneous group of homogeneous dimension $Q\geq 3$. Let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Here, we are going to discuss weighted Hardy inequalities on $\mathbb{G}$ which are the consequences of exact equalities.

Theorem 2.1.5 (Weighted Hardy identity in $L^2(\mathbb{G})$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q\geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then for every complex-valued function $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$ and for any $\alpha\in\mathbb{R}$ we have the equality*

$$\left\|\frac{1}{|x|^\alpha}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}=\left(\frac{Q-2}{2}-\alpha\right)^2\left\|\frac{f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{1}{|x|^\alpha}\mathcal{R}f+\frac{Q-2-2\alpha}{2|x|^{\alpha+1}}f\right\|^2_{L^2(\mathbb{G})}. \quad (2.16)$$

The equality (2.16) implies many different inequalities. For instance, by taking $\alpha=1$ and simplifying its coefficient, for any $Q\geq 3$ we obtain the identity

$$\left\|\frac{1}{|x|}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}=\left(\frac{Q-4}{2}\right)^2\left\|\frac{f}{|x|^2}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{1}{|x|}\mathcal{R}f+\frac{Q-4}{2|x|^2}f\right\|^2_{L^2(\mathbb{G})}. \quad (2.17)$$

By dropping the last term in (2.16) which is non-negative we obtain:

Corollary 2.1.6 (Weighted Hardy inequality in $L^2(\mathbb{G})$)**.** *Let $Q \geq 3$ and let $\alpha \in \mathbb{R}$ be such that $Q - 2 - 2\alpha \neq 0$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\left\| \frac{f}{|x|^{\alpha+1}} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{|Q-2-2\alpha|} \left\| \frac{1}{|x|^\alpha} \mathcal{R} f \right\|_{L^2(\mathbb{G})}. \tag{2.18}$$

Here the constant $\frac{2}{|Q-2-2\alpha|}$ is sharp and it is attained if and only if $f = 0$.

Remark 2.1.7.

1. It is interesting to note that the constant $\frac{2}{|Q-2-2\alpha|}$ in (2.18) is sharp for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$.
2. When $\alpha = 1$, (2.17) or (2.18) also imply that

$$\left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-4} \left\| \frac{1}{|x|} \mathcal{R} f \right\|_{L^2(\mathbb{G})}, \quad Q \geq 5, \tag{2.19}$$

again with $\frac{2}{Q-4}$ being the sharp constant.
3. If $\alpha = 0$, the identity (2.16) recovers Part (iii) of Theorem 2.1.1. However, we will use Part (iii) of Theorem 2.1.1 in the proof of Theorem 2.1.5.
4. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 3$, we have $Q = n$, so for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{R}^n$ identity (2.16) implies the following inequality with the optimal constant:

$$\frac{|n-2-2\alpha|}{2} \left\| \frac{f}{|x|^{\alpha+1}} \right\|_{L^2(\mathbb{R}^n)} \leq \left\| \frac{1}{|x|^\alpha} \frac{x}{|x|} \cdot \nabla f \right\|_{L^2(\mathbb{R}^n)} \quad \text{for all } \alpha \in \mathbb{R}.$$

In the case of the Euclidean distance $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$, by the Cauchy–Schwarz inequality we obtain the following estimate:

$$\frac{|n-2-2\alpha|}{2} \left\| \frac{f}{|x|_E^{\alpha+1}} \right\|_{L^2(\mathbb{R}^n)} \leq \left\| \frac{1}{|x|_E^\alpha} \nabla f \right\|_{L^2(\mathbb{R}^n)} \tag{2.20}$$

for all $\alpha \in \mathbb{R}$ and for any $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. The sharpness of the constant $\frac{|n-2-2\alpha|}{2}$ in the Euclidean case of $\mathbb{R}^n$ with the Euclidean norm, (2.20) was shown in [CW01, Theorem 1.1. (ii)].
5. Hardy inequalities with homogeneous weights have been also considered by Hoffmann-Ostenhof and Laptev [HOL15]. There are also further many-particle versions of such inequalities, see [HOHOLT08] and many further references therein. We will discuss some of such inequalities in Section 6.11 and Section 6.12.
6. Theorem 2.1.5 was established in [RS17b]. Its extension from L^2 to L^p spaces presented in Theorem 2.1.8 was made in [Ngu17].

Proof of Theorem 2.1.5. We first observe the equality

$$\frac{1}{|x|^{\alpha}}\mathcal{R}f=\mathcal{R}\frac{f}{|x|^{\alpha}}+\alpha\frac{f}{|x|^{\alpha+1}} \tag{2.21}$$

for any $\alpha\in\mathbb{R}$, which follows from

$$\mathcal{R}\frac{f}{|x|^{\alpha}}=\frac{1}{|x|^{\alpha}}\mathcal{R}f+f\mathcal{R}\frac{1}{|x|^{\alpha}}$$

and hence, by using (1.30), we have

$$\mathcal{R}\frac{1}{|x|^{\alpha}}=\frac{d}{dr}\frac{1}{r^{\alpha}}=-\alpha\frac{1}{r^{\alpha+1}}=-\alpha\frac{1}{|x|^{\alpha+1}},\quad r=|x|.$$

Then using (2.21) we can write

$$\begin{aligned}\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}&=\left\|\mathcal{R}\frac{f}{|x|^{\alpha}}+\frac{\alpha f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})}\\&=\left\|\mathcal{R}\frac{f}{|x|^{\alpha}}\right\|^2_{L^2(\mathbb{G})}+2\alpha\mathrm{Re}\int_{\mathbb{G}}\mathcal{R}\left(\frac{f}{|x|^{\alpha}}\right)\frac{\overline{f}}{|x|^{\alpha+1}}dx+\left\|\frac{\alpha f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

By applying (2.5) to the function $\frac{f}{|x|^{\alpha}}$ and using (2.21) we have that

$$\left\|\mathcal{R}\frac{f}{|x|^{\alpha}}\right\|^2_{L^2(\mathbb{G})}=\left(\frac{Q-2}{2}\right)^2\left\|\frac{f}{|x|^{1+\alpha}}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f+\frac{Q-2-2\alpha}{2|x|^{\alpha+1}}f\right\|^2_{L^2(\mathbb{G})}.$$

In addition, a direct calculation using the polar decomposition in Proposition 1.2.10 shows that

$$\begin{aligned}2\alpha\mathrm{Re}\int_{\mathbb{G}}\mathcal{R}\left(\frac{f}{|x|^{\alpha}}\right)\frac{\overline{f}}{|x|^{\alpha+1}}dx&=2\alpha\mathrm{Re}\int_0^{\infty}r^{Q-2}\int_{\wp}\frac{d}{dr}\left(\frac{f(ry)}{r^{\alpha}}\right)\frac{\overline{f(ry)}}{r^{\alpha}}d\sigma(y)dr\\&=\alpha\int_0^{\infty}r^{Q-2}\int_{\wp}\frac{d}{dr}\left(\frac{|f(ry)|^2}{r^{2\alpha}}\right)d\sigma(y)dr=-\alpha(Q-2)\left\|\frac{f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

In conclusion, combining these identities we arrive at

$$\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f\right\|^2_{L^2(\mathbb{G})}=\left(\frac{Q-2}{2}-\alpha\right)^2\left\|\frac{f}{|x|^{\alpha+1}}\right\|^2_{L^2(\mathbb{G})}+\left\|\frac{1}{|x|^{\alpha}}\mathcal{R}f+\frac{Q-2-2\alpha}{2|x|^{\alpha+1}}f\right\|^2_{L^2(\mathbb{G})},$$

yielding (2.16). ∎

To present a weighted L^p-Hardy inequality on $\mathbb{G}$ we will use the following function R_p in analogy to I_p in (2.4). For $\xi,\eta\in\mathbb{C}$ we denote

$$R_p(\xi,\eta):=\frac{1}{p}|\eta|^p+\frac{p-1}{p}|\xi|^p-\mathrm{Re}(|\xi|^{p-2}\xi\overline{\eta}). \tag{2.22}$$

By the convexity of the function $z \mapsto |z|^p$ we see that $R_p(\xi,\eta) \geq 0$ is non-negative and $R_p(\xi,\eta) = 0$ and if and only if $\xi = \eta$. If $\xi, \eta \in \mathbb{R}$, we then have

$$R_p(\xi,\eta) = (p-1)\int_0^1 |t\xi + (1-t)\eta|^{p-2} t dt\, |\xi - \eta|^2,$$

analogous to I_p in (2.4).

Theorem 2.1.8 (Weighted Hardy identity in $L^p(\mathbb{G})$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $1 < p < Q$ and $\alpha \in \mathbb{R}$. Then for any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ and for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\begin{aligned}\left\|\frac{\mathcal{R}f}{|x|^\alpha}\right\|^p_{L^p(\mathbb{G})} = &\left|\frac{Q-p(1+\alpha)}{p}\right|^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(1+\alpha)}}dx \\ &+ p\int_{\mathbb{G}} \frac{1}{|x|^{p\alpha}} R_p\left(-\frac{Q-p(1+\alpha)}{p}\frac{f}{|x|}, \mathcal{R}f\right)dx.\end{aligned} \tag{2.23}$$

By dropping the last term in (2.23) which is non-negative we obtain:

Corollary 2.1.9 (Weighted Hardy inequality in $L^p(\mathbb{G})$)**.** *Let $1 < p < Q$ and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\frac{|Q-p(1+\alpha)|}{p}\left\|\frac{f}{|x|^{1+\alpha}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\mathcal{R}f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}. \tag{2.24}$$

If $Q - p(1+\alpha) \neq 0$, then the constant $\frac{|Q-p(1+\alpha)|}{p}$ in (2.24) is sharp and it is attained if and only if $f = 0$.

Proof of Theorem 2.1.8. We can assume that $Q - p(1+\alpha) \neq 0$, otherwise there is nothing to prove. A direct calculation gives

$$\begin{aligned}\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(1+\alpha)}}dx &= \int_0^\infty r^{Q-p(1+\alpha)-1}\int_{\wp} |f(ry)|^p d\sigma(y)dr \\ &= \frac{1}{Q-p(1+\alpha)}\int_0^\infty (r^{Q-p(1+\alpha)})'\int_{\wp} |f(ry)|^p d\sigma(y)dr \\ &= -\frac{1}{Q-p(1+\alpha)}\mathrm{Re}\int_0^\infty r^{Q-p(1+\alpha)}\int_{\wp} |f(ry)|^{p-2}f(ry)\overline{\mathcal{R}f(ry)}d\sigma(y)dr \\ &= -\frac{1}{Q-p(1+\alpha)}\mathrm{Re}\int_{\mathbb{G}} \frac{|f(x)|^{p-2}f(x)}{|x|^{(p-1)(1+\alpha)}}\frac{\overline{\mathcal{R}f(x)}}{|x|^\alpha}dx \\ &= \frac{p-1}{p}\int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(1+\alpha)}}dx + \frac{1}{p}\left(\frac{p}{|Q-p(1+\alpha)|}\right)^p\int_{\mathbb{G}} \frac{|\mathcal{R}f|^p}{|x|^{p\alpha}}dx \\ &\quad - \int_{\mathbb{G}} R_p\left(\frac{f}{|x|^{1+\alpha}}, -\frac{p}{Q-p(1+\alpha)}\frac{\mathcal{R}f}{|x|^\alpha}\right)dx,\end{aligned}$$

which implies the identity (2.23). $\square$

Proof of Corollary 2.1.9. The equality (2.23) implies inequality (2.24) since the last term in (2.23) is non-negative. Let us now show the sharpness of the constant. For this we approximate the function $r^{-(Q-p(1+\alpha))/p}$ by smooth compactly supported functions, for details of such an argument see also the proof of Theorem 3.1.4. Using (2.23) it follows that the equality in (2.24) holds if and only if

$$R_p\left(-\frac{Q-p(1+\alpha)}{p}\frac{f}{|x|}, \mathcal{R}f\right) = 0,$$

or, equivalently,

$$\mathcal{R}f = -\frac{Q-p(1+\alpha)}{p}\frac{f}{|x|}.$$

In turn, this is equivalent to

$$\mathbb{E}f = -\frac{Q-p(1+\alpha)}{p}f.$$

By Proposition 1.3.1 it follows that f is positively homogeneous of order $-(Q-p(1+\alpha))/p$. Since $|f|/|x|^{1+\alpha}$ is in $L^p(\mathbb{G})$, it follows that $f=0$. □

2.1.3 Hardy inequalities with super weights

In this section we discuss sharp L^p-Hardy type inequalities with *super weights*, i.e., with weights of the form

$$\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}. \tag{2.25}$$

Such weights are sometimes called the super weights because of the arbitrariness of the choice of any homogeneous quasi-norm as well as a wide range of parameters. However, all the inequalities can be obtained with best constants.

Theorem 2.1.10 (Hardy inequalities with super weights)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 1$. Let $a, b > 0$ and $1 < p < \infty$. Then we have the following inequalities:*

(i) *If $\alpha\beta > 0$ and $pm \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\frac{Q-pm-p}{p}\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}}f\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}f\right\|_{L^p(\mathbb{G})}. \tag{2.26}$$

If $Q \neq pm+p$ then the constant $\frac{Q-pm-p}{p}$ is sharp.

(ii) *If $\alpha\beta < 0$ and $pm - \alpha\beta \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\frac{Q-pm+\alpha\beta-p}{p}\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}}f\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}f\right\|_{L^p(\mathbb{G})}. \tag{2.27}$$

If $Q \neq pm+p-\alpha\beta$ then the constant $\frac{Q-pm+\alpha\beta-p}{p}$ is sharp.

Proof of Theorem 2.1.10. Proof of Part (i). We can assume $Q \neq pm + p$ since in the case $Q = pm + p$ there is nothing to prove. As usual with $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \wp$ on $\mathbb{G}$, where

$$\wp := \{x \in \mathbb{G} : |x| = 1\},$$

using the polar decomposition in Proposition 1.2.10 and integrating by parts, we obtain

$$\int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta}}{|x|^{pm+p}} |f(x)|^p dx = \int_0^{\infty} \int_{\wp} \frac{(a + br^{\alpha})^{\beta}}{r^{pm+p}} |f(ry)|^p r^{Q-1} d\sigma(y) dr. \tag{2.28}$$

Since $a, b > 0$, $\alpha\beta > 0$ and $m < \frac{Q-p}{p}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta}}{|x|^{pm+p}} |f(x)|^p dx \\
&\quad \leq \int_0^{\infty} \int_{\wp} (a + br^{\alpha})^{\beta} r^{Q-1-pm-p} \left(\frac{\alpha\beta b r^{\alpha}}{(a + br^{\alpha})(Q - pm - p)} + 1 \right) |f(ry)|^p d\sigma(y) dr \\
&\quad = \int_0^{\infty} \int_{\wp} \frac{d}{dr} \left(\frac{(a + br^{\alpha})^{\beta} r^{Q-pm-p}}{Q - pm - p} \right) |f(ry)|^p d\sigma(y) dr \\
&\quad = -\frac{p}{Q - pm - p} \int_0^{\infty} (a + br^{\alpha})^{\beta} r^{Q-pm-p} \operatorname{Re} \int_{\wp} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr \\
&\quad \leq \left| \frac{p}{Q - pm - p} \right| \int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta} |\mathcal{R}f(x)| |f(x)|^{p-1}}{|x|^{pm+p-1}} dx \\
&\quad = \frac{p}{Q - pm - p} \int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\frac{\beta(p-1)}{p}} |f(x)|^{p-1}}{|x|^{(m+1)(p-1)}} \frac{(a + b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^m} |\mathcal{R}f(x)| dx.
\end{aligned}$$

Now by using Hölder's inequality we arrive at the inequality

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta}}{|x|^{pm+p}} |f(x)|^p dx \\
&\quad \leq \frac{p}{Q - pm - p} \left(\int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta}}{|x|^{pm+p}} |f(x)|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{G}} \frac{(a + b|x|^{\alpha})^{\beta}}{|x|^{pm}} |\mathcal{R}f(x)|^p dx \right)^{\frac{1}{p}},
\end{aligned}$$

which gives (2.26).

We need to check the equality condition in the above Hölder inequality in order to show the sharpness of the constant. Setting

$$g(x) = |x|^C,$$

where $C \in \mathbb{R}, C \neq 0$ and $Q \neq pm + p$ a direct calculation shows

$$\left| \frac{1}{C} \right|^p \left(\frac{(a + b|x|^{\alpha})^{\frac{\beta}{p}} |\mathcal{R}g(x)|}{|x|^m} \right)^p = \left(\frac{(a + b|x|^{\alpha})^{\frac{\beta(p-1)}{p}} |g(x)|^{p-1}}{|x|^{(m+1)(p-1)}} \right)^{\frac{p}{p-1}},$$

which satisfies the equality condition of the Hölder inequality. This gives the sharpness of the constant $\frac{Q-pm-p}{p}$ in the inequality (2.26).

Proof of Part (ii). Here we can assume that $Q \neq pm + p - \alpha\beta$ since for $Q = pm + p - \alpha\beta$ there is nothing to prove. Using the polar decomposition in Proposition 1.2.10, as before we have the equality (2.28). Since $\alpha\beta < 0$ and $pm - \alpha\beta < Q - p$ we obtain

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(a+b|x|^{\alpha})^{\beta}}{|x|^{pm+p}}|f(x)|^p dx \\
&\quad \leq \int_0^{\infty}\int_{\wp}(a+br^{\alpha})^{\beta}r^{Q-1-pm-p} \\
&\qquad \times \left(\frac{br^{\alpha}}{a+br^{\alpha}} + \frac{a}{a+br^{\alpha}}\cdot\frac{Q-pm-p}{Q-pm-p+\alpha\beta}\right)|f(ry)|^p d\sigma(y)dr \\
&\quad = \int_0^{\infty}\int_{\wp}\frac{(a+br^{\alpha})^{\beta}r^{Q-1-pm-p}}{Q-pm-p+\alpha\beta}\left(\frac{\alpha\beta br^{\alpha}}{a+br^{\alpha}} + Q-pm-p\right)|f(ry)|^p d\sigma(y)dr \\
&\quad = \int_0^{\infty}\int_{\wp}\frac{d}{dr}\left(\frac{(a+br^{\alpha})^{\beta}r^{Q-pm-p}}{Q-pm-p+\alpha\beta}\right)|f(ry)|^p d\sigma(y)dr \\
&\quad = -\frac{p}{Q-pm-p+\alpha\beta} \\
&\qquad\quad \times \int_0^{\infty}(a+br^{\alpha})^{\beta}r^{Q-pm-p}\,\mathrm{Re}\int_{\wp}|f(ry)|^{p-2}f(ry)\overline{\frac{df(ry)}{dr}}d\sigma(y)dr \\
&\quad \leq \left|\frac{p}{Q-pm-p+\alpha\beta}\right|\int_{\mathbb{G}}\frac{(a+b|x|^{\alpha})^{\beta}|\mathcal{R}f(x)||f(x)|^{p-1}}{|x|^{pm+p-1}}dx \\
&\quad = \frac{p}{Q-pm-p+\alpha\beta}\int_{\mathbb{G}}\frac{(a+b|x|^{\alpha})^{\frac{\beta(p-1)}{p}}|f(x)|^{p-1}}{|x|^{(m+1)(p-1)}}\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^m}|\mathcal{R}f(x)|dx.
\end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{G}}\frac{(a+b|x|^{\alpha})^{\beta}}{|x|^{pm+p}}|f(x)|^p dx \leq \frac{p}{Q-pm-p+\alpha\beta}&\left(\int_{\mathbb{G}}\frac{(a+b|x|^{\alpha})^{\beta}}{|x|^{pm+p}}|f(x)|^p dx\right)^{(p-1)/p} \\
\times &\left(\int_{\mathbb{G}}\frac{(a+b|x|^{\alpha})^{\beta}}{|x|^{pm}}|\mathcal{R}f(x)|^p dx\right)^{1/p},
\end{aligned}$$

which gives (2.27).

To show the sharpness of the constant we will check the equality condition in the above Hölder inequality. Thus, by taking

$$h(x) = |x|^C,$$

where $C \in \mathbb{R}, C \neq 0$ and $Q \neq pm + p - \alpha\beta$, we get

$$\left|\frac{1}{C}\right|^p\left(\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}|\mathcal{R}h(x)|}{|x|^m}\right)^p = \left(\frac{(a+b|x|^{\alpha})^{\frac{\beta(p-1)}{p}}|h(x)|^{p-1}}{|x|^{(m+1)(p-1)}}\right)^{p/(p-1)},$$

which satisfies the equality condition in Hölder's inequality. This gives the sharpness of the constant $\frac{Q-pm-p+\alpha\beta}{p}$ in (2.27). □

2.1.4 Hardy inequalities of higher order with super weights

The iteration process gives the following higher-order L^p-Hardy type inequalities with super weights. Here as before, $\mathbb{G}$ is a homogeneous group of homogeneous dimension $Q \geq 1$ and $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$.

Theorem 2.1.11 (Higher-order Hardy inequalities with super weights)**.** *Let $a, b > 0$ and $1 < p < \infty$, $Q \geq 1$, $k \in \mathbb{N}$. Then we have the following inequalities.*

(i) *If $\alpha\beta > 0$ and $pm \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\begin{aligned}\left[\prod_{j=0}^{k-1}\left(\frac{Q-p}{p}-(m+j)\right)\right]&\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+k}}f\right\|_{L^p(\mathbb{G})}\\ &\leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}^k f\right\|_{L^p(\mathbb{G})}.\end{aligned} \tag{2.29}$$

(ii) *If $\alpha\beta < 0$ and $pm - \alpha\beta \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\begin{aligned}\left[\prod_{j=0}^{k-1}\left(\frac{Q-p+\alpha\beta}{p}-(m+j)\right)\right]&\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+k}}f\right\|_{L^p(\mathbb{G})}\\ &\leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}^k f\right\|_{L^p(\mathbb{G})}.\end{aligned} \tag{2.30}$$

Remark 2.1.12.

1. In the case of $k = 1$, (2.29) gives inequality (2.26), and (2.30) gives inequality (2.27).
2. In the Euclidean case $\mathbb{G} = \mathbb{R}^n$ and $|\cdot| = |\cdot|_E$ the Euclidean norm, the super weights in the form (2.25) have appeared in [GM08], together with some applications to problems for differential equations. The case of homogeneous groups, as well as the iterative higher order estimates as in Theorem 2.1.11 were analysed in [RSY17b, RSY18b].

Proof of Theorem 2.1.11*.* We can iterate (2.26). That is, we start with

$$\frac{Q-pm-p}{p}\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}}f\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}f\right\|_{L^p(\mathbb{G})}. \tag{2.31}$$

In (2.31) replacing f by $\mathcal{R}f$ we obtain

$$\frac{Q-pm-p}{p}\left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}\mathcal{R}^2 f\right\|_{L^p(\mathbb{G})}. \tag{2.32}$$

On the other hand, replacing m by $m+1$, (2.31) gives

$$\frac{Q-p(m+1)-p}{p}\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m+2}}f\right\|_{L^p(\mathbb{G})}\leq\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m+1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})}.$$

Combining this with (2.32) we obtain

$$\left(\frac{Q-pm-p}{p}\right)\left(\frac{Q-p(m+1)-p}{p}\right)\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m+2}}f\right\|_{L^p(\mathbb{G})}$$
$$\leq\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m}}\mathcal{R}^2f\right\|_{L^p(\mathbb{G})}.$$

This iteration process gives

$$\prod_{j=0}^{k-1}\left(\frac{Q-p}{p}-(m+j)\right)\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m+k}}f\right\|_{L^p(\mathbb{G})}\leq\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m}}\mathcal{R}^kf\right\|_{L^p(\mathbb{G})}.$$

Similarly, we have for $\alpha\beta<0$, $pm-\alpha\beta\leq Q-2$ and $f\in C_0^{\infty}(\mathbb{G}\backslash\{0\})$ that

$$\prod_{j=0}^{k-1}\left(\frac{Q-p+\alpha\beta}{p}-(m+j)\right)\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m+k}}f\right\|_{L^p(\mathbb{G})}$$
$$\leq\left\|\frac{(a+b|x|^{\alpha})^{\frac{\beta}{p}}}{|x|^{m}}\mathcal{R}^kf\right\|_{L^p(\mathbb{G})},$$

completing the proof. □

2.1.5 Two-weight Hardy inequalities

In this section, using the method of factorization of differential expressions, we obtain Hardy type inequalities with two general weights $\phi(x)$ and $\psi(x)$. The idea of the factorization method can be best illustrated by the following example of an estimate due to Gesztesy and Littlejohn [GL17].

Example 2.1.13 (Gesztesy and Littlejohn two-parameter inequality)**.** Let $\alpha,\beta\in\mathbb{R}$, $x\in\mathbb{R}^n\backslash\{0\}$ and $n\geq 2$. Let us define the operator

$$T_{\alpha,\beta}:=-\Delta+\alpha|x|_E^{-2}x\cdot\nabla+\beta|x|_E^{-2}.$$

One readily checks that its formal adjoint is given by

$$T^+_{\alpha,\beta}:=-\Delta-\alpha|x|_E^{-2}x\cdot\nabla+(\beta-\alpha(n-2))|x|_E^{-2}.$$

Using the non-negativity of the operator $T^+_{\alpha,\beta}T_{\alpha,\beta}$ on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$, for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 dx \geq &((n-4)\alpha - 2\beta) \int_{\mathbb{R}^n} |x|_E^{-2} |(\nabla f)(x)|^2 dx \\ &- \alpha(\alpha-4) \int_{\mathbb{R}^n} |x|_E^{-4} |x \cdot (\nabla f)(x)|^2 dx \\ &+ \beta((n-4)(\alpha-2) - \beta) \int_{\mathbb{R}^n} |x|_E^{-4} |f(x)|^2 dx.\end{aligned} \tag{2.33}$$

By choosing particular values of α and β it can be checked that this inequality yields classical Rellich and Hardy–Rellich type inequalities as special cases, see [GL17] for details.

On the other hand, using the non-negativity of the operator $T_{\alpha,\beta}T^+_{\alpha,\beta}$, it was shown in [RY17] that for $\alpha, \beta \in \mathbb{R}$ and $n \geq 2$, and for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ we can deduce another two-parameter inequality

$$\begin{aligned}&\int_{\mathbb{R}^n} |(\Delta f)(x)|^2 dx \\ &\geq (n\alpha - 2\beta) \int_{\mathbb{R}^n} |x|_E^{-2} |(\nabla f)(x)|^2 dx - \alpha(\alpha+4) \int_{\mathbb{R}^n} |x|_E^{-4} |x \cdot (\nabla f)(x)|^2 dx \\ &\quad + (2(n-4)(\alpha(n-2) - \beta) - 2\alpha^2(n-2) + \alpha\beta n - \beta^2) \int_{\mathbb{R}^n} |x|_E^{-4} |f(x)|^2 dx.\end{aligned} \tag{2.34}$$

The following result is a two-weight inequality on general homogeneous groups with general weights.

Theorem 2.1.14 (Two-weight Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $\phi, \psi \in L^2_{\mathrm{loc}}(\mathbb{G}\backslash\{0\})$ be any real-valued functions such that $\mathcal{R}\phi, \mathcal{R}\psi \in L^2_{\mathrm{loc}}(\mathbb{G}\backslash\{0\})$. Let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have inequalities*

$$\begin{aligned}&\int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}f(x)|^2 dx \\ &\quad\geq \alpha \int_{\mathbb{G}} (\phi(x)\mathcal{R}\psi(x) + \psi(x)\mathcal{R}\phi(x)) |f(x)|^2 dx \\ &\qquad + \alpha(Q-1) \int_{\mathbb{G}} \frac{\phi(x)\psi(x)}{|x|} |f(x)|^2 dx - \alpha^2 \int_{\mathbb{G}} (\psi(x))^2 |f(x)|^2 dx\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}&\int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}f(x)|^2 dx \\ &\quad\geq \alpha \int_{\mathbb{G}} (\psi(x)\mathcal{R}\phi(x) - \phi(x)\mathcal{R}\psi(x)) |f(x)|^2 dx\end{aligned}$$

$$+\alpha(Q-1)\int_{\mathbb{G}}\frac{\phi(x)\psi(x)}{|x|}|f(x)|^2dx-\alpha^2\int_{\mathbb{G}}(\psi(x))^2|f(x)|^2dx$$
$$-(Q-1)\int_{\mathbb{G}}\frac{(\phi(x))^2}{|x|^2}|f(x)|^2dx+(Q-1)\int_{\mathbb{G}}\frac{\phi(x)\mathcal{R}\phi(x)}{|x|}|f(x)|^2dx$$
$$+\int_{\mathbb{G}}\phi(x)\mathcal{R}^2\phi(x)|f(x)|^2dx. \tag{2.36}$$

From Theorem 2.1.14 one can get different weighted Hardy inequalities. Let us point out several examples.

Remark 2.1.15.

1. If we take $\phi(x)\equiv 1$ in Theorem 2.1.14, we obtain for all $\alpha\in\mathbb{R}$ and for all $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$ families of inequalities

$$\int_{\mathbb{G}}|\mathcal{R}f(x)|^2dx\geq\int_{\mathbb{G}}\left(\alpha\mathcal{R}\phi(x)+\alpha(Q-1)\frac{\psi(x)}{|x|}-\alpha^2(\psi(x))^2\right)|f(x)|^2dx$$

and

$$\int_{\mathbb{G}}|\mathcal{R}f(x)|^2dx$$
$$\geq\int_{\mathbb{G}}\left(-\alpha\mathcal{R}\phi(x)+\alpha(Q-1)\frac{\psi(x)}{|x|}-\alpha^2(\psi(x))^2-\frac{Q-1}{|x|^2}\right)|f(x)|^2dx.$$

2. If we take $\phi(x)=|x|^{-a}$ and $\psi(x)=|x|^{-b}$ for $a,b\in\mathbb{R}$, then (2.35) implies that

$$\int_{\mathbb{G}}\frac{|\mathcal{R}f(x)|^2}{|x|^{2a}}dx\geq\alpha(Q-a-b-1)\int_{\mathbb{G}}\frac{|f(x)|^2}{|x|^{a+b+1}}dx-\alpha^2\int_{\mathbb{G}}\frac{|f(x)|^2}{|x|^{2b}}dx.$$

In the case when we take $b=a+1$, we get

$$\int_{\mathbb{G}}\frac{|\mathcal{R}f(x)|^2}{|x|^{2a}}dx\geq(\alpha(Q-2a-2)-\alpha^2)\int_{\mathbb{G}}\frac{|f(x)|^2}{|x|^{2a+2}}dx.$$

Then, by maximizing the constant $(\alpha(Q-2a-2)-\alpha^2)$ with respect to α we obtain the weighted Hardy inequality from Corollary 2.1.6, namely,

$$\int_{\mathbb{G}}\frac{|\mathcal{R}f(x)|^2}{|x|^{2a}}dx\geq\frac{(Q-2a-2)^2}{4}\int_{\mathbb{G}}\frac{|f(x)|^2}{|x|^{2a+2}}dx, \tag{2.37}$$

for which it is known that the constant in (2.37) is sharp.

3. If we take $\phi(x) = |x|^{-a}(\log|x|)^c$ and $\psi(x) = |x|^{-b}(\log|x|)^d$ for $a, b, c, d \in \mathbb{R}$, then we obtain from (2.35) the inequality

$$\begin{aligned}
&\int_{\mathbb{G}} \frac{(\log|x|)^{2c}}{|x|^{2a}} |\mathcal{R}f(x)|^2 dx \\
&\quad \geq \alpha \int_{\mathbb{G}} \left(\frac{(c+d)(\log|x|)^{c+d-1} + (Q-1-a-b)(\log|x|)^{c+d}}{|x|^{a+b+1}} \right) |f(x)|^2 dx \\
&\quad\quad - \alpha^2 \int_{\mathbb{G}} \frac{(\log|x|)^{2d}}{|x|^{2b}} |f(x)|^2 dx.
\end{aligned}$$

   If we take $a = \frac{Q-2}{2}$, $b = \frac{Q}{2}$, $c = 1$ and $d = 0$, it follows that

$$\int_{\mathbb{G}} \frac{(\log|x|)^2}{|x|^{Q-2}} |\mathcal{R}f(x)|^2 dx \geq (\alpha - \alpha^2) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^Q} dx.$$

   After maximizing the above constant with respect to α we obtain the critical Hardy inequality

$$\int_{\mathbb{G}} \frac{(\log|x|)^2}{|x|^{Q-2}} |\mathcal{R}f(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^Q} dx, \tag{2.38}$$

   recovering the critical inequality in Theorem 2.2.4. There, it is shown that the constant $\frac{1}{4}$ in (2.38) is sharp.

4. We can refer to [GL17] for a thorough discussion of the factorization method, its history and different features. We also refer to [GP80] for obtaining the Hardy inequality and to [Ges84] for logarithmic refinements by this factorization method.

   The inequalities in Theorem 2.1.14 were obtained in [RY17] which we follow for the proof.

Proof of Theorem 2.1.14. Let us introduce the one-parameter differential expression

$$T_\alpha := \phi(x)\mathcal{R} + \alpha\psi(x).$$

One can readily calculate for the formal adjoint operator of T_α on $C_0^\infty(\mathbb{G}\backslash\{0\})$:

$$\begin{aligned}
&\int_{\mathbb{G}} \phi(x)\mathcal{R}f(x)\overline{g(x)}dx + \alpha \int_{\mathbb{G}} \psi(x)f(x)\overline{g(x)}dx \\
&\quad = \int_0^\infty \int_{\wp} \phi(ry)\frac{d}{dr}(f(ry))\overline{g(ry)}r^{Q-1}d\sigma(y)dr + \alpha \int_{\mathbb{G}} \psi(x)f(x)\overline{g(x)}dx \\
&\quad = -\int_0^\infty \int_{\wp} \phi(ry)f(ry)\overline{\frac{d}{dr}(g(ry))}r^{Q-1}d\sigma(y)dr \\
&\quad\quad - (Q-1)\int_0^\infty \int_{\wp} \phi(ry)f(ry)\overline{g(ry)}r^{Q-2}d\sigma(y)dr + \alpha \int_{\mathbb{G}} \psi(x)f(x)\overline{g(x)}dx
\end{aligned}$$

$$= \int_{\mathbb{G}} f(x)\left(-\phi(x)\overline{\mathcal{R}g(x)}\right)dx - (Q-1)\int_{\mathbb{G}} f(x)\left(\frac{\phi(x)}{|x|}\overline{g(x)}\right)dx$$
$$- \int_{\mathbb{G}} f(x)\left(\mathcal{R}\phi(x)\overline{g(x)}\right)dx + \alpha\int_{\mathbb{G}} f(x)\left(\psi(x)\overline{g(x)}\right)dx.$$

Thus, the formal adjoint operator of T_α has the form

$$T_\alpha^+ := -\phi(x)\mathcal{R} - \frac{Q-1}{|x|}\phi(x) - \mathcal{R}\phi(x) + \alpha\psi(x),$$

where $x \neq 0$. Then we have

$$\begin{aligned}(T_\alpha^+ T_\alpha f)(x) = & -\phi(x)\mathcal{R}(\phi(x)\mathcal{R}f(x)) - \alpha\phi(x)\mathcal{R}(f(x)\psi(x)) - \frac{Q-1}{|x|}(\phi(x))^2\mathcal{R}f(x)\\ & - \frac{\alpha(Q-1)}{|x|}\phi(x)\psi(x)f(x) - \phi(x)\mathcal{R}\phi(x)\mathcal{R}f(x) - \alpha\psi(x)f(x)\mathcal{R}\phi(x)\\ & + \alpha\phi(x)\psi(x)\mathcal{R}f(x) + \alpha^2(\psi(x))^2 f(x).\end{aligned}$$

By the non-negativity of $T_\alpha^+ T_\alpha$, introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times \wp$ on $\mathbb{G}$, where $\wp$ is the quasi-sphere as in (1.12), and using Proposition 1.2.10 one calculates

$$\begin{aligned}0 &\le \int_{\mathbb{G}} |(T_\alpha f)(x)|^2 dx = \int_{\mathbb{G}} f(x)\overline{(T_\alpha^+ T_\alpha f)(x)}dx\\ &= \operatorname{Re}\int_{\mathbb{G}} f(x)\overline{(T_\alpha^+ T_\alpha f)(x)}dx =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6,\end{aligned} \tag{2.39}$$

where we set

$$I_1 = -\operatorname{Re}\int_0^\infty \int_\wp f(ry)\phi(ry)\overline{\frac{d}{dr}\left(\phi(ry)\frac{d}{dr}(f(ry))\right)}r^{Q-1}d\sigma(y)dr, \tag{2.40}$$

$$I_2 = -\alpha\operatorname{Re}\int_0^\infty \int_\wp f(ry)\phi(ry)\overline{\frac{d}{dr}(f(ry)\psi(ry))}r^{Q-1}d\sigma(y)dr,$$

$$I_3 = -(Q-1)\operatorname{Re}\int_0^\infty \int_\wp f(ry)\frac{(\phi(ry))^2\overline{\frac{d}{dr}(f(ry))}}{r}r^{Q-1}d\sigma(y)dr,$$

$$\begin{aligned}I_4 = & -\alpha(Q-1)\int_{\mathbb{G}} \frac{\phi(x)\psi(x)}{|x|}|f(x)|^2dx - \alpha\int_{\mathbb{G}} \mathcal{R}\phi(x)\psi(x)|f(x)|^2dx\\ & + \alpha^2\int_{\mathbb{G}} (\psi(x))^2|f(x)|^2dx,\end{aligned}$$

$$I_5 = -\operatorname{Re}\int_0^\infty \int_\wp f(ry)\frac{d}{dr}(\phi(ry))\phi(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr$$

and

$$I_6 = \alpha\operatorname{Re}\int_0^\infty \int_\wp f(ry)\phi(ry)\psi(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr.$$

Now we simplify the sum of the terms I_1, I_2, I_3, I_5 and I_6. By a direct calculation we obtain

$$\begin{aligned}
I_1 = &(Q-1)\mathrm{Re}\int_0^\infty\int_\wp (\phi(ry))^2 f(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-2}d\sigma(y)dr \\
&+\mathrm{Re}\int_0^\infty\int_\wp (\phi(ry))^2\frac{d}{dr}(f(ry))\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr \\
&+\mathrm{Re}\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\phi(ry))\overline{\frac{d}{dr}f(ry)}f(ry)r^{Q-1}d\sigma(y)dr \\
=&\frac{Q-1}{2}\int_0^\infty\int_\wp (\phi(ry))^2\frac{d}{dr}|f(ry)|^2 r^{Q-2}d\sigma(y)dr \\
&+\int_0^\infty\int_\wp (\phi(ry))^2\left|\frac{d}{dr}(f(ry))\right|^2 r^{Q-1}d\sigma(y)dr \\
&+\frac{1}{2}\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\phi(ry))\frac{d}{dr}|f(ry)|^2 r^{Q-1}d\sigma(y)dr \\
=&\int_0^\infty\int_\wp (\phi(ry))^2\left|\frac{d}{dr}(f(ry))\right|^2 r^{Q-1}d\sigma(y)dr \\
&-\frac{(Q-1)(Q-2)}{2}\int_0^\infty\int_\wp (\phi(ry))^2|f(ry)|^2 r^{Q-3}d\sigma(y)dr \\
&-(Q-1)\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\phi(ry))|f(ry)|^2 r^{Q-2}d\sigma(y)dr \\
&-\frac{1}{2}\int_0^\infty\int_\wp \left(\frac{d}{dr}(\phi(ry))\right)^2|f(ry)|^2 r^{Q-1}d\sigma(y)dr \\
&-\frac{1}{2}\int_0^\infty\int_\wp \phi(ry)\frac{d^2}{dr^2}(\phi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr \\
&-\frac{Q-1}{2}\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\phi(ry))|f(ry)|^2 r^{Q-2}d\sigma(y)dr \\
=&\int_{\mathbb{G}}(\phi(x))^2|\mathcal{R}f(x)|^2dx-\frac{(Q-1)(Q-2)}{2}\int_{\mathbb{G}}\frac{(\phi(x))^2}{|x|^2}|f(x)|^2dx \\
&-(Q-1)\int_{\mathbb{G}}\frac{\phi(x)\mathcal{R}\phi(x)}{|x|}|f(x)|^2dx-\frac{1}{2}\int_{\mathbb{G}}(\mathcal{R}\phi(x))^2|f(x)|^2dx \\
&-\frac{1}{2}\int_{\mathbb{G}}\mathcal{R}^2\phi(x)\phi(x)|f(x)|^2dx-\frac{Q-1}{2}\int_{\mathbb{G}}\frac{\phi(x)\mathcal{R}\phi(x)}{|x|}|f(x)|^2dx.
\end{aligned}$$

Now we calculate I_2 as follows,

$$I_2 = -\alpha\mathrm{Re}\int_0^\infty\int_\wp \phi(ry)\psi(ry)f(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr$$

$$
\begin{aligned}
&\quad -\alpha\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\psi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&= -\frac{\alpha}{2}\int_0^\infty\int_\wp \phi(ry)\psi(ry)\frac{d}{dr}|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&\quad -\alpha\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\psi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&= -\alpha\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\psi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&\quad +\frac{\alpha}{2}\int_0^\infty\int_\wp \psi(ry)\frac{d}{dr}(\phi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&\quad +\frac{\alpha}{2}\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\psi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&\quad +\frac{\alpha(Q-1)}{2}\int_0^\infty\int_\wp \phi(ry)\psi(ry)|f(ry)|^2 r^{Q-2}d\sigma(y)dr\\
&= \frac{\alpha}{2}\int_{\mathbb{G}} \psi(x)\mathcal{R}\phi(x)|f(x)|^2dx+\frac{\alpha}{2}\int_{\mathbb{G}}\phi(x)\mathcal{R}\psi(x)|f(x)|^2dx\\
&\quad +\frac{\alpha(Q-1)}{2}\int_{\mathbb{G}}\frac{\phi(x)\psi(x)}{|x|}|f(x)|^2dx-\alpha\int_{\mathbb{G}}\phi(x)\mathcal{R}\psi(x)|f(x)|^2dx.
\end{aligned}
$$

For I_3, one has

$$
\begin{aligned}
I_3 &= -(Q-1)\mathrm{Re}\int_0^\infty\int_\wp (\phi(ry))^2 f(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-2}d\sigma(y)dr\\
&= -\frac{Q-1}{2}\int_0^\infty\int_\wp (\phi(ry))^2\frac{d}{dr}|f(ry)|^2 r^{Q-2}d\sigma(y)dr\\
&= (Q-1)\int_0^\infty\int_\wp \phi(ry)\frac{d}{dr}(\phi(ry))|f(ry)|^2 r^{Q-2}d\sigma(y)dr\\
&\quad +\frac{(Q-1)(Q-2)}{2}\int_0^\infty\int_\wp (\phi(ry))^2|f(ry)|^2 r^{Q-3}d\sigma(y)dr\\
&= (Q-1)\int_{\mathbb{G}}\frac{\phi(x)\mathcal{R}\phi(x)}{|x|}|f(x)|^2dx+\frac{(Q-1)(Q-2)}{2}\int_{\mathbb{G}}\frac{(\phi(x))^2}{|x|^2}|f(x)|^2dx.
\end{aligned}
$$

For I_5, we have

$$
\begin{aligned}
I_5 &= -\mathrm{Re}\int_0^\infty\int_\wp f(ry)\frac{d}{dr}(\phi(ry))\phi(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr\\
&= -\frac{1}{2}\int_0^\infty\int_\wp \frac{d}{dr}(\phi(ry))\phi(ry)\frac{d}{dr}|f(ry)|^2 r^{Q-1}d\sigma(y)dr\\
&= \frac{1}{2}\int_0^\infty\int_\wp \phi(ry)\frac{d^2}{dr^2}(\phi(ry))|f(ry)|^2 r^{Q-1}d\sigma(y)dr
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^\infty\int_\wp\left(\frac{d}{dr}(\phi(ry))\right)^2|f(ry)|^2r^{Q-1}d\sigma(y)dr\\
&+\frac{Q-1}{2}\int_0^\infty\int_\wp\frac{\phi(ry)\frac{d}{dr}(\phi(ry))}{r}|f(ry)|^2r^{Q-1}d\sigma(y)dr\\
=&\frac{1}{2}\int_{\mathbb{G}}\mathcal{R}^2\phi(x)\phi(x)|f(x)|^2dx+\frac{1}{2}\int_{\mathbb{G}}(\mathcal{R}\phi(x))^2|f(x)|^2dx\\
&+\frac{Q-1}{2}\int_{\mathbb{G}}\frac{\mathcal{R}\phi(x)\phi(x)}{|x|}|f(x)|^2dx.
\end{aligned}
$$

Finally, for I_6 we obtain

$$
\begin{aligned}
I_6 =& \alpha\mathrm{Re}\int_0^\infty\int_\wp f(ry)\phi(ry)\psi(ry)\overline{\frac{d}{dr}(f(ry))}r^{Q-1}d\sigma(y)dr\\
=&\frac{\alpha}{2}\int_0^\infty\int_\wp\phi(ry)\psi(ry)\frac{d}{dr}|f(ry)|^2r^{Q-1}d\sigma(y)dr\\
=&-\frac{\alpha}{2}\int_0^\infty\int_\wp\frac{d}{dr}(\phi(ry))\psi(ry)|f(ry)|^2r^{Q-1}d\sigma(y)dr\\
&-\frac{\alpha}{2}\int_0^\infty\int_\wp\frac{d}{dr}(\psi(ry))\phi(ry)|f(ry)|^2r^{Q-1}d\sigma(y)dr\\
&-\frac{\alpha(Q-1)}{2}\int_0^\infty\int_\wp\phi(ry)\psi(ry)\frac{|f(ry)|^2}{r}r^{Q-1}d\sigma(y)dr\\
=&-\frac{\alpha}{2}\int_{\mathbb{G}}\mathcal{R}\phi(x)\psi(x)|f(x)|^2dx-\frac{\alpha}{2}\int_{\mathbb{G}}\phi(x)\mathcal{R}\psi(x)|f(x)|^2dx\\
&-\frac{\alpha(Q-1)}{2}\int_{\mathbb{G}}\frac{\phi(x)\psi(x)}{|x|}|f(x)|^2dx.
\end{aligned}
\tag{2.41}
$$

Putting (2.40)–(2.41) in (2.39), we obtain that

$$
\begin{aligned}
&\int_{\mathbb{G}}(\phi(x))^2|\mathcal{R}f(x)|^2dx\\
&-\alpha\int_{\mathbb{G}}\left(\phi(x)\mathcal{R}\psi(x)+\psi(x)\mathcal{R}\phi(x)+(Q-1)\frac{\phi(x)\psi(x)}{|x|}\right)|f(x)|^2dx\\
&+\alpha^2\int_{\mathbb{G}}(\psi(x))^2|f(x)|^2dx\geq 0,
\end{aligned}
$$

which implies (2.35).

Thus, we have obtained (2.35) using the non-negativity of $T_\alpha^+T_\alpha$. Now we can obtain (2.36) using the non-negativity of $T_\alpha T_\alpha^+$. Similar to the above we calculate

$$
(T_\alpha T_\alpha^+f)(x) = -\phi(x)\mathcal{R}(\phi(x)\mathcal{R}f(x))+\alpha\phi(x)\mathcal{R}(f(x)\psi(x))-\frac{Q-1}{|x|}(\phi(x))^2\mathcal{R}f(x)
$$

$$- \frac{\alpha(Q-1)}{|x|}\phi(x)\psi(x)f(x) - \phi(x)\mathcal{R}\phi(x)\mathcal{R}f(x) - \alpha\psi(x)f(x)\mathcal{R}\phi(x)$$
$$- \alpha\phi(x)\psi(x)\mathcal{R}f(x) + \alpha^2(\psi(x))^2 f(x)$$
$$- (Q-1)\phi(x)f(x)\mathcal{R}\left(\frac{\phi(x)}{|x|}\right) - \phi(x)f(x)\mathcal{R}^2\phi(x).$$

Using the non-negativity of $T_\alpha T_\alpha^+$ we get

$$0 \le \int_{\mathbb{G}} |(T_\alpha f)(x)|^2 dx = \int_{\mathbb{G}} f(x)\overline{(T_\alpha T_\alpha^+ f)(x)}dx$$
$$= \mathrm{Re}\int_{\mathbb{G}} f(x)\overline{(T_\alpha T_\alpha^+ f)(x)}dx = \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \widetilde{I}_4 + \widetilde{I}_5 + \widetilde{I}_6,$$

where

$$\widetilde{I}_1 = I_1,\ \widetilde{I}_2 = -I_2,\ \widetilde{I}_3 = I_3,\ \widetilde{I}_4 = I_4 + A,\ \widetilde{I}_5 = I_5,\ \widetilde{I}_6 = -I_6$$

with

$$A = -(Q-1)\int_{\mathbb{G}} \phi(x)|f(x)|^2\mathcal{R}\left(\frac{\phi(x)}{|x|}\right)dx - \int_{\mathbb{G}} \phi(x)\mathcal{R}^2\phi(x)|f(x)|^2 dx.$$

Taking into account these and (2.40)–(2.41), we obtain (2.36). □

To finish this section, we observe another version of weighted Hardy inequality with the radial derivative. Anticipating the material presented in later chapters, the proof will be based on the integral Hardy inequality from Theorem 5.1.1.

Theorem 2.1.16 (Weighted Hardy inequality for radial functions)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $\phi > 0$, $\psi > 0$ be positive weight functions on $\mathbb{G}$ and let $1 < p \le q < \infty$. Then there exists a positive constant $C > 0$ such that*

$$\left(\int_{\mathbb{G}} \phi(x)|f(x)|^q dx\right)^{1/q} \le C\left(\int_{\mathbb{G}} \psi(x)|\mathcal{R}f(x)|^p dx\right)^{1/p} \tag{2.42}$$

holds for all radial functions f with $f(0) = 0$ if and only if

$$\sup_{R>0}\left(\int_{|x|\ge R} \phi(x)dx\right)^{1/q}\left(\int_0^R\left(\int_{\wp} r^{Q-1}\psi(ry)d\sigma(y)\right)^{1-p'} dr\right)^{1/p'} < \infty. \tag{2.43}$$

Remark 2.1.17. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (2.42) was obtained in [DHK97] and in [Saw84]. On homogeneous groups it was observed in [RY18a] and we follow the proof there.

Proof of Theorem 2.1.16*.* For $r = |x|$, let us denote $\tilde{f}(r) = f(x)$. We also denote

$$\Phi(r) := \int_{\wp} r^{Q-1}\phi(ry)d\sigma(y), \quad \Psi(r) := \int_{\wp} r^{Q-1}\psi(ry)d\sigma(y).$$

Consequently, using $\tilde{f}(0)=0$, we have

$$\begin{aligned}\left(\int_{\mathbb{G}} \phi(x)|f(x)|^q dx\right)^{1/q} &= \left(\int_{\wp}\int_0^\infty r^{Q-1}\phi(ry)|\tilde{f}(r)|^q dr d\sigma(y)\right)^{1/q} \\ &= \left(\int_0^\infty \Phi(r)|\tilde{f}(r)|^q dr\right)^{1/q} \\ &= \left(\int_0^\infty \Phi(r)\left|\int_0^r \mathcal{R}\tilde{f}(r)dr\right|^q dr\right)^{1/q} \\ &\le C\left(\int_0^\infty \Psi(r)\left|\mathcal{R}\tilde{f}(r)\right|^p dr\right)^{1/p} \\ &= C\left(\int_{\mathbb{G}} \psi(x)|\mathcal{R}f(x)|^p dx\right)^{1/p}\end{aligned}$$

if and only if the condition (2.43) holds by Theorem 5.1.1, namely by (5.2) and (5.3). □

2.2 Critical Hardy inequalities

In this section we discuss critical Hardy inequalities. The critical behaviour may be manifested with respect to different parameters. For example, one major critical case arises when we have $p=Q$ in (2.2). In this case, in the Euclidean case of $\mathbb{R}^n$ with $p=Q=n$ it is known that the Hardy inequality (2.1) fails for any constant, see, e.g., [ET99] and [IIO16a], and references therein. In such critical cases it is natural to expect the appearance of the logarithmic terms.

One version of such a critical case is the inequality

$$\sup_{R>0}\left\|\frac{f-f_R}{|x|\log\frac{R}{|x|}}\right\|_{L^Q(\mathbb{G})} \le \frac{Q}{Q-1}\left\|\mathcal{R}f\right\|_{L^Q(\mathbb{G})}, \quad Q\ge 2, \tag{2.44}$$

for all $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$, where we denote

$$f_R(x) := f\left(R\frac{x}{|x|}\right) \quad \text{for } \ x\in\mathbb{G} \ \text{ and } \ R>0.$$

In fact, this inequality is a special case (with $p=Q$) of the following more general family of critical inequalities derived in Theorem 2.2.1, namely,

$$\sup_{R>0}\left\|\frac{f-f_R}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right\|_{L^p(\mathbb{G})} \le \frac{p}{p-1}\left\|\frac{1}{|x|^{\frac{Q}{p}-1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})}, \tag{2.45}$$

for all $1<p<\infty$. Here the constant $\frac{p}{p-1}$ in (2.45) is sharp but is in general unattainable.

The inequalities (2.45) are all critical with respect to their weight $|x|^{-\frac{Q}{p}}$ in L^p-space because its pth power gives $|x|^{-Q}$ which is the critical order for the integrability at zero and at infinity in $L^p(\mathbb{G})$.

Moreover, we show another type of a critical Hardy inequality on general homogeneous groups with the logarithm being on the right-hand side:

$$\left\| \frac{f}{|x|} \right\|_{L^Q(\mathbb{G})} \leq Q \|(\log|x|)\mathcal{R}f\|_{L^Q(\mathbb{G})}. \tag{2.46}$$

Furthermore, we also give improved versions of the Hardy inequality (2.1) on quasi-balls of homogeneous (Lie) groups, the so-called Hardy–Sobolev type inequalities.

2.2.1 Critical Hardy inequalities

First we discuss a family of generalized critical Hardy inequalities on the homogeneous group $\mathbb{G}$ mentioned in (2.45). In this section, $\mathbb{G}$ is a homogeneous group of homogeneous dimension $Q \geq 2$ and $|\cdot|$ is an arbitrary homogeneous quasi-norm on $\mathbb{G}$.

Theorem 2.2.1 (A family of critical Hardy inequalities)**.** *Let $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ and denote $f_R(x) := f(R\frac{x}{|x|})$ for $x \in \mathbb{G}$ and $R > 0$. Then we have the inequalities*

$$\sup_{R>0} \left\| \frac{f - f_R}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{p-1} \left\| \frac{1}{|x|^{\frac{Q}{p}-1}} \mathcal{R}f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \tag{2.47}$$

where the constant $\frac{p}{p-1}$ is sharp. Moreover, denoting

$$u_R(x) := \frac{f(x) - f_R(x)}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}} \quad \text{and} \quad v(x) := \frac{1}{|x|^{\frac{Q}{p}-1}} \mathcal{R}f(x),$$

for each $R > 0$ we have the following expression for the remainder:

$$\begin{aligned} \left\| \frac{f - f_R}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}} \right\|^p_{L^p(\mathbb{G})} = \left(\frac{p}{p-1} \right)^p \left\| \frac{1}{|x|^{\frac{Q}{p}-1}} \mathcal{R}f \right\|^p_{L^p(\mathbb{G})} \\ - p \int_{\mathbb{G}} I(u_R, -\frac{p}{p-1}v) \left| \frac{p}{p-1}v + u_R \right|^2 dx, \end{aligned} \tag{2.48}$$

where I is defined by

$$I(u,g) := \left(\frac{1}{p}|g|^p + \left(1 - \frac{1}{p}\right)|u|^p - |u|^{p-2}\operatorname{Re}(u\overline{g}) \right)|u-g|^{-2} \geq 0, \quad u \neq g,$$
$$I(g,g) := \frac{p-1}{2}|g|^{p-2}.$$

Remark 2.2.2.

1. For $p = Q$ the inequality (2.47) becomes

$$\sup_{R>0} \left\| \frac{f - f_R}{|x| \log \frac{R}{|x|}} \right\|_{L^Q(\mathbb{G})} \leq \frac{Q}{Q-1} \|\mathcal{R} f\|_{L^Q(\mathbb{G})}, \quad Q \geq 2, \tag{2.49}$$

   which gives the mentioned critical estimate (2.44).

2. In the Euclidean isotropic case $\mathbb{G} = \mathbb{R}^n$ with the Euclidean norm $|\cdot|_E$, similar to Remark 2.1.2, Part 1, inequalities (2.47) yields inequalities for the usual gradient ∇ in $\mathbb{R}^n$ for all $1 < p < \infty$:

$$\sup_{R>0} \left\| \frac{f - f_R}{|x|_E^{\frac{n}{p}} \log \frac{R}{|x|_E}} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{p-1} \left\| \frac{1}{|x|_E^{\frac{n}{p}-1}} \nabla f \right\|_{L^p(\mathbb{R}^n)}, \tag{2.50}$$

   where $f_R(x) := f(R\frac{x}{|x|_E})$ for $x \in \mathbb{R}^n$ and $R > 0$. The critical inequality (2.49) becomes

$$\sup_{R>0} \left\| \frac{f - f_R}{|x|_E \log \frac{R}{|x|_E}} \right\|_{L^n(\mathbb{R}^n)} \leq \frac{n}{n-1} \|\nabla f\|_{L^n(\mathbb{R}^n)}, \quad n \geq 2. \tag{2.51}$$

3. For the sharpness of the constant $\frac{p}{p-1}$ in (2.47) we refer to the Euclidean case with the Euclidean norm where this constant is known to be sharp but is in general unattainable. This was shown in [IIO16a]. In the case of general homogeneous groups this follows from the expression (2.48) for the remainder.

4. For $p = 2$, in the Euclidean case $\mathbb{G} = \mathbb{R}^n$ with the Euclidean norm, the estimate (2.50) was shown in [MOW17a]. In principle, this case can be also obtained from [MOW15a, Theorem 1.1]. Inequality (2.51) in bounded domains of $\mathbb{R}^n$ was analysed in [II15].

Proof of Theorem 2.2.1. Using the polar decomposition in Proposition 1.2.10, a straightforward calculation shows

$$\begin{aligned}
&\int_{B(0,R)} \frac{|f(x) - f_R(x)|^p}{|x|^Q |\log \frac{R}{|x|}|^p} dx \\
&= \int_0^R \int_{\wp} \frac{|f(ry) - f(Ry)|^p}{r^Q \left(\log \frac{R}{r}\right)^p} r^{Q-1} d\sigma(y) dr \\
&= \int_0^R \frac{d}{dr} \left(\frac{1}{p-1} \frac{1}{\left(\log \frac{R}{r}\right)^{p-1}} \int_{\wp} |f(ry) - f(Ry)|^p d\sigma(y) \right) dr
\end{aligned}$$

$$-\frac{p}{p-1}\mathrm{Re}\int_0^R\left(\frac{1}{\left(\log\frac{R}{r}\right)^{p-1}}\int_{\wp}|f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)\right)dr$$
$$=-\frac{p}{p-1}\mathrm{Re}\int_0^R\left(\frac{1}{\left(\log\frac{R}{r}\right)^{p-1}}\int_{\wp}|f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)\right)dr,$$

where σ is the Borel measure on $\wp$ and the contribution on the boundary at $r=R$ vanishes due to the inequalities

$$|f(ry)-f(Ry)|\le C(R-r),\qquad \frac{R-r}{R}\le\log\frac{R}{r}.$$

By using the formula (1.30) we obtain

$$\int_{B(0,R)}\frac{|f(x)-f_R(x)|^p}{|x|^Q|\log\frac{R}{|x|}|^p}dx$$
$$=-\frac{p}{p-1}\mathrm{Re}\int_0^R\frac{1}{\left(\log\frac{R}{r}\right)^{p-1}}\int_{\wp}|f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)dr$$
$$=-\frac{p}{p-1}\mathrm{Re}\int_{B(0,R)}\left|\frac{f(x)-f_R(x)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right|^{p-2}\frac{(f(x)-f_R(x))}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\frac{1}{|x|^{\frac{Q}{p}-1}}\overline{\frac{df(x)}{d|x|}}dx.$$

Similarly, one has

$$\int_{B^c(0,R)}\frac{|f(x)-f_R(x)|^p}{|x|^Q|\log\frac{R}{|x|}|^p}dx=\int_R^\infty\int_{\wp}\frac{|f(ry)-f(Ry)|^p}{r^Q\left(\log\frac{r}{R}\right)^p}r^{Q-1}d\sigma(y)dr$$
$$=-\int_R^\infty\frac{d}{dr}\left(\frac{1}{p-1}\frac{1}{\left(\log\frac{r}{R}\right)^{p-1}}\int_{\wp}|f(ry)-f(Ry)|^pd\sigma(y)\right)dr$$
$$+\frac{p}{p-1}\mathrm{Re}\int_R^\infty\left(\frac{1}{\left(\log\frac{r}{R}\right)^{p-1}}\int_{\wp}|f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)\right)dr$$
$$=-\frac{p}{p-1}\mathrm{Re}\int_{B^c(0,R)}\left|\frac{f(x)-f_R(x)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right|^{p-2}\frac{(f(x)-f_R(x))}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\frac{1}{|x|^{\frac{Q}{p}-1}}\overline{\frac{df(x)}{d|x|}}dx.$$

This implies that

$$\int_{\mathbb{G}}\frac{|f(x)-f_R(x)|^p}{|x|^Q|\log\frac{R}{|x|}|^p}dx$$
$$=-\frac{p}{p-1}\mathrm{Re}\int_{\mathbb{G}}\left|\frac{f(x)-f_R(x)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right|^{p-2}\frac{(f(x)-f_R(x))}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\frac{1}{|x|^{\frac{Q}{p}-1}}\overline{\frac{df(x)}{d|x|}}dx$$
$$=\left(\frac{p}{p-1}\right)^p\|v\|^p_{L^p(\mathbb{G})}-p\int_{\mathbb{G}}I(u,-\frac{p}{p-1}v)\left|\frac{p}{p-1}v+u\right|^2dx,$$

with
$$u = \frac{f(x) - f_R(x)}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}}, \qquad v = \frac{1}{|x|^{\frac{Q}{p}-1}} \frac{df(x)}{d|x|},$$
and I is defined by
$$I(f,g) := \left(\frac{1}{p}|g|^p + \frac{1}{p'}|f|^p - |f|^{p-2}\operatorname{Re}(f\overline{g})\right)|f-g|^{-2} \geq 0, \ f \neq g, \ \frac{1}{p} + \frac{1}{p'} = 1,$$
$$I(g,g) := \frac{p-1}{2}|g|^{p-2}.$$
Thus, we establish
$$\|u\|^p_{L^p(\mathbb{G})} = \left(\frac{p}{p-1}\right)^p \|v\|^p_{L^p(\mathbb{G})} - p\int_{\mathbb{G}} I\left(u, -\frac{p}{p-1}v\right)\left|\frac{p}{p-1}v + u\right|^2 dx.$$
This proves the equality (2.48) and the inequality (2.47) since the last term is non-positive. □

We have the following consequence of Theorem 2.2.1:

Corollary 2.2.3 (Critical uncertainty type principles)**.** *Let $1 < p < \infty$ and $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Then for any $R > 0$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ with $q > 1$, we have*
$$\left\|\frac{1}{|x|^{\frac{Q}{p}-1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} \geq \frac{p-1}{p}\left\|\frac{f(f-f_R)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right\|_{L^2(\mathbb{G})} \tag{2.52}$$
and also
$$\left\|\frac{1}{|x|^{\frac{Q}{p}-1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})} \left\|\frac{f-f_R}{|x|^{\frac{Q}{p'}}\log\frac{R}{|x|}}\right\|_{L^{p'}(\mathbb{G})} \geq \frac{p-1}{p}\left\|\frac{f-f_R}{|x|^{\frac{Q}{2}}\log\frac{R}{|x|}}\right\|^2_{L^2(\mathbb{G})} \tag{2.53}$$
for $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof of Corollary 2.2.3. By Theorem 2.2.1 and Hölder's inequality we have
$$\begin{aligned}
\left\|\frac{1}{|x|^{\frac{Q}{p}-1}}\mathcal{R}f\right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} &\geq \frac{p-1}{p}\left\|\frac{f-f_R}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} \\
&= \frac{p-1}{p}\left(\int_{\mathbb{G}}\left|\frac{f-f_R}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right|^{2\frac{p}{2}} dx\right)^{\frac{1}{2}\frac{2}{p}} \left(\int_{\mathbb{G}}|f|^{2\frac{q}{2}}dx\right)^{\frac{1}{2}\frac{2}{q}} \\
&\geq \frac{p-1}{p}\left(\int_{\mathbb{G}}\left|\frac{f(f-f_R)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right|^2 dx\right)^{1/2} = \frac{p-1}{p}\left\|\frac{f(f-f_R)}{|x|^{\frac{Q}{p}}\log\frac{R}{|x|}}\right\|_{L^2(\mathbb{G})}.
\end{aligned}$$
The formula (2.52) is proved. The proof of (2.53) is similar so we can omit it. □

2.2.2 Another type of critical Hardy inequality

In this section we analyse another type of a critical Hardy inequality when the logarithmic term appears on the other side of the inequality. This extends inequality (2.38) that was obtained for $p = 2$ by a factorization method.

Theorem 2.2.4 (Critical Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 1$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < \infty$. Then for any complex-valued function $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx \leq p^p \int_{\mathbb{G}} \frac{|\log|x||^p}{|x|^{Q-p}} |\mathcal{R}f(x)|^p dx, \tag{2.54}$$

and the constant p^p in this inequality is sharp.

Remark 2.2.5.

1. Inequality (2.54) was mentioned in (2.46) as another type of a critical Hardy inequality in the critical case of $p = Q$, in which case we have

$$\int_{\mathbb{G}} \frac{|f(x)|^Q}{|x|^Q} dx \leq Q^Q \int_{\mathbb{G}} |(\log|x|)\mathcal{R}f(x)|^Q dx. \tag{2.55}$$

2. In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$ we have $Q = n$, so for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$ the inequality (2.54) implies a new inequality with the optimal constant: For each $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, we have

$$\left\|\frac{f}{|x|}\right\|_{L^n(\mathbb{R}^n)} \leq n \left\|(\log|x|)\,\frac{x}{|x|} \cdot \nabla f\right\|_{L^n(\mathbb{R}^n)}. \tag{2.56}$$

   If we take now the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$, it follows that we have

$$\left\|\frac{f}{|x|_E}\right\|_{L^n(\mathbb{R}^n)} \leq n \left\|(\log|x|_E)\nabla f\right\|_{L^n(\mathbb{R}^n)}, \tag{2.57}$$

   for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, where ∇ is the standard gradient in $\mathbb{R}^n$. The constants in the above inequalities are sharp.

3. The inequality (2.57) and its consequence are analogous to the critical Hardy inequality of Edmunds and Triebel [ET99] that they showed in $\mathbb{R}^n$ for the Euclidean norm $|\cdot|_E$ in bounded domains $B \subset \mathbb{R}^n$:

$$\left\|\frac{f(x)}{|x|_E(1 + \log\frac{1}{|x|_E})}\right\|_{L^n(B)} \leq \frac{n}{n-1} \left\|\nabla f\right\|_{L^n(B)}, \quad n \geq 2, \tag{2.58}$$

   with sharp constant $\frac{n}{n-1}$, which was also discussed in [AS06]. This inequality was also shown to be equivalent to the critical case of the Sobolev–Lorentz inequality. However, a different feature of (2.57) compared to (2.58) is that the logarithmic term enters the other side of the inequality. Inequalities of this type have been investigated in [RS16a].

Proof of Theorem 2.2.4. Let $R > 0$ be such that $\operatorname{supp} f \subset B(0,R)$. A direct calculation using the polar decomposition in Proposition 1.2.10 with integration by parts yields

$$\begin{aligned}
\int_{B(0,R)} \frac{|f(x)|^p}{|x|^Q} dx &= \int_0^R \int_{\wp} |f(\delta_r(y))|^p r^{Q-1-Q} d\sigma(y) dr \\
&= -p \int_0^R \log r \operatorname{Re} \int_{\wp} |f(\delta_r(y))|^{p-2} f(\delta_r(y)) \overline{\frac{df(\delta_r(y))}{dr}} d\sigma(y) dr \\
&\leq p \int_{B(0,R)} \frac{|\mathcal{R} f(x)||f(x)|^{p-1}}{|x|^{Q-1}} |\log|x|| dx \\
&= p \int_{B(0,R)} \frac{|\mathcal{R} f(x)||\log|x||}{|x|^{\frac{Q}{p}-1}} \frac{|f(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} dx,
\end{aligned}$$

and by using the Hölder inequality, we obtain that

$$\int_{B(0,R)} \frac{|f(x)|^p}{|x|^Q} dx \leq p \left(\int_{B(0,R)} \frac{|\mathcal{R} f(x)|^p |\log|x||^p}{|x|^{Q-p}} dx \right)^{\frac{1}{p}} \left(\int_{B(0,R)} \frac{|f(x)|^p}{|x|^Q} dx \right)^{\frac{p-1}{p}},$$

which gives (2.54).

Now it remains to show the optimality of the constant, so we need to check the equality condition in the above Hölder inequality. Let us consider the test function

$$h(x) = \log|x|.$$

Thus, we have

$$\left(\frac{|\mathcal{R} h(x)||\log|x||}{|x|^{\frac{Q}{p}-1}} \right)^p = \left(\frac{|h(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} \right)^{\frac{p}{p-1}},$$

which satisfies the equality condition in Hölder's inequality. This gives the optimality of the constant p^p in (2.54). □

As usual, the Hardy inequality implies the corresponding uncertainty principle:

Corollary 2.2.6 (Another type of critical uncertainty principle)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then for each $f \in C_0^\infty(\mathbb{G} \backslash \{0\})$ we have*

$$\left(\int_{\mathbb{G}} |(\log|x|) \mathcal{R} f|^Q \, dx \right)^{\frac{1}{Q}} \left(\int_{\mathbb{G}} |x|^{\frac{Q}{Q-1}} |f|^{\frac{Q}{Q-1}} dx \right)^{\frac{Q-1}{Q}} \geq \frac{1}{Q} \int_{\mathbb{G}} |f|^2 dx. \tag{2.59}$$

Proof. From the inequality (2.54) we get

$$\left(\int_{B(0,R)} |(\log|x|)\mathcal{R}f|^Q\, dx\right)^{\frac{1}{Q}} \left(\int_{B(0,R)} |x|^{\frac{Q}{Q-1}}|f|^{\frac{Q}{Q-1}} dx\right)^{\frac{Q-1}{Q}}$$
$$\geq \frac{1}{Q}\left(\int_{B(0,R)} \frac{|f|^Q}{|x|^Q}\, dx\right)^{\frac{1}{Q}} \left(\int_{B(0,R)} |x|^{\frac{Q}{Q-1}}|f|^{\frac{Q}{Q-1}} dx\right)^{\frac{Q-1}{Q}} \geq \frac{1}{Q}\int_{B(0,R)} |f|^2 dx,$$

where we have used the Hölder inequality in the last line. This shows (2.59). □

2.2.3 Critical Hardy inequalities of logarithmic type

In this section we present yet another logarithmic type of critical Hardy inequalities on the homogeneous group $\mathbb{G}$ of homogeneous dimension $Q \geq 1$. As usual, let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$.

Theorem 2.2.7 (Another family of logarithmic Hardy inequalities)**.** *Let* $1 < \gamma < \infty$ *and* $\max\{1, \gamma-1\} < p < \infty$*. Then for all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *and all* $R > 0$ *we have the inequality*

$$\left\| \frac{f - f_R}{|x|^{\frac{Q}{p}}\left(\log\frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{\gamma-1}\left\| |x|^{\frac{p-Q}{p}}\left(\log\frac{R}{|x|}\right)^{\frac{p-\gamma}{p}} \mathcal{R}f \right\|_{L^p(\mathbb{G})}, \tag{2.60}$$

where $f_R(x) := f\left(R\frac{x}{|x|}\right)$*, and the constant* $\frac{p}{\gamma-1}$ *is sharp.*

Proof of Theorem 2.2.7. For a quasi-ball $B(0,R)$ we have using the polar decomposition in Proposition 1.2.10 that

$$\int_{B(0,R)} \frac{|f(x)-f_R(x)|^p}{|x|^Q\left|\log\frac{R}{|x|}\right|^\gamma}dx = \int_0^R \int_\wp \frac{|f(ry)-f(Ry)|^p}{r^Q\left(\log\frac{R}{r}\right)^\gamma} r^{Q-1} d\sigma(y) dr$$
$$= \int_0^R \frac{d}{dr}\left(\frac{1}{(\gamma-1)\left(\log\frac{R}{r}\right)^{\gamma-1}}\int_\wp |f(ry)-f(Ry)|^p d\sigma(y)\right) dr$$
$$- \frac{p}{\gamma-1}\mathrm{Re}\int_0^R \left(\log\frac{R}{r}\right)^{-\gamma+1}$$
$$\times \int_\wp |f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}} d\sigma(y) dr$$
$$= -\frac{p}{\gamma-1}\mathrm{Re}\int_0^R \left(\log\frac{R}{r}\right)^{-\gamma+1}$$
$$\times \int_\wp |f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}} d\sigma(y) dr,$$

where $p-\gamma+1>0$, so that the boundary term at $r=R$ vanishes due to inequalities

$$|f(ry)-f(Ry)|\le C(R-r), \qquad \log\frac{R}{r}\ge\frac{R-r}{R}.$$

Then by the Hölder inequality we get

$$\begin{aligned}
&\int_0^R\int_{\wp}\frac{|f(ry)-f(Ry)|^p}{r\left(\log\frac{R}{r}\right)^{\gamma}}d\sigma(y)dr\\
&=-\frac{p}{\gamma-1}\mathrm{Re}\int_0^R\left(\log\frac{R}{r}\right)^{-\gamma+1}\int_{\wp}|f(ry)-f(Ry)|^{p-2}(f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)dr\\
&\le\frac{p}{\gamma-1}\int_0^R\left(\log\frac{R}{r}\right)^{-\gamma+1}\int_{\wp}|f(ry)-f(Ry)|^{p-1}\left|\frac{df(ry)}{dr}\right|d\sigma(y)dr\\
&\le\frac{p}{\gamma-1}\left(\int_0^R\int_{\wp}\frac{|f(ry)-f(Ry)|^p}{r\left(\log\frac{R}{r}\right)^{\gamma}}d\sigma(y)dr\right)^{(p-1)/p}\\
&\quad\times\left(\int_0^R\int_{\wp}r^{p-1}\left(\log\frac{R}{r}\right)^{p-\gamma}\left|\frac{df(ry)}{dr}\right|^p d\sigma(y)dr\right)^{1/p}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left(\int_{B(0,R)}\frac{|f(x)-f_R(x)|^p}{|x|^Q\left|\log\frac{R}{|x|}\right|^{\gamma}}dx\right)^{1/p}\\
&\qquad\le\frac{p}{\gamma-1}\left(\int_{B(0,R)}|x|^{p-Q}\left|\log\frac{R}{|x|}\right|^{p-\gamma}|\mathcal{R}f(x)|^p\,dx\right)^{1/p}.
\end{aligned}\tag{2.61}$$

Similarly, we have

$$\begin{aligned}
&\left(\int_{B^c(0,R)}\frac{|f(x)-f_R(x)|^p}{|x|^Q\left|\log\frac{R}{|x|}\right|^{\gamma}}dx\right)^{1/p}\\
&\qquad\le\frac{p}{\gamma-1}\left(\int_{B^c(0,R)}|x|^{p-Q}\left|\log\frac{R}{|x|}\right|^{p-\gamma}|\mathcal{R}f(x)|^p\,dx\right)^{1/p}.
\end{aligned}\tag{2.62}$$

The inequalities (2.61) and (2.62) imply (2.60). Furthermore, the optimality of the constant in (2.60) is proved exactly in the same way as in the Euclidean case (see [MOW15b, Section 3]). □

Corollary 2.2.8 (Another family of logarithmic uncertainty type principles)**.** *Let* $1<p<\infty$ *and* $q>1$ *be such that* $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. *Let* $1<\gamma<\infty$ *and* $\max\{1,\gamma-1\}<p<\infty$. *Then for any* $R>0$ *and* $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have*

$$\left\||x|^{\frac{p-Q}{p}}\left(\log\frac{R}{|x|}\right)^{(p-\gamma)/p}\mathcal{R}f\right\|_{L^p(\mathbb{G})}\|f\|_{L^q(\mathbb{G})}\ge\frac{\gamma-1}{p}\left\|\frac{f(f-f_R)}{|x|^{\frac{Q}{p}}\left(\log(R/|x|)\right)^{\gamma/p}}\right\|_{L^2(\mathbb{G})}.$$

Moreover,

$$\begin{aligned}\left\| |x|^{\frac{p-Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{p-\gamma}{p}} \mathcal{R}f \right\|_{L^p(\mathbb{G})} \left\| \frac{f-f_R}{|x|^{\frac{Q}{p'}} \left(\log \frac{R}{|x|}\right)^{2-\frac{\gamma}{p}}} \right\|_{L^{p'}(\mathbb{G})} \\ \geq \frac{\gamma-1}{p} \left\| \frac{f-f_R}{|x|^{\frac{Q}{2}} \log \frac{R}{|x|}} \right\|^2_{L^2(\mathbb{G})}\end{aligned} \tag{2.63}$$

holds for $\frac{1}{p}+\frac{1}{p'}=1$.

Proof of Corollary 2.2.8. By (2.60), we have

$$\begin{aligned}&\left\| |x|^{\frac{p-Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{p-\gamma}{p}} \mathcal{R}f \right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} \geq \frac{\gamma-1}{p} \left\| \frac{f-f_R}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} \\ &= \frac{\gamma-1}{p} \left(\int_{\mathbb{G}} \left| \frac{f(x)-f_R(x)}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right|^{2\frac{p}{2}} dx \right)^{\frac{1}{2}\frac{2}{p}} \left(\int_{\mathbb{G}} |f(x)|^{2\frac{q}{2}} dx \right)^{\frac{1}{2}\frac{2}{q}},\end{aligned}$$

and using Hölder's inequality, we obtain

$$\begin{aligned}&\left\| |x|^{\frac{p-Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{p-\gamma}{p}} \mathcal{R}f \right\|_{L^p(\mathbb{G})} \|f\|_{L^q(\mathbb{G})} \\ &\geq \frac{\gamma-1}{p} \left(\int_{\mathbb{G}} \left| \frac{f(x)(f(x)-f_R(x))}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\gamma/p}} \right|^2 dx \right)^{1/2} = \frac{\gamma-1}{p} \left\| \frac{f(f-f_R)}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\gamma/p}} \right\|_{L^2(\mathbb{G})}.\end{aligned}$$

Similarly, one can prove (2.63). □

Remark 2.2.9. When $\gamma = p$, the statement in Theorem 2.2.7 appeared in [RS16a, Theorem 3.1] in the form

$$\left\| \frac{f-f_R}{|x|^{\frac{Q}{p}} \log \frac{R}{|x|}} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{p-1} \left\| |x|^{\frac{p-Q}{p}} \mathcal{R}f \right\|_{L^p(\mathbb{G})}, \quad 1<p<\infty, \tag{2.64}$$

for all $R>0$. For general p and γ it was analysed in [RS16a]. In the Euclidean case with the Euclidean distance such inequalities have been analysed in [MOW15b, Section 3].

2.3 Remainder estimates

In this section we analyse the remainder estimates for L^p-weighted Hardy inequalities with sharp constants on homogeneous groups. In addition, other refined versions involving a distance and the critical case $p = Q = 2$ on the quasi-ball are discussed.

The analysis of remainder terms in Hardy inequalities has a long history initiated by Brézis and Nirenberg in [BN83], with subsequent works by Brézis and Lieb [BL85] for Hardy–Sobolev inequalities, Brézis and Vázquez in [BV97, Section 4]. Nowadays there is a lot of literature on this subject and this section will contain some further references on this subject.

2.3.1 Remainder estimates for L^p-weighted Hardy inequalities

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We now present a family of remainder estimates for the weighted L^p-Hardy inequalities, with a freedom of choosing the parameter $b \in \mathbb{R}$.

Theorem 2.3.1 (Remainder estimates for L^p-weighted Hardy inequalities)**.** *Let*

$$2 \leq p < Q, \quad -\infty < \alpha < \frac{Q-p}{p},$$

and let

$$\delta_1 = Q - p - \alpha p - \frac{Q+pb}{p},$$
$$\delta_2 = Q - p - \alpha p - \frac{bp}{p-1},$$

for $b \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\begin{aligned} \int_{\mathbb{G}} \frac{|\mathcal{R}f(x)|^p}{|x|^{\alpha p}} dx - \left(\frac{Q-p-\alpha p}{p}\right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx \\ \geq C_p \frac{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_1} dx\right)^p}{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_2} dx\right)^{p-1}}, \end{aligned} \tag{2.65}$$

where the constant $C_p = c_p \left|\frac{Q(p-1)-pb}{p^2}\right|^p$ is sharp, with

$$c_p = \min_{0<t\leq 1/2}((1-t)^p - t^p + pt^{p-1}). \tag{2.66}$$

Due to the positivity of the last remainder term, Theorem 2.3.1 implies the L^p-weighted Hardy inequalities with the radial derivative. In the case of the Euclidean norm, they reduce to the usual L^p-weighted Hardy estimates:

Remark 2.3.2 (L^p-weighted Hardy inequalities)**.**

1. If we take $b = \frac{Q(p-1)}{p}$ we have $C_p = 0$, and then inequality (2.65) gives the L^p-weighted Hardy inequalities with sharp constant on $\mathbb{G}$:
$$\int_{\mathbb{G}} \frac{|\mathcal{R}f(x)|^p}{|x|^{\alpha p}} dx \geq \left(\frac{Q-p-\alpha p}{p}\right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx, \quad -\infty < \alpha < \frac{Q-p}{p}, \;\; 2 \leq p < Q, \tag{2.67}$$
for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Such inequalities on homogeneous groups have been investigated in [RS17b], and their remainders have been analysed in [RSY18b].
2. If $\mathbb{G} = (\mathbb{R}^n, +)$ with $Q = n$, the inequality (2.67) gives the L^p-weighted Hardy inequalities with sharp constant for any quasi-norm on $\mathbb{R}^n$: For any complex-valued function $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ we obtain
$$\int_{\mathbb{R}^n} \left|\frac{x}{|x|} \cdot \nabla f(x)\right|^p |x|^{-\alpha p} dx \geq \left(\frac{n-p-\alpha p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx,$$
where $-\infty < \alpha < \frac{n-p}{p}$ and $2 \leq p < n$, and where ∇ is the standard gradient in $\mathbb{R}^n$. Now, if we take the Euclidean norm $|x|_E = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, by using the Schwarz inequality we obtain the Euclidean form of the L^p-weighted Hardy inequalities with sharp constants:
$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x|_E^{\alpha p}} dx \geq \left(\frac{n-p-\alpha p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|_E^{p(\alpha+1)}} dx, \quad -\infty < \alpha < \frac{n-p}{p}, \;\; 2 \leq p < n, \tag{2.68}$$
for any complex-valued function $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$.
3. Moreover, for any function $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and for any $b \in \mathbb{R}$, we have
$$\begin{aligned}\int_{\mathbb{R}^n} \left|\frac{x}{|x|} \cdot \nabla f(x)\right|^p |x|^{-\alpha p} dx - \left(\frac{n-p-\alpha p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx \\ \geq C_p \frac{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\delta_1} dx\right)^p}{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\delta_2} dx\right)^{p-1}}, \quad 2 \leq p < n, \; -\infty < \alpha < \frac{n-p}{p},\end{aligned} \tag{2.69}$$
where ∇ is the standard gradient in $\mathbb{R}^n$. As in Part 2 above, by the Schwarz inequality with the usual Euclidean distance $|\cdot|_E$, we obtain
$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x|_E^{\alpha p}} dx - \left(\frac{n-p-\alpha p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|_E^{p(\alpha+1)}} dx \\ \geq C_p \frac{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|_E^{\delta_1} dx\right)^p}{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|_E^{\delta_2} dx\right)^{p-1}}, \quad 2 \leq p < n, \; -\infty < \alpha < \frac{n-p}{p},\end{aligned} \tag{2.70}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and for any $b \in \mathbb{R}$, where the constant C_p is sharp.

4. In $\mathbb{R}^n$ a variant of inequality (2.65) is known for radially symmetric functions. Let $n \geq 3$, $2 \leq p < n$ and $-\infty < \alpha < \frac{n-p}{p}$. Let $N \in \mathbb{N}$, $t \in (0,1)$, $\gamma < \min\{1-t, \frac{p-N}{p}\}$ and $\delta = N - n + \frac{N}{1-t-\gamma}\left(\gamma + \frac{n-p-\alpha p}{p}\right)$. Then there exists a constant $C > 0$ such that the inequality
$$\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|_E^{\alpha p}} dx - \left(\frac{n-p-\alpha p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|_E^{p(\alpha+1)}} dx$$
$$\geq C \frac{\left(\int_{\mathbb{R}^n} |f|^{\frac{N}{1-t-\gamma}} |x|_E^\delta dx\right)^{\frac{p(1-t-\gamma)}{Nt}}}{\left(\int_{\mathbb{R}^n} |f|^p |x|_E^{-\alpha p} dx\right)^{\frac{1-t}{t}}}$$
holds for all radially symmetric functions $f \in W_{0,\alpha}^{1,p}(\mathbb{R}^n)$, $f \neq 0$, where $W_{0,\alpha}^{1,p}(\mathbb{R}^n)$ is an appropriate Sobolev type space. For $\alpha = 0$ this was shown by Sano and Takahashi [ST17] and then extended in [ST18a] for any $-\infty < \alpha < \frac{n-p}{p}$.

5. The constant c_p in (2.66) appears in view of the following result that will be also of use in the determination of this constant:

   Lemma 2.3.3 ([FS08])**.** *Let $p \geq 2$ and let a, b be real numbers. Then there exists $c_p > 0$ such that*
   $$|a-b|^p \geq |a|^p - p|a|^{p-2}ab + c_p|b|^p$$
   holds, where $c_p = \min_{0<t\leq 1/2}((1-t)^p - t^p + pt^{p-1})$ is sharp in this inequality.

6. Remainder estimates of different forms are possible. In general, it is known from Ghoussoub and Moradifam [GM08] that there are no strictly positive functions $V \in C^1(0,\infty)$ such that the inequality
$$\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f|^2}{|x|_E^2} dx + \int_{\mathbb{R}^n} V(|x|_E)|f|^2 dx$$
holds for all Sobolev space functions $f \in W^{1,2}(\mathbb{R}^n)$. At the same time, Cianchi and Ferone showed in [CF08] that for all $1 < p < n$ there exists a constant $C = C(p,n)$ such that
$$\int_{\mathbb{R}^n} |\nabla f|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|_E^p} dx \,(1 + C d_p(f)^{2p^*})$$
holds for all real-valued weakly differentiable functions f in $\mathbb{R}^n$ such that f and $|\nabla f| \in L^p(\mathbb{R}^n)$ go to zero at infinity, where
$$d_p(f) = \inf_{c\in\mathbb{R}} \frac{\|f - c|x|_E^{-\frac{n-p}{p}}\|_{L^{p^*,\infty}(\mathbb{R}^n)}}{\|f\|_{L^{p^*,p}(\mathbb{R}^n)}}$$

with $p^* = \frac{np}{n-p}$, and $L^{\tau,\sigma}(\mathbb{R}^n)$ is the Lorentz space for $0 < \tau \leq \infty$ and $1 \leq \sigma \leq \infty$. In the case of a bounded domain Ω, Wang and Willem [WW03] for $p = 2$ and Abdellaoui, Colorado and Peral [ACP05] for $1 < p < \infty$ investigated other expressions of remainders, see also [ST17] and [ST18a] for more details.

In the following proof we will rely on a useful feature that some estimates involving radial derivatives of the Euler operator can be proved first for radial functions, and then extended to non-radial ones by a more abstract argument, see Section 1.3.3.

Proof of Theorem 2.3.1. Let $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ be a radial function, then f can be represented as $f(x) = \widetilde{f}(|x|)$. By using Brézis–Vázquez' idea ([BV97]), we define

$$\widetilde{g}(r) := r^{\frac{Q-p-\alpha p}{p}} \widetilde{f}(r). \tag{2.71}$$

Since $\widetilde{f} = \widetilde{f}(r) \in C_0^\infty(0,\infty)$ and $\alpha < \frac{Q-p}{p}$, we have $\widetilde{g}(0) = 0$ and $\widetilde{g}(+\infty) = 0$. We set

$$g(x) := \widetilde{g}(|x|)$$

for $x \in \mathbb{G}$. Introducing polar coordinates $(r,y) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times \wp$ on $\mathbb{G}$, by Proposition 1.2.10 we have

$$\begin{aligned}
J &:= \int_{\mathbb{G}} \frac{|\mathcal{R}f(x)|^p}{|x|^{\alpha p}} dx - \left(\frac{Q-p-\alpha p}{p}\right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx \\
&= |\wp| \int_0^\infty \left|\frac{d}{dr}\widetilde{f}(r)\right|^p r^{-\alpha p + Q - 1} dr - |\wp| \left(\frac{Q-p-\alpha p}{p}\right)^p \int_0^\infty |\widetilde{f}(r)|^p r^{-p(\alpha+1)+Q-1} dr \\
&= |\wp| \int_0^\infty \left| \left(\frac{Q-p-\alpha p}{p}\right) r^{-\frac{Q-\alpha p}{p}} \widetilde{g}(r) - r^{-\frac{Q-p-\alpha p}{p}} \frac{d}{dr}\widetilde{g}(r)\right|^p r^{Q-1-\alpha p} dr \\
&\quad - |\wp| \left(\frac{Q-p-\alpha p}{p}\right)^p \int_0^\infty |\widetilde{g}(r)|^p r^{-1} dr,
\end{aligned}$$

where $|\wp|$ is the $Q-1$-dimensional surface measure of the unit sphere. Here applying Lemma 2.3.3 to the integrand of the first term in the last expression above, we get

$$\begin{aligned}
&\left| \left(\frac{Q-p-\alpha p}{p}\right) r^{-\frac{Q-\alpha p}{p}} \widetilde{g}(r) - r^{-\frac{Q-p-\alpha p}{p}} \frac{d}{dr}\widetilde{g}(r)\right|^p r^{Q-1-\alpha p} \\
&\quad \geq \left(\left(\frac{Q-p-\alpha p}{p}\right)^p r^{-Q+\alpha p} |\widetilde{g}(r)|^p \right) r^{Q-1-\alpha p} \\
&\qquad - p \left(\frac{Q-p-\alpha p}{p}\right)^{p-1} |\widetilde{g}(r)|^{p-2} \widetilde{g}(r) \frac{d}{dr}\widetilde{g}(r) r^{-(\frac{Q-\alpha p}{p})(p-1)} r^{-(\frac{Q-p-\alpha p}{p})} r^{Q-1-\alpha p} \\
&\qquad + c_p \left|\frac{d}{dr}\widetilde{g}(r)\right|^p r^{-Q+p+\alpha p} r^{Q-1-\alpha p}
\end{aligned}$$

$$= \left(\frac{Q-p-\alpha p}{p}\right)^p r^{-1}|\widetilde{g}(r)|^p - p\left(\frac{Q-p-\alpha p}{p}\right)^{p-1}|\widetilde{g}(r)|^{p-2}\widetilde{g}(r)\frac{d}{dr}\widetilde{g}(r)$$
$$+ c_p\left|\frac{d}{dr}\widetilde{g}(r)\right|^p r^{p-1}.$$

Since $\widetilde{g}(0) = \widetilde{g}(+\infty) = 0$ and $p \geq 2$, we note that

$$p\int_0^\infty |\widetilde{g}(r)|^{p-2}\widetilde{g}(r)\frac{d}{dr}\widetilde{g}(r)dr = \int_0^\infty \frac{d}{dr}(|\widetilde{g}(r)|^p)dr = 0.$$

This gives a so-called ground state representation of the Hardy difference J:

$$J \geq c_p|\wp|\int_0^\infty \left|\frac{d}{dr}\widetilde{g}(r)\right|^p r^{p-1}dr = c_p\int_{\mathbb{G}} |\mathcal{R}g(x)|^p|x|^{p-Q}dx. \tag{2.72}$$

Putting $a = \frac{Q-p}{p}$ in Lemma 2.3.3, we obtain for any $b \in \mathbb{R}$ that

$$\left|\frac{Q(p-1)-pb}{p^2}\right|\int_{\mathbb{G}} |g|^p|x|^{-\frac{Q+pb}{p}}dx$$
$$\leq \left(\int_{\mathbb{G}} |\mathcal{R}g|^p|x|^{p-Q}dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{G}} |g|^p|x|^{-\frac{bp}{p-1}}dx\right)^{\frac{p-1}{p}}.$$

It gives the estimate

$$\begin{aligned} J &\geq c_p\int_{\mathbb{G}} |\mathcal{R}g(x)|^p|x|^{p-Q}dx \\ &\geq c_p\left|\frac{Q(p-1)-pb}{p^2}\right|^p \frac{\left(\int_{\mathbb{G}} |g|^p|x|^{-\frac{Q+pb}{p}}dx\right)^p}{\left(\int_{\mathbb{G}} |g|^p|x|^{-\frac{bp}{p-1}}dx\right)^{p-1}}. \end{aligned} \tag{2.73}$$

Taking into account that $g(x) = \widetilde{g}(|x|)$, $x \in \mathbb{G}$, and (2.71), one calculates

$$\int_{\mathbb{G}} |x|^{-\frac{Q+pb}{p}}|g(x)|^p dx = |\wp|\int_0^\infty r^{Q-p-\alpha p}|\widetilde{f}(r)|^p r^{-\frac{Q+pb}{p}}r^{Q-1}dr$$
$$= \int_{\mathbb{G}} |f(x)|^p|x|^{Q-p-\alpha p-\frac{Q+pb}{p}}dx = \int_{\mathbb{G}} |f(x)|^p|x|^{\delta_1}dx.$$

On the other hand,

$$\int_{\mathbb{G}} |x|^{-\frac{bp}{p-1}}|g(x)|^p dx = |\wp|\int_0^\infty r^{Q-p-\alpha p}|\widetilde{f}(r)|^p r^{-\frac{bp}{p-1}}r^{Q-1}dr$$
$$= \int_{\mathbb{G}} |f(x)|^p|x|^{Q-p-\alpha p-\frac{bp}{p-1}}dx = \int_{\mathbb{G}} |f(x)|^p|x|^{\delta_2}dx.$$

Putting these equalities into (2.73), we obtain

$$J \geq c_p \left| \frac{Q(p-1)-pb}{p^2} \right|^p \frac{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_1} dx\right)^p}{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_2} dx\right)^{p-1}}.$$

Now let us prove the statement for non-radial functions. For a non-radial function f we consider the radial one obtained as its spherical average with respect to the homogeneous quasi-norm $|\cdot|$:

$$U(r) := \left(\frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}}. \tag{2.74}$$

Using Hölder's inequality, we calculate

$$\begin{aligned}
\frac{d}{dr}U(r) &= \frac{1}{p}\left(\frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\wp|} \int_{\wp} p|f(ry)|^{p-2} f(ry) \overline{\frac{d}{dr} f(ry)} d\sigma(y) \\
&\leq \left(\frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\wp|} \int_{\wp} |f(ry)|^{p-1} \left| \frac{d}{dr} f(ry) \right| d\sigma(y) \\
&\leq \left(\frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\wp|} \left(\int_{\wp} \left| \frac{d}{dr} f(ry) \right|^p d\sigma(y) \right)^{\frac{1}{p}} \left(\int_{\wp} |f(ry)|^p d\sigma(y) \right)^{\frac{p-1}{p}} \\
&= \left(\frac{1}{|\wp|} \int_{\wp} \left| \frac{d}{dr} f(ry) \right|^p d\sigma(y) \right)^{\frac{1}{p}}.
\end{aligned}$$

Here we note that since there exists the function $h(x) = e^{-|x|}$ which satisfies the equality

$$\left| \frac{d}{dr} h(x) \right|^p = (|h(x)|^{p-1})^{\frac{p}{p-1}},$$

the equality condition in the above Hölder inequality holds. Thus, we have

$$\frac{d}{dr}U(r) \leq \left(\frac{1}{|\wp|} \int_{\wp} \left| \frac{d}{dr} f(ry) \right|^p d\sigma(y) \right)^{\frac{1}{p}}.$$

It follows that

$$\begin{aligned}
|\wp| \int_0^\infty \left| \frac{d}{dr} U(r) \right|^p r^{Q-1-\alpha p} dr &\leq |\wp| \int_0^\infty \frac{1}{|\wp|} \int_{\wp} \left| \frac{d}{dr} f(ry) \right|^p r^{Q-1-\alpha p} d\sigma(y) dr \\
&= \int_{\mathbb{G}} |\mathcal{R} f|^p \, |x|^{-\alpha p} dx,
\end{aligned}$$

that is,

$$\int_{\mathbb{G}} |\mathcal{R} U|^p \, |x|^{-\alpha p} dx \leq \int_{\mathbb{G}} |\mathcal{R} f|^p \, |x|^{-\alpha p} dx. \tag{2.75}$$

In view of (2.74), we obtain the equality

$$\begin{aligned}\int_{\mathbb{G}} |U(|x|)|^p |x|^\theta dx &= |\wp| \int_0^\infty |U(r)|^p r^{\theta+Q-1} dr \\ &= |\wp| \int_0^\infty \frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) r^{\theta+Q-1} dr = \int_{\mathbb{G}} |f(x)|^p |x|^\theta dx,\end{aligned} \tag{2.76}$$

for any $\theta \in \mathbb{R}$. Then, it is easy to see that (2.75) and (2.76) imply that (2.65) holds also for all non-radial functions. $\square$

2.3.2 Critical and subcritical Hardy inequalities

Here we discuss the relation between the critical and the subcritical Hardy inequalities on homogeneous groups. We formulate this relation for functions that are radially symmetric with respect to a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$.

Proposition 2.3.4 (Critical and subcritical Hardy inequalities)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $\widetilde{\mathbb{G}}$ be a homogeneous group of homogeneous dimension $m \geq 2$, and assume that $Q \geq m+1$. Let $|\cdot|$ denote homogeneous quasi-norms on $\mathbb{G}$ and on $\widetilde{\mathbb{G}}$. Then for any non-negative radially symmetric function $g \in C_0^1(B^m(0,R)\backslash\{0\})$, there exists a non-negative radially symmetric function $f \in C_0^1(B^Q(0,1)\backslash\{0\})$ such that*

$$\begin{aligned}&\int_{B^Q(0,1)} |\mathcal{R}f(x)|^m dx - \left(\frac{Q-m}{m}\right)^m \int_{B^Q(0,1)} \frac{|f(x)|^m}{|x|^m} dx \\ &= \frac{|\wp|}{|\widetilde{\wp}|} \left(\frac{Q-m}{m-1}\right)^{m-1} \\ &\times \left(\int_{B^m(0,R)} |\mathcal{R}g|^m dz - \left(\frac{m-1}{m}\right)^m \int_{B^m(0,R)} \frac{|g|^m}{|z|^m \left(\log \frac{Re}{|z|}\right)^m} dz \right)\end{aligned} \tag{2.77}$$

holds true, where $|\wp|$ and $|\widetilde{\wp}|$ are $Q-1$- and $m-1$-dimensional surface measures of the unit sphere, respectively.

Proof of Proposition 2.3.4. Let $r = |x|$, $x \in \mathbb{G}$ and $s = |z|$, $z \in \widetilde{\mathbb{G}}$, where $\widetilde{\mathbb{G}}$ is a homogeneous group of homogeneous dimension m. Let us define a radial function $f = f(x) \in C_0^1(B^Q(0,1)\backslash\{0\})$ for a non-negative radial function $g = g(z) \in C_0^1(B^m(0,R)\backslash\{0\})$:

$$f(r) = g(s(r)),$$

where $s(r) = R\exp(1 - r^{-\frac{Q-m}{m-1}})$, that is,

$$r^{-\frac{Q-m}{m-1}} = \log \frac{Re}{s}, \quad s'(r) = \frac{Q-m}{m-1} r^{-\frac{Q-m}{m-1}-1} s(r).$$

Here we see that $s'(r) > 0$ for $r \in [0,1]$ and $s(0) = 0$, $s(1) = R$. Since $g(s) \equiv 0$ near $s = R$, we also note that $f \equiv 0$ near $r = 1$.

Then a direct calculation shows

$$
\begin{aligned}
&\int_{B^Q(0,1)} |\mathcal{R}f|^m dx - \left(\frac{Q-m}{m}\right)^m \int_{B^Q(0,1)} \frac{|f|^m}{|x|^m} dx \\
&\quad = |\wp| \int_0^1 |f'(r)|^m r^{Q-1} dr - \left(\frac{Q-m}{m}\right)^m |\wp| \int_0^1 f^m(r) r^{Q-m-1} dr \\
&\quad = |\wp| \int_0^R |g'(s)s'(r(s))|^m r^{Q-1}(s) \frac{ds}{s'(r(s))} \\
&\qquad - \left(\frac{Q-m}{m}\right)^m |\wp| \int_0^R g^m(s) r^{Q-m-1}(s) \frac{ds}{s'(r(s))} \\
&\quad = |\wp| \left(\frac{Q-m}{m-1}\right)^{m-1} \int_0^R |g'(s)|^m s^{m-1} ds \\
&\qquad - \left(\frac{Q-m}{m}\right)^m \frac{m-1}{Q-m} |\wp| \int_0^R \frac{g^m(s)}{s\left(\log \frac{Re}{s}\right)^m} ds \\
&\quad = \frac{|\wp|}{|\widetilde{\wp}|} \left(\frac{Q-m}{m-1}\right)^{m-1} \\
&\qquad \times \left(\int_{B^m(0,R)} |\mathcal{R}g|^m dz - \left(\frac{m-1}{m}\right)^m \int_{B^m(0,R)} \frac{|g|^m}{|z|^m \left(\log \frac{Re}{|z|}\right)^m} dz \right),
\end{aligned}
$$

yielding (2.77). ∎

2.3.3 A family of Hardy–Sobolev type inequalities on quasi-balls

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$. It will be convenient to denote the dilations by $\delta_r(x) = rx$ in the following formulations. Here we discuss another type of Hardy–Sobolev inequalities for functions supported in balls of radius R. As usual, we denote by $B(0,R)$ a quasi-ball of radius R around 0 with respect to the quasi-norm $|\cdot|$.

Theorem 2.3.5 (Another type of Hardy inequalities for $Q \geq 3$). *For each $f \in C_0^\infty(B(0,R)\backslash\{0\})$ and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ we have*

$$
\left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right) \right|^2 dx \right)^{1/2} \leq \frac{2}{Q-2} \left(\int_{B(0,R)} |\mathcal{R}f|^2 dx \right)^{1/2}, \tag{2.78}
$$

and

$$
\begin{aligned}
\left(\int_{B(0,R)} \frac{1}{|x|^2} |f(x)|^2 dx \right)^{\frac12} \leq \left(\frac{Q}{Q-2}\right)^{\frac12} \frac{1}{R} \left(\int_{B(0,R)} |f(x)|^2 dx \right)^{\frac12} \\
+ \frac{2}{Q-2} \left(1 + \left(\frac{Q}{Q-2}\right)^{\frac12}\right) \left(\int_{B(0,R)} |\mathcal{R}f|^2 dx \right)^{\frac12}.
\end{aligned} \tag{2.79}
$$

Remark 2.3.6.

1. Theorem 2.3.5 could have been formulated for functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ choosing $R > 0$ such that supp$f \subset B(0,R)$. The introduction of R into the notation is essential here since the dilated function $f\left(\frac{\delta_R(x)}{|x|}\right)$ appears in the inequality (2.78).
2. In the Euclidean setting with the Euclidean norm inequalities in Theorem 2.3.5 have been studied in [MOW13b].

   In the case $Q = 2$ we have the following inequalities:

Theorem 2.3.7 (Another type of critical Hardy inequality for $Q = 2$)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q = 2$. Then for each $f \in C_0^\infty(B(0,R)\backslash\{0\})$ and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ we have*

$$\left(\int_{B(0,R)} \frac{1}{|x|^2\left|\log\frac{R}{|x|}\right|^2}\left|f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{1/2} \leq 2\left(\int_{B(0,R)} |\mathcal{R}f|^2 dx\right)^{1/2}, \tag{2.80}$$

and

$$\left(\int_{B(0,R)} \frac{|f(x)|^2}{|x|^2\left(1+\left|\log\frac{R}{|x|}\right|^2\right)^2} dx\right)^{1/2} \leq \frac{\sqrt{2}}{R}\left(\int_{B(0,R)} |f(x)|^2\, dx\right)^{1/2} + 2\left(1+\sqrt{2}\right)\left(\int_{B(0,R)} |\mathcal{R}f|^2 dx\right)^{1/2}. \tag{2.81}$$

Proof of Theorem 2.3.5. By the polar decomposition from Proposition 1.2.10, we write $(r,y) = (|x|, \frac{x}{|x|}) \in (0,\infty)\times\wp$ on $\mathbb{G}$, where $\wp$ is the unit quasi-sphere, so that

$$\begin{aligned}
&\int_{B(0,R)} \frac{1}{|x|^2}\left|f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \\
&= \int_0^R \int_\wp |f(\delta_r(y)) - f(\delta_R(y))|^2 r^{Q-3} d\sigma(y) dr \\
&= \frac{1}{Q-2} r^{Q-2} \int_\wp |f(\delta_r(y)) - f(\delta_R(y))|^2 d\sigma(y)\Big|_{r=0}^{r=R} \\
&\quad - \frac{1}{Q-2}\int_0^R r^{Q-2}\left(\frac{d}{dr}\int_\wp |f(\delta_r(y)) - f(\delta_R(y))|^2 d\sigma(y)\right) dr \\
&= -\frac{2}{Q-2}\int_0^R r^{Q-2}\mathrm{Re}\int_\wp (f(\delta_r(y)) - f(\delta_R(y)))\overline{\frac{df(\delta_r(y))}{dr}} d\sigma(y) dr.
\end{aligned}$$

Now using Schwarz' inequality, we obtain

$$\begin{aligned}
&\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \\
&\quad \le \frac{2}{Q-2} \left(\int_0^R \int_{\wp} |f(\delta_r(y)) - f(\delta_R(y))|^2 r^{Q-3} d\sigma(y) dr \right)^{1/2} \\
&\qquad \times \left(\int_0^R \int_{\wp} \left| \frac{df(\delta_r(y))}{dr} \right|^2 r^{Q-1} d\sigma(y) dr \right)^{1/2} \\
&\quad = \frac{2}{Q-2} \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{1/2} \left(\int_{B(0,R)} |\mathcal{R} f|^2 dx \right)^{1/2}.
\end{aligned}$$

This implies that

$$\left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{1/2} \le \frac{2}{Q-2} \left(\int_{B(0,R)} |\mathcal{R} f|^2 dx \right)^{1/2},$$

that is, the inequality (2.78) is proved. The triangle inequality gives

$$\begin{aligned}
&\left(\int_{B(0,R)} \frac{1}{|x|^2} |f|^2 dx \right)^{\frac{1}{2}} = \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right) + f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{\frac{1}{2}} \\
&\le \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.82}$$

Moreover, we have

$$\begin{aligned}
&\left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{\frac{1}{2}} = \left(\int_0^R \int_{\wp} |f(\delta_R(y))|^2 r^{Q-3} d\sigma(y) dr \right)^{\frac{1}{2}} \\
&\quad = \left(\frac{R^{Q-2}}{Q-2} \int_{\wp} |f(\delta_R(y))|^2 d\sigma(y) \right)^{\frac{1}{2}} \\
&\quad = \left(\frac{R^{Q-2}}{Q-2} \frac{Q}{R^Q} \int_0^R \int_{\wp} |f(\delta_R(y))|^2 r^{Q-1} d\sigma(y) dr \right)^{\frac{1}{2}} \\
&\quad = \left(\frac{Q}{Q-2} \right)^{\frac{1}{2}} \frac{1}{R} \left(\int_{B(0,R)} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le \left(\frac{Q}{Q-2} \right)^{\frac{1}{2}} \frac{1}{R} \left(\left(\int_{B(0,R)} \left| f\left(\frac{\delta_R(x)}{|x|}\right) - f(x)\right|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B(0,R)} |f|^2 dx \right)^{\frac{1}{2}} \right)
\end{aligned}$$

$$\leq \left(\frac{Q}{Q-2}\right)^{\frac{1}{2}} \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right) - f(x)\right|^2 dx\right)^{\frac{1}{2}}$$
$$+ \left(\frac{Q}{Q-2}\right)^{\frac{1}{2}} \frac{1}{R} \left(\int_{B(0,R)} |f|^2 dx\right)^{\frac{1}{2}},$$

thus,

$$\left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{1/2}$$
$$\leq \left(\frac{Q}{Q-2}\right)^{\frac{1}{2}} \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right) - f(x)\right|^2 dx\right)^{1/2} \tag{2.83}$$
$$+ \left(\frac{Q}{Q-2}\right)^{\frac{1}{2}} \frac{1}{R} \left(\int_{B(0,R)} |f|^2 dx\right)^{1/2}.$$

Combining (2.83) with (2.82) we arrive at

$$\left(\int_{B(0,R)} \frac{1}{|x|^2} |f|^2 dx\right)^{1/2}$$
$$\leq \left(1 + \left(\frac{Q}{Q-2}\right)^{1/2}\right) \left(\int_{B(0,R)} \frac{1}{|x|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{1/2}$$
$$+ \left(\frac{Q}{Q-2}\right)^{1/2} \frac{1}{R} \left(\int_{B(0,R)} |f|^2 dx\right)^{1/2}.$$

Now by using (2.78) we arrive at (2.79). $\square$

Proof of Theorem 2.3.7. By using polar coordinates $(r,y) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times \wp$ on $\mathbb{G}$, where $\wp$ is the unit quasi-sphere, one calculates

$$\int_{B(0,R)} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx$$
$$= \int_0^R \int_{\wp} |f(\delta_r(y)) - f(\delta_R(y))|^2 \frac{1}{r\,(\log(R/r))^2} d\sigma(y) dr$$
$$= \frac{1}{\log(R/r)} \int_{\wp} |f(\delta_r(y)) - f(\delta_R(y))|^2 d\sigma(y) \Big|_{r=0}^{r=R}$$
$$- \int_0^R \frac{1}{\log(R/r)} \left(\frac{d}{dr} \int_{\wp} |f(\delta_r(y)) - f(\delta_R(y))|^2 d\sigma(y)\right) dr$$

$$= -2\int_0^R \frac{1}{\log(R/r)} \mathrm{Re} \int_{\wp} (f(\delta_r(y)) - f(\delta_R(y))) \overline{\frac{df(\delta_r(y))}{dr}} d\sigma(y) dr.$$

Here we have used the fact that

$$\log(R/r) = \log\left(1 + \left(\frac{R}{r} - 1\right)\right) \geq \frac{R}{r} - 1 = \frac{R-r}{r},$$

and that

$$|f(\delta_r(y)) - f(\delta_R(y))|^2 \leq C|R-r|^2.$$

Using Schwarz's inequality we obtain

$$\begin{aligned}
&\int_{B(0,R)} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx \\
&\leq 2\left(\int_0^R \int_{\wp} \frac{1}{r\left(\log(R/r)\right)^2} |f(\delta_r(y)) - f(\delta_R(y))|^2 d\sigma(y) dr\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^R \int_{\wp} \left|\frac{df(\delta_r(y))}{dr}\right|^2 r d\sigma(y) dr\right)^{\frac{1}{2}} \\
&= 2\left(\int_{B(0,R)} \frac{1}{|x|^2 |\log(R/|x|)|^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{B(0,R)} |\mathcal{R} f|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

It completes the proof of (2.80). To show (2.81) we calculate

$$\begin{aligned}
&\left(\int_{B(0,R)} \frac{1}{|x|^2\left(1 + |\log(R/|x|)|\right)^2} |f(x)|^2 dx\right)^{1/2} \\
&\leq \left(\int_{B(0,R)} \frac{1}{|x|^2\left(1 + |\log(R/|x|)|\right)^2} \left| f(x) - f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{1/2} \\
&\quad + \left(\int_{B(0,R)} \frac{1}{|x|^2\left(1 + |\log(R/|x|)|\right)^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{1/2}.
\end{aligned} \tag{2.84}$$

Moreover, we have

$$\begin{aligned}
&\left(\int_{B(0,R)} \frac{1}{|x|^2\left(1 + |\log(R/|x|)|\right)^2} \left| f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^R \int_{\wp} \frac{1}{r\left(1 + |\log(R/r)|\right)^2} |f(\delta_R(y))|^2 d\sigma(y) dr\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^R \frac{1}{r\left(1+|\log(R/r)|\right)^2} dr \int_{\wp} |f(\delta_R(y))|^2 d\sigma(y) \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{1+|\log(R/r)|}\Bigg|_0^R \int_{\wp} |f(\delta_R(y))|^2 d\sigma(y) \right)^{\frac{1}{2}} \\
&= \left(\int_{\wp} |f(\delta_R(y))|^2 d\sigma(y) \right)^{\frac{1}{2}} = \left(\frac{2}{R^2} \int_0^R \int_{\wp} |f(\delta_R(y))|^2 r d\sigma(y) dr \right)^{\frac{1}{2}} \\
&= \left(\frac{2}{R^2} \right)^{\frac{1}{2}} \left(\int_{B(0,R)} \left| f\left(\frac{\delta_R(x)}{|x|} \right) \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}}{R} \left(\left(\int_{B(0,R)} \left| f\left(\frac{\delta_R(x)}{|x|} \right) - f(x) \right|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B(0,R)} |f(x)|^2 dx \right)^{\frac{1}{2}} \right) \\
&\le \sqrt{2} \left(\int_{B(0,R)} \frac{1}{|x|^2\left(1+|\log(R/|x|)|\right)^2} \left| f\left(\frac{\delta_R(x)}{|x|} \right) - f(x) \right|^2 dx \right)^{\frac{1}{2}} \\
&+ \frac{\sqrt{2}}{R} \left(\int_{B(0,R)} |f(x)|^2 dx \right)^{\frac{1}{2}},
\end{aligned}
$$

where we use the simple inequality

$$
\frac{1}{R^2} \le \frac{1}{r^2(1+\log(R/r))^2}, \quad r \in (0,R).
$$

Therefore, we obtain

$$
\begin{aligned}
&\left(\int_{B(0,R)} \frac{1}{|x|^2\left(1+|\log(R/|x|)|\right)^2} \left| f\left(\frac{\delta_R(x)}{|x|} \right) \right|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le \sqrt{2} \left(\int_{B(0,R)} \frac{1}{|x|^2\left(1+|\log(R/|x|)|\right)^2} \left| f\left(\frac{\delta_R(x)}{|x|} \right) - f(x) \right|^2 dx \right)^{\frac{1}{2}} \\
&\qquad + \frac{\sqrt{2}}{R} \left(\int_{B(0,R)} |f(x)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.85}
$$

Combining (2.85) with (2.84) we arrive at

$$
\left(\int_{B(0,R)} \frac{1}{|x|^2\left(1+|\log(R/|x|)|\right)^2} |f(x)|^2 dx \right)^{\frac{1}{2}}
$$

$$\leq (1+\sqrt{2})\left(\int_{B(0,R)} \frac{1}{|x|^2\left(1+|\log(R/|x|)|\right)^2}\left|f(x)-f\left(\frac{\delta_R(x)}{|x|}\right)\right|^2 dx\right)^{\frac{1}{2}}$$
$$+\frac{\sqrt{2}}{R}\left(\int_{B(0,R)}|f(x)|^2 dx\right)^{\frac{1}{2}}.$$

Finally, using (2.80) we obtain (2.81). □

2.3.4 Improved Hardy inequalities on quasi-balls

For $p = Q = 2$ and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ we have the following refinement of Theorem 2.3.7 with an estimate for a remainder.

Theorem 2.3.8 (Remainder estimate in critical Hardy inequality for $Q = 2$)**.** *We have*

$$\int_{B^2(0,R)}|\mathcal{R}f|^2 dx-\frac{1}{4}\int_{B^2(0,R)}\frac{|f(|x|)|^2}{|x|^2\left(\log\frac{R}{|x|}\right)^2}dx \tag{2.86}$$
$$\geq \frac{4}{R^2|\widehat{\sigma}|}\sup_{\nu>0}\nu^{-4}\left|\int_{B^2(0,R)} f(|x|)\frac{\nu-\log\frac{R}{|x|}}{|x|\left(\log\frac{R}{|x|}\right)^{\frac{1}{2}}}\left(\frac{R}{|x|}\right)^{1-\frac{1}{\nu}}dx\right|^2,\ \forall\nu>0,$$

for all real-valued radial functions $f \in C_0^\infty(B^2(0,R)\backslash\{0\})$, where $B^2(0,R)$ and $|\widehat{\sigma}|$ are 2-dimensional quasi-ball with radius R and 1-dimensional surface measure of the unit sphere, respectively.

Proof of Theorem 2.3.8. Let us define the new function $g = g(x)$ on $B^2(0,R)$ as

$$g(r) := \left(\log\frac{R}{r}\right)^{-\frac{1}{2}} f(r),\ r=|x|,\ x\in B^2(0,R).$$

One calculates

$$I_1 := \int_{B^2(0,R)}|\mathcal{R}f|^2 dx-\frac{1}{4}\int_{B^2(0,R)}\frac{|f|^2}{|x|^2\left(\log\frac{R}{|x|}\right)^2}dx$$
$$=|\widehat{\sigma}|\int_0^R (f'(r))^2 r dr-\frac{|\widehat{\sigma}|}{4}\int_0^R\frac{|f(r)|^2}{r^2\left(\log\frac{R}{r}\right)^2}rdr$$
$$=|\widehat{\sigma}|\int_0^R\left(-\frac{1}{2}\left(\log\frac{R}{r}\right)^{-\frac{1}{2}}\frac{g(r)}{r}+\left(\log\frac{R}{r}\right)^{\frac{1}{2}}g'(r)\right)^2 rdr-\frac{|\widehat{\sigma}|}{4}\int_0^R\frac{|g(r)|^2}{r\log\frac{R}{r}}dr$$
$$=-|\widehat{\sigma}|\int_0^R g(r)g'(r)dr+|\widehat{\sigma}|\int_0^R|g'(r)|^2 r\log\frac{R}{r}dr$$

$$= -\frac{|\widehat{\sigma}|}{2}\int_0^R (g^2(r))' dr + \frac{R^2|\widehat{\sigma}|}{4}\left(\int_0^R |g'(r)|^2 \frac{4}{R^2} r\log\frac{R}{r}dr\right),$$

and applying $g(0) = g(R) = 0$, we obtain

$$I_1 = \frac{R^2|\widehat{\sigma}|}{4}\left(\int_0^R |g'(r)|^2 \frac{4}{R^2} r\log\frac{R}{r}dr\right).$$

Taking into account that $\frac{4}{R^2}\int_0^R r\log\frac{R}{r}dr = 1$ and using Hölder's inequality, we get

$$\begin{aligned}\left(\int_0^R |g'(r)|^2 \frac{4}{R^2} r\log\frac{R}{r}dr\right)^{\frac{1}{2}} &= \left(\int_0^R |g'(r)|^2 \frac{4}{R^2} r\log\frac{R}{r}dr\right)^{\frac{1}{2}}\left(\frac{4}{R^2}\int_0^R r\log\frac{R}{r}dr\right)^{\frac{1}{2}}\\ &\geq \int_0^R |g'(r)|\frac{4}{R^2} r\log\frac{R}{r}dr.\end{aligned}$$

It follows that

$$I_1 \geq \frac{R^2|\widehat{\sigma}|}{4}\left(\int_0^R |g'(r)|\frac{4}{R^2} r\log\frac{R}{r}dr\right)^2 \geq \frac{4|\widehat{\sigma}|}{R^2}\left|\int_0^R g'(r) r\log\frac{R}{r}dr\right|^2. \tag{2.87}$$

Using $g(0) = g(R) = 0$, we have

$$\begin{aligned}\int_0^R g'(r) r\log\frac{R}{r}dr &= -\int_0^R g(r)\left(\log\frac{R}{r} - 1\right)dr = \int_0^R f(r)\frac{1-\log\frac{R}{r}}{\left(\log\frac{R}{r}\right)^{\frac{1}{2}}}dr\\ &= \frac{1}{|\widehat{\sigma}|}\int_{B^2(0,R)} f(x)\frac{1-\log\frac{R}{|x|}}{|x|\left(\log\frac{R}{|x|}\right)^{\frac{1}{2}}}dx.\end{aligned}$$

Putting this in (2.87), we arrive at

$$I_1 \geq \frac{4}{R^2|\widehat{\sigma}|}\left|\int_{B^2(0,R)} f(x)\frac{1-\log\frac{R}{|x|}}{|x|\left(\log\frac{R}{|x|}\right)^{\frac{1}{2}}}dx\right|^2.$$

Now we note that I_1 is invariant under the scaling $f \longmapsto f_\nu(r) = \nu^{-\frac{1}{2}} f(R^{1-\nu}r^\nu)$. Then setting

$$I_2(f) := \int_{B^2(0,R)} f(x)\frac{1-\log\frac{R}{|x|}}{|x|\left(\log\frac{R}{|x|}\right)^{\frac{1}{2}}}dx = |\widehat{\sigma}|\int_0^R f(r)\frac{1-\log\frac{R}{r}}{\left(\log\frac{R}{r}\right)^{\frac{1}{2}}}dr,$$

one obtains

$$I_1 \geq \frac{4}{R^2|\widehat{\sigma}|}|I_2(f_\nu)|^2 \tag{2.88}$$

for any $\nu > 0$. On the other hand, we have

$$I_2(f_\nu) = |\widehat{\sigma}| \int_0^R f_\nu(r) \frac{1-\log\frac{R}{r}}{\left(\log\frac{R}{r}\right)^{\frac{1}{2}}} dr = |\widehat{\sigma}|\nu^{-\frac{1}{2}} \int_0^R f(r^\nu R^{1-\nu}) \frac{1-\log\frac{R}{r}}{\left(\log\frac{R}{r}\right)^{\frac{1}{2}}} dr.$$

Using a change of variable $s = r^\nu R^{1-\nu}$, $dr = \frac{1}{\nu}\left(\frac{R}{s}\right)^{\frac{\nu-1}{\nu}} ds$, one calculates

$$\begin{aligned} I_2(f_\nu) &\geq |\widehat{\sigma}|\nu^{-\frac{1}{2}} \int_0^R f(s) \frac{1-\log\left(\frac{R}{s}\right)^{\frac{1}{\nu}}}{\left(\log\left(\frac{R}{s}\right)^{\frac{1}{\nu}}\right)^{\frac{1}{2}}} \frac{1}{\nu}\left(\frac{R}{s}\right)^{1-\frac{1}{\nu}} ds \\ &= |\widehat{\sigma}|\nu^{-2} \int_0^R f(s) \frac{\nu-\log\frac{R}{s}}{\left(\log\frac{R}{s}\right)^{\frac{1}{2}}} \left(\frac{R}{s}\right)^{1-\frac{1}{\nu}} ds \\ &= \nu^{-2} \int_{B^2(0,R)} f(x) \frac{\nu - \log\frac{R}{|x|}}{|x|\left(\log\frac{R}{|x|}\right)^{\frac{1}{2}}} \left(\frac{R}{|x|}\right)^{1-\frac{1}{\nu}} dx. \end{aligned} \tag{2.89}$$

The estimates (2.88) and (2.89) imply (2.86). □

Theorem 2.3.9 (Remainder estimate in critical Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 2$. Let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $q > 0$ be such that*

$$\alpha = \alpha(q, L) := \frac{Q-1}{Q}q + L + 2 \leq Q,$$

for $-1 < L < Q-2$. Then for all real-valued positive non-increasing radial functions $u \in C_0^\infty(B(0,R))$ we have

$$\begin{aligned} &\int_{B(0,R)} |\mathcal{R}u|^Q dx - \left(\frac{Q-1}{Q}\right)^Q \int_{B(0,R)} \frac{|u(x)|^Q}{|x|^Q\left(\log\frac{Re}{|x|}\right)^Q} dx \\ &\qquad \geq |\wp|^{1-\frac{Q}{q}} C^{\frac{Q}{q}} \left(\int_{B(0,R)} \frac{|u(x)|^q}{|x|^Q\left(\log\frac{Re}{|x|}\right)^\alpha} dx\right)^{\frac{Q}{q}}, \end{aligned} \tag{2.90}$$

where $|\wp|$ is the measure of the unit quasi-sphere in $\mathbb{G}$ and

$$\begin{aligned} C^{-1} = C(L,Q,q)^{-1} &:= \int_0^1 s^L \left(\log\frac{1}{s}\right)^{\frac{Q-1}{Q}q} ds \\ &= (L+1)^{-\left(\frac{Q-1}{Q}q+1\right)} \Gamma\left(\frac{Q-1}{Q}q+1\right), \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

Proof of Theorem 2.3.9. As in previous proofs we set

$$v(s) = \left(\log \frac{Re}{r}\right)^{-\frac{Q-1}{Q}} u(r), \quad \text{where} \quad r = |x|, s = s(r) = \left(\log \frac{Re}{r}\right)^{-1},$$

$$s'(r) = \frac{s(r)}{r \log \frac{Re}{r}} \geq 0.$$

We have $v(0) = v(1) = 0$ since $u(R) = 0$ and, moreover,

$$u'(r) = -\left(\frac{Q-1}{Q}\right)\left(\log \frac{Re}{r}\right)^{-\frac{1}{Q}} \frac{v(s(r))}{r} + \left(\log \frac{Re}{r}\right)^{\frac{Q-1}{Q}} v'(s(r))s'(r) \leq 0.$$

It is straightforward to calculate that

$$\begin{aligned}
I &:= \int_{B(0,R)} |\mathcal{R}u|^Q dx - \left(\frac{Q-1}{Q}\right)^Q \int_{B(0,R)} \frac{|u|^Q}{|x|^Q \left(\log \frac{Re}{|x|}\right)^Q} dx \\
&= |\wp| \int_0^R |u'(r)|^Q r^{Q-1} dr - \left(\frac{Q-1}{Q}\right)^Q |\wp| \int_0^R \frac{|u(r)|^Q}{r \left(\log \frac{Re}{r}\right)^Q} dr \\
&= |\wp| \int_0^R \left(\frac{Q-1}{Q}\left(\log \frac{Re}{r}\right)^{-\frac{1}{Q}} \frac{v(s(r))}{r} - \left(\log \frac{Re}{r}\right)^{\frac{Q-1}{Q}} v'(s(r))s'(r)\right)^Q r^{Q-1} dr \\
&\quad - \left(\frac{Q-1}{Q}\right)^Q |\wp| \int_0^R \frac{|u(r)|^Q}{r \left(\log \frac{Re}{r}\right)^Q} dr.
\end{aligned}$$

By applying the third relation in Lemma 2.4.2 with

$$a = \frac{Q-1}{Q}\left(\log \frac{Re}{r}\right)^{-\frac{1}{Q}} \frac{v(s(r))}{r} \quad \text{and} \quad b = \left(\log \frac{Re}{r}\right)^{\frac{Q-1}{Q}} v'(s(r))s'(r),$$

and dropping $a^Q \geq 0$ as well as using the boundary conditions $v(0) = v(1) = 0$, we get

$$\begin{aligned}
I \geq & -|\wp| Q \left(\frac{Q-1}{Q}\right)^{Q-1} \int_0^R v(s(r))^{Q-1} v'(s(r))s'(r) dr \\
& + |\wp| \int_0^R |v'(s(r))|^Q (s'(r))^Q \left(r \log \frac{Re}{r}\right)^{Q-1} dr \\
= & -|\wp| Q \left(\frac{Q-1}{Q}\right)^{Q-1} \int_0^R v(s(r))^{Q-1} v'(s(r))s'(r) dr \\
& + |\wp| \int_0^R |v'(s(r))|^Q \frac{1}{r^Q \left(\log \frac{Re}{r}\right)^{2Q}} \left(r \log \frac{Re}{r}\right)^{Q-1} dr
\end{aligned} \tag{2.91}$$

$$
\begin{aligned}
&= -|\wp|Q\left(\frac{Q-1}{Q}\right)^{Q-1}\int_0^R v(s(r))^{Q-1}v'(s(r))s'(r)dr \\
&\quad + |\wp|\int_0^R |v'(s(r))|^Q s(r)^{Q-1}s'(r)dr \\
&= -|\wp|Q\left(\frac{Q-1}{Q}\right)^{Q-1}\int_0^1 v(s)^{Q-1}v'(s)ds + |\wp|\int_0^1 |v'(s)|^Q s^{Q-1}ds \\
&= |\wp|\int_0^1 |v'(s)|^Q s^{Q-1}ds.
\end{aligned}
$$

Moreover, by using the inequality

$$
\begin{aligned}
|v(s)| = \left|\int_s^1 v'(t)dt\right| &= \left|\int_s^1 v'(t)t^{\frac{Q-1}{Q}-\frac{Q-1}{Q}}dt\right| \\
&\le \left(\int_0^1 |v'(t)|^Q t^{Q-1}dt\right)^{\frac{1}{Q}}\left(\log\frac{1}{s}\right)^{\frac{Q-1}{Q}},
\end{aligned}
$$

we obtain

$$
\int_0^1 |v(s)|^q s^L ds \le \left(\int_0^1 |v'(s)|^Q s^{Q-1}ds\right)^{\frac{q}{Q}}\int_0^1 s^L\left(\log\frac{1}{s}\right)^{\frac{Q-1}{Q}q}ds
$$

for $-1 < L < Q-2$. Thus, we have

$$
\int_0^1 |v'(s)|^Q s^{Q-1}ds \ge C^{\frac{q}{Q}}\left(\int_0^1 |v(s)|^q s^L ds\right)^{\frac{Q}{q}}. \tag{2.92}
$$

Now it follows from (2.91) and (2.92) that

$$
\begin{aligned}
I &\ge |\wp|C^{\frac{Q}{q}}\left(\int_0^1 |v(s)|^q s^L ds\right)^{\frac{Q}{q}} = |\wp|C^{\frac{Q}{q}}\left(\int_0^R \frac{|u(r)|^q}{r\left(\log\frac{Re}{r}\right)^{\alpha}}dr\right)^{\frac{Q}{q}} \\
&= |\wp|^{1-\frac{Q}{q}}C^{\frac{Q}{q}}\left(\int_0^R \frac{|u(x)|^q}{|x|^Q\left(\log\frac{Re}{|x|}\right)^{\alpha}}dx\right)^{\frac{Q}{q}},
\end{aligned}
$$

where $\alpha = \alpha(q, L) = \frac{Q-1}{Q}q + L + 2$. The proof is complete. $\square$

2.4 Stability of Hardy inequalities

Expressions for the remainder in an estimate in terms of the distance function to the set of extremisers are sometimes called the stability estimates in the literature. The purpose of this section is to discuss such estimates for the Hardy inequalities. Several results of such a type have been established in the Euclidean space $\mathbb{R}^n$ in [San18, ST17, ST18a, ST15, ST16]. In the following section our presentation on homogeneous groups follows [RS18].

2.4.1 Stability of Hardy inequalities for radial functions

Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. We note that although sharp constants in the Hardy inequalities are not achieved the formal extremisers of such inequalities are given by homogeneous functions. To this end let us denote

$$f_\alpha(x) := |x|^{-\frac{Q-p-\alpha p}{p}} \tag{2.93}$$

for $-\infty < \alpha < \frac{Q-p}{p}$, and

$$d_R(f,g) := \left(\int_{\mathbb{G}} \frac{|f(x)-g(x)|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(\alpha+1)}} dx \right)^{\frac{1}{p}} \tag{2.94}$$

for functions f, g for which the integral in (2.94) is finite. We start with the case of radially symmetric functions.

Theorem 2.4.1 (Stability of Hardy inequalities for radially symmetric functions)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let*

$$2 \leq p < Q \quad \textit{and} \quad -\infty < \alpha < \frac{Q-p}{p}.$$

Then for all radial complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have

$$\begin{aligned} \int_{\mathbb{G}} \frac{|\mathcal{R}f(x)|^p}{|x|^{\alpha p}} dx - \left(\frac{Q-p-\alpha p}{p}\right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx \\ \geq c_p \left(\frac{p-1}{p}\right)^p \sup_{R>0} d_R(f, c_f(R) f_\alpha)^p, \end{aligned} \tag{2.95}$$

where $c_f(R) := R^{\frac{Q-p-\alpha p}{p}} \widetilde{f}(R)$ with $f(x) = \widetilde{f}(r)$, $|x| = r$, $\mathcal{R} := \frac{d}{d|x|}$ is the radial derivative, c_p is defined in Lemma 2.3.3, i.e.,

$$c_p = \min_{0<t\leq 1/2} ((1-t)^p - t^p + pt^{p-1}),$$

and f_α and $d_R(\cdot,\cdot)$ are defined in (2.93) and (2.94), respectively.

Proof of Theorem 2.4.1. Since $p \geq 2$, as in (2.72) in the proof of Theorem 2.3.1, we have

$$\begin{aligned} J(f) &= \int_{\mathbb{G}} \frac{|\mathcal{R}f|^p}{|x|^{\alpha p}} dx - \left(\frac{Q-p-\alpha p}{p}\right)^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(\alpha+1)}} dx \\ &\geq c_p |\wp| \int_0^\infty \left|\frac{d}{dr}\widetilde{g}\right|^p r^{p-1} dr = c_p \int_{\mathbb{G}} \left|\frac{d}{dr} g\right|^p |x|^{p-Q} dx. \end{aligned}$$

By Corollary 2.2.8 with $\gamma = p$, we obtain

$$\begin{aligned} J(f) &\geq c_p \int_{\mathbb{G}} |\mathcal{R}g|^p |x|^{p-Q} dx \geq c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{G}} \frac{\left|g(x) - g(\frac{Rx}{|x|})\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^Q} dx \\ &= c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{G}} \frac{\left||x|^{\frac{Q-p-\alpha p}{p}} f(x) - R^{\frac{Q-p-\alpha p}{p}} f(\frac{Rx}{|x|})\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^Q} dx \end{aligned}$$

for any $R > 0$. Here using $f(x) = \widetilde{f}(r)$, $r = |x|$, we can estimate

$$\begin{aligned} J(f) &\geq c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{G}} \frac{\left|f(x) - R^{\frac{Q-p-\alpha p}{p}} \widetilde{f}(R) |x|^{-\frac{Q-p-\alpha p}{p}}\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(\alpha+1)}} dx \\ &= c_p \left(\frac{p-1}{p}\right)^p \int_{\mathbb{G}} \frac{\left|f(x) - c_f(R) |x|^{-\frac{Q-p-\alpha p}{p}}\right|^p}{\left|\log \frac{R}{|x|}\right|^p |x|^{p(\alpha+1)}} dx, \end{aligned}$$

yielding (2.95). □

2.4.2 Stability of Hardy inequalities for general functions

Here we discuss the stability of L^p-Hardy inequalities for functions which do not have to be radially symmetric. Before discussing the stability estimates for non-radial functions let us recall the following relations.

Lemma 2.4.2. *Let $a, b \in \mathbb{R}$. Then*

(i) *We have*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab, \quad p \geq 1.$$

(ii) *There exists a constant $C = C(p) > 0$ such that*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab + C|b|^p, \quad p \geq 2.$$

(iii) *If $a \geq 0$ and $a - b \geq 0$, then*

$$(a-b)^p + pa^{p-1}b - a^p \geq |b|^p, \quad p \geq 2.$$

Proof of Lemma 2.4.2. One has (see, e.g., [Lin90]) the inequality

$$|d|^p \geq |c|^p + p|c|^{p-2}c(d-c), \quad p \geq 1,$$

which is the condition for the convexity of the function $|x|^p$. Now, taking $a = c$ and $d = a - b$ we get the first relation. Furthermore, consider the function

$$g(s) := |1-s|^p - |s|^p + p|s|^{p-2}s, \quad s \in \mathbb{R}.$$

To show (ii), it is sufficient to prove that $g(s) \geq C > 0$, and then take $s = \frac{a}{b}$. Let $s \geq 1$. Then for $p \geq 2$ we have

$$g'(s) = p((t-1)^{p-1} - t^{p-1}) + p(p-1)s^{p-2} = p(p-1)(s^{p-2} - t^{p-2}) \geq 0,$$

by the mean value theorem for the function x^{p-2}, $x \geq 0$. Thus, we have

$$g(s) \geq g(1) = p - 1$$

for all $s \geq 1$. Similarly, we get $g(s) \geq g(0) = 1$ for all $s \leq 1$. Let now $0 \leq s \leq 1$. Setting

$$C_p := \min_{0 \leq t \leq 1} f(t),$$

we assume that $C_p = f(s_0)$ for some $0 \leq s_0 \leq 1$. The first relation implies that $C_p \geq 0$. If $C_p = 0$, then we have $f(s_0) = 0$ and $\frac{s_0 - 1}{p} f'(s_0) = 0$ which implies $s_0 = 0$, contradicting that $f(0) = 1$. Therefore, we have $g(s) \geq C_p > 0$. This proves (ii). The Taylor formula yields that

$$(a-b)^p + pa^{p-1}b - a^p \geq |b|^p = p(p-1)b^2 \int_0^1 (1-t)(a-\tau b)^{p-2} d\tau.$$

Thus, if $b \leq 0$, then $a - \tau b \geq \tau |b|$, which implies (iii) in this case. On the other hand, if $0 \leq b$, then $a - \tau b \geq (1-\tau)|b|$ which implies (iii) in this case as well. $\square$

To formulate the following result, for $R > 0$, let us set

$$d_H(u;R) := \left(\int_{\mathbb{G}} \frac{\left| u(x) - R^{\frac{Q-p}{p}} u\left(R\frac{x}{|x|}\right) |x|^{-\frac{Q-p}{p}} \right|^p}{|x|^p |\log \frac{R}{|x|}|^p} dx \right)^{\frac{1}{p}}.$$

Then we have the following stability property.

Theorem 2.4.3 (Stability of Hardy inequalities for general real-valued functions)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $2 \leq p < Q$. Then there exists a constant $C > 0$ such that for all real-valued functions $u \in C_0^\infty(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} |\mathcal{R}u|^p \, dx - \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{G}} \frac{|u|^p}{|x|^p} dx \geq C_p \sup_{R>0} d_H^p(u;R). \tag{2.96}$$

Proof of Theorem 2.4.3. Let $x=(r,y)=(|x|,\frac{x}{|x|})\in(0,\infty)\times\wp$ on $\mathbb{G}$, where $\wp$ is the unit quasi-sphere $\wp:=\{x\in\mathbb{G}:\,|x|=1\}$, and

$$v(ry):=r^{\frac{Q-p}{p}}u(ry),$$

where $u\in C_0^\infty(\mathbb{G})$. It follows that $v(0)=0$ and that $\lim_{r\to\infty}v(ry)=0$ for $y\in\wp$ since u is compactly supported. Using the polar decomposition from Proposition 1.2.10 and integrating by parts, we get

$$\begin{aligned}
D:&=\int_{\mathbb{G}}|\mathcal{R}u|^p\,dx-\left(\frac{Q-p}{p}\right)^p\int_{\mathbb{G}}\frac{|u|^p}{|x|^p}dx\\
&=\int_\wp\int_0^\infty\left|-\frac{\partial}{\partial r}u(ry)\right|^p r^{Q-1}-\left(\frac{Q-p}{p}\right)^p|u(ry)|^p r^{Q-p-1}drdy\\
&=\int_\wp\int_0^\infty\left|\frac{Q-p}{p}r^{-\frac{Q}{p}}v(ry)-r^{-\frac{Q-p}{p}}\frac{\partial}{\partial r}v(ry)\right|^p r^{Q-1}\\
&\quad-\left(\frac{Q-p}{p}\right)^p|v(ry)|^p r^{-1}drdy.
\end{aligned}$$

Now using relation (ii) in Lemma 2.4.2 with the choice $a=\frac{Q-p}{p}r^{-\frac{Q}{p}}v(ry)$ and $b=r^{-\frac{Q-p}{p}}\frac{\partial}{\partial r}v(ry)$, and using the fact that $\int_0^\infty|v|^{p-2}v\left(\frac{\partial}{\partial r}v\right)dr=0$, we obtain

$$\begin{aligned}
D&\geq\int_\wp\int_0^\infty-p\left(\frac{Q-p}{p}\right)^{p-1}|v(ry)|^{p-2}v(ry)\frac{\partial}{\partial r}v(ry)\\
&\quad+C\left|\frac{\partial}{\partial r}v(ry)\right|^p r^{p-1}drdy\\
&=C\int_{\mathbb{G}}|x|^{p-Q}\,|\mathcal{R}v|^p\,dx.
\end{aligned}\tag{2.97}$$

Finally, combining (2.97) and Remark 2.2.9, we arrive at

$$\begin{aligned}
D&\geq C_p\int_{\mathbb{G}}\frac{|v(x)-v(R\frac{x}{|x|})|^p}{|x|^Q|\log\frac{R}{|x|}|^p}dx=C\int_\wp\int_0^\infty\frac{|v(ry)-v(Ry)|^p}{r\left|\log\frac{R}{r}\right|^p}drdy\\
&=C_p\int_\wp\int_0^\infty\frac{|u(ry)-R^{\frac{Q-p}{p}}u(Ry)r^{-\frac{Q-p}{p}}|^p}{r^{1+p-Q}|\log\frac{R}{r}|^p}drdy,
\end{aligned}$$

for any $R>0$. This proves the desired result. $\square$

2.4.3 Stability of critical Hardy inequality

Here we discuss the stability of the critical Hardy inequality, i.e., the L^p-Hardy inequality in the case $p=Q$. For this, let us denote

$$f_{T,R}(x):=T^{\frac{Q-1}{Q}}u\left(Re^{-\frac{1}{T}}\frac{x}{|x|}\right)\left(\log\frac{R}{|x|}\right)^{\frac{Q-1}{Q}}.$$

We also introduce the following 'distance' function:

$$d_{cH}(u;T,R) := \left(\int_{B(0,R)} \frac{|u(x) - f_{T,R}(x)|^Q}{|x|^Q \left|\log \frac{R}{|x|}\right|^Q \left|T \log \frac{R}{|x|}\right|^Q} dx \right)^{\frac{1}{Q}}, \tag{2.98}$$

for some parameter $T > 0$, functions u and $f_{T,R}$ for which the integral in (2.98) is finite.

Theorem 2.4.4 (Stability of critical Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then there exists a constant $C > 0$ such that for all real-valued functions $u \in C_0^\infty(B(0,R))$ we have*

$$\begin{aligned} \int_{B(0,R)} |\mathcal{R}u(x)|^Q \, dx - \left(\frac{Q-1}{Q}\right)^Q \int_{B(0,R)} \frac{|u(x)|^Q}{|x|^Q (\log \frac{R}{|x|})^Q} dx \\ \geq C_p \sup_{T>0} d_{cH}^Q(u;T,R). \end{aligned} \tag{2.99}$$

Proof of Theorem 2.4.4. With polar coordinates $(r,y) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times \wp$ on $\mathbb{G}$, where $\wp$ is the sphere as in (1.12), we have $u(x) = u(ry) \in C_0^\infty(B(0,R))$. In addition, let us set

$$v(sy) := \left(\log \frac{R}{r}\right)^{-\frac{Q-1}{Q}} u(ry), \quad y \in \wp,$$

where

$$s = s(r) := \left(\log \frac{R}{r}\right)^{-1}.$$

Since $u \in C_0^\infty(B(0,R))$ we have $v(0) = 0$ and v has a compact support. Moreover, it is straightforward that

$$\frac{\partial}{\partial r} u(ry) = -\left(\frac{Q-1}{Q}\right)\left(\log \frac{R}{r}\right)^{-\frac{1}{Q}} \frac{v(sy)}{r} + \left(\log \frac{R}{r}\right)^{\frac{Q-1}{Q}} \frac{\partial}{\partial s} v(sy) s'(r).$$

A direct calculation using the polar decomposition in Proposition 1.2.10 gives

$$\begin{aligned} S &:= \int_{B(0,R)} |\mathcal{R}u|^Q \, dx - \left(\frac{Q-1}{Q}\right)^Q \int_{B(0,R)} \frac{|u|^Q}{|x|^Q \left(\log \frac{R}{|x|}\right)^Q} dx \\ &= \int_\wp \left(\int_0^R \left| \frac{\partial}{\partial r} u(ry) \right|^Q r^{Q-1} - \left(\frac{Q-1}{Q}\right)^Q \frac{|u(ry)|^Q}{r \left(\log \frac{R}{r}\right)^Q} dr \right) dy \end{aligned}$$

$$= \int_{\wp}\int_0^R \left(\left| \left(\frac{Q-1}{Q}\right) \left(r\log\frac{R}{r}\right)^{-\frac{1}{Q}} v(sy) + \left(r\log\frac{R}{r}\right)^{\frac{Q-1}{Q}} \frac{\partial}{\partial s} v(sy)s'(r) \right|^Q \right.$$
$$\left. - \left(\frac{Q-1}{Q}\right)^Q \frac{|v(sy)|^Q}{r\log\frac{R}{r}} \right) drdy.$$

Now by applying relation (ii) from Lemma 2.4.2 with the choice

$$a = \frac{Q-1}{Q}\left(r\log\frac{R}{r}\right)^{-\frac{1}{Q}} v(sy) \quad \text{and} \quad b = \left(r\log\frac{R}{r}\right)^{\frac{Q-1}{Q}} \frac{\partial}{\partial s} v(sy)s'(r),$$

and by using the properties that $v(0) = 0$ and $\lim\limits_{r\to\infty} v(ry) = 0$, we obtain

$$\begin{aligned}
S &\geq \int_{\wp}\int_0^R \Bigg(-Q\left(\frac{Q-1}{Q}\right)^{Q-1} |v(sy)|^{Q-2} v(sy) \frac{\partial}{\partial s} v(sy) s'(r) \\
&\qquad + C\left|\frac{\partial}{\partial s} v(sy)\right|^Q (s'(r))^Q \left(r\log\frac{R}{r}\right)^{Q-1} \Bigg) drdy \\
&= \int_{\wp}\int_0^R \Bigg(-Q\left(\frac{Q-1}{Q}\right)^{Q-1} |v(sy)|^{Q-2} v(sy) \frac{\partial}{\partial s} v(sy) s'(r) \\
&\qquad + C\left|\frac{\partial}{\partial s} v(sy)\right|^Q \frac{1}{r^Q\left(\log\frac{R}{r}\right)^{2Q}} \left(r\log\frac{R}{r}\right)^{Q-1} \Bigg) drdy \\
&= \int_{\wp}\int_0^R \Bigg(-Q\left(\frac{Q-1}{Q}\right)^{Q-1} |v(sy)|^{Q-2} v(sy) \frac{\partial}{\partial s} v(sy) s'(r) \\
&\qquad + C\left|\frac{\partial}{\partial s} v(sy)\right|^Q \frac{1}{\left(\log\frac{R}{r}\right)^{Q-1}} s'(r) \Bigg) drdy \\
&= \int_{\wp}\int_0^R \Bigg(-Q\left(\frac{Q-1}{Q}\right)^{Q-1} |v(sy)|^{Q-2} v(sy) \frac{\partial}{\partial s} v(s) \\
&\qquad + C\left|\frac{\partial}{\partial s} v(sy)\right|^Q s^{Q-1} \Bigg) dsdy \\
&= C\int_{\mathbb{G}} |\mathcal{R}v|^Q \, dx,
\end{aligned}$$

that is,

$$S \geq C\int_{\mathbb{G}} |\mathcal{R}v|^Q \, dx. \tag{2.100}$$

According to Remark 2.2.9 for $v \in C_0^\infty(\mathbb{G}\backslash\{0\})$ with $p = Q$ and (2.100), it follows that

$$S \geq C_p \int_{\mathbb{G}} \frac{|v(x) - v(T\frac{x}{|x|})|^Q}{|x|^Q|\log\frac{T}{|x|}|^Q} dx = C_p \int_{\wp}\int_0^\infty \frac{|v(sy) - v(Ty)|^Q}{s|\log\frac{T}{s}|^Q} dsdy$$

$$= C_p \int_{\wp} \int_0^R \frac{\left| \left(\log \frac{R}{r}\right)^{-\frac{Q-1}{Q}} u(ry) - T^{\frac{Q-1}{Q}} u(Re^{-\frac{1}{T}} y) \right|^Q}{r(\log \frac{R}{r}) |\log(T \log \frac{R}{r})|^Q} drdy$$

$$= C_p \int_{\wp} \int_0^R \frac{\left| u(ry) - T^{\frac{Q-1}{Q}} u(Re^{-\frac{1}{T}} y) (\log \frac{R}{r})^{\frac{Q-1}{Q}} \right|^Q}{r(\log \frac{R}{r})^Q |\log(T \log \frac{R}{r})|^Q} drdy.$$

Thus, we arrive at

$$S \geq C_p \int_{B(0,R)} \frac{\left| u(x) - T^{\frac{Q-1}{Q}} u\left(Re^{-\frac{1}{T}} \frac{x}{|x|}\right) \left(\log \frac{R}{|x|}\right)^{\frac{Q-1}{Q}} \right|^Q}{|x|^Q \left|\log \frac{R}{|x|}\right|^Q \left|\log \left(T \log \frac{R}{|x|}\right)\right|^Q} dx$$

for all $T > 0$. This completes the proof of Theorem 2.4.4. □