Chapter 1

Analysis on Homogeneous Groups

In this chapter we provide preliminaries for the analysis on homogeneous groups to make the use of the monograph more self-sufficient. We make a selection of topics which will be playing a role in the subsequent analysis. Thus, we first discuss relevant properties of general Lie groups and algebras and then concentrate on properties of homogeneous groups required for our further analysis. Lastly, we introduce the notion of the Euler operator on homogeneous groups and establish its main properties.

This chapter is not intended to be a comprehensive treatise of homogeneous groups but rather a description of a collection of tools used throughout the book. The theory of homogeneous groups for their use in analysis was developed by Folland and Stein [FS82]. A recent rather comprehensive description of homogeneous groups and their place among nilpotent Lie groups have appeared in [FR16]. We refer to both books for the expositions devoted specifically to homogeneous groups. For some related information we may also refer to Ricci's notes [Ric].

There are many sources with rather comprehensive and deep treatments of nilpotent Lie groups, for example the books by Goodman [Goo76] or Corwin and Greenleaf [CG90]. There are also many books on groups or Lie groups, we can refer for example to [RT10, Part III] for a basic introduction. Therefore, we assume the reader to have some familiarity with the concepts of the Lie groups and Lie algebras.

1.1 Homogeneous groups

In this section we discuss nilpotent Lie algebras and groups in the spirit of Folland and Stein's book [FS82] as well as introduce homogeneous (Lie) groups. For more analysis and details in this direction we refer to the recent open access book [FR16].

Let $\mathfrak{g}$ be a Lie algebra (always assumed real and finite-dimensional), and let $\mathbb{G}$ be the corresponding connected and simply-connected Lie group. The lower

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central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} :=\mathfrak{g}, \quad \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}].$$

If the lower central series of a Lie algebra $\mathfrak{g}$ terminates at 0 in a finite number of steps then this Lie algebra is called *nilpotent*. Moreover, if $\mathfrak{g}_{(s+1)} = \{0\}$ and $\mathfrak{g}_{(s)} \neq \{0\}$, then $\mathfrak{g}$ is said to be *nilpotent of step s*. A Lie group $\mathbb{G}$ is *nilpotent* (of step s) whenever its Lie algebra is nilpotent (of step s). If $\exp : \mathfrak{g} \to \mathbb{G}$ is the exponential map, by the Campbell–Hausdorff formula for $X, Y \in \mathfrak{g}$ sufficiently close to 0 we have

$$\exp X \exp Y = \exp H(X, Y),$$

where $H(X, Y)$, the Campbell–Hausdorff series, is an infinite linear combination of X and Y and their iterated commutators and H is universal, i.e., independent of $\mathfrak{g}$, and that

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \cdots ,$$

where the dots indicate terms of order ≥ 3.

If $\mathfrak{g}$ is nilpotent, the Campbell–Hausdorff series terminates after finitely many terms and defines a polynomial map from $V \times V$ to V, where V is the underlying vector space of $\mathfrak{g}$.

Altogether, we have the following useful properties:

Proposition 1.1.1 (Exponential mapping and Haar measure)**.** *Let* $\mathbb{G}$ *be a connected and simply-connected nilpotent Lie group with Lie algebra* $\mathfrak{g}$*. Then:*

(i) *The exponential map* exp *is a diffeomorphism from* $\mathfrak{g}$ *to* $\mathbb{G}$*. Moreover, if* $\mathbb{G}$ *is identified with* $\mathfrak{g}$ *via* exp*, then the group law* $(x, y) \mapsto xy$ *is a polynomial map.*

(ii) *If* λ *denotes a Lebesgue measure on* $\mathfrak{g}$*, then* $\lambda \circ \exp^{-1}$ *is a bi-invariant Haar measure on* $\mathbb{G}$*.*

Proof of Proposition 1.1.1*.* Part (i) is a direct consequence of the fact that $\mathbb{G}$ is uniquely (up to isomorphism) determined by $\mathfrak{g}$, see, e.g., [FS82, Proposition 1.2], [CG90, Section 1.2] or [FR16, Proposition 1.6.6].

Let us give an argument for Part (ii). Let us denote the lower central series for $\mathfrak{g}$ by

$$\mathfrak{g}_{(1)}, \ldots, \mathfrak{g}_{(m)}, \mathfrak{g}_{(m+1)} = \{0\}$$

and denote

$$n := \dim \mathfrak{g} \quad \text{and} \quad n_j := \dim \mathfrak{g}_{(j)}.$$

Let $X_{n-n_m+1}, \ldots, X_n$ be a basis for $\mathfrak{g}_{(m)}$, and we extend it to a basis

$$X_{n-n_{m-1}+1}, \ldots, X_n$$

for $\mathfrak{g}_{(m-1)}$, and so forth obtaining eventually a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Let $\xi_1, \ldots, \xi_n$ be the dual basis for $\mathfrak{g}^*$, and let $\eta_k := \xi_k \circ \exp^{-1}$. These $\eta_1, \ldots, \eta_n$ are a system of global coordinates on $\mathbb{G}$. By using the Campbell–Hausdorff formula and the construction of the η_k's we obtain

$$\eta_k(xy) = \eta_k(x) + \eta_k(y) + P_k(x,y),$$

where $P_k(x,y)$ depends only on the coordinates $\eta_i(x), \eta_i(y)$ with $i < k$. Thus, with respect to the coordinates η_k, the differentials of the maps $x \mapsto xy$ with fixed y and $y \mapsto xy$ with fixed x are given by lower triangular matrices with only 1 elements on the diagonal, and therefore, each of the determinants is equal to one. This implies that the volume form $d\eta_1 \cdots d\eta_n$ on $\mathbb{G}$, which corresponds to the Lebesgue measure on $\mathfrak{g}$, is left and right invariant. $\square$

Definition 1.1.2 (Dilations on a Lie algebra)**.** A family of *dilations* of a Lie algebra $\mathfrak{g}$ is a family of linear mappings

$$\{\delta_r : r > 0\}$$

from $\mathfrak{g}$ to itself which satisfies:

- the mappings are of the form

  $$\delta_r = \exp(A \log r),$$

  where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues.
- In particular, $\delta_{rs} = \delta_r\delta_s$ for all $r, s > 0$. If $\alpha > 0$ and $\{\delta_r\}$ is a family of dilations on $\mathfrak{g}$, then so is $\{\widetilde{\delta}_r\}$, where

  $$\widetilde{\delta}_r := \delta_{r^\alpha} = \exp(\alpha A \log r).$$

  By adjusting α we can always assume that the minimum eigenvalue of A is equal to 1.

Let $\mathcal{A}$ be the set of eigenvalues of A and denote by $W_a \subset \mathfrak{g}$ the corresponding eigenfunction space of A, where $a \in \mathcal{A}$. Then we have

$$\delta_r X = r^a X \quad \text{for} \quad X \in W_a.$$

If $X \in W_a$ and $Y \in W_b$, then

$$\delta_r[X,Y] = [\delta_r X, \delta_r Y] = r^{a+b}[X,Y]$$

and thus $[W_a, W_b] \subset W_{a+b}$. In particular, since $a \geq 1$ for $a \in \mathcal{A}$, we see that $\mathfrak{g}_{(j)} \subset \bigoplus_{a \geq j} W_a$. Since the set $\mathcal{A}$ is finite, it follows that $\mathfrak{g}_{(j)} = \{0\}$ for j sufficiently large. Thus, we obtain:

Proposition 1.1.3 (Lie algebras with dilations are nilpotent)**.** *If a Lie algebra $\mathfrak{g}$ admits a family of dilations then it is nilpotent.*

However, not all nilpotent Lie algebras admit a dilation structure: an example of a (nine-dimensional) nilpotent Lie algebra that does not allow any compatible family of dilations was constructed by Dyer [Dye70].

Definition 1.1.4 (Graded Lie algebras and groups)**.** A Lie algebra $\mathfrak{g}$ is called *graded* if it is endowed with a vector space decomposition (where all but finitely many of the V_k's are 0)

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j \quad \text{such that} \quad [V_i, V_j] \subset V_{i+j}.$$

Consequently, a Lie group is called *graded* if it is a connected simply-connected Lie group whose Lie algebra is graded.

Definition 1.1.5 (Stratified Lie algebras and groups)**.** A graded Lie algebra $\mathfrak{g}$ is called *stratified* if V_1 generates $\mathfrak{g}$ as an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step m we have

$$\mathfrak{g} = \oplus_{j=1}^{m} V_j, \quad [V_j, V_1] = V_{j+1},$$

and the natural dilations of $\mathfrak{g}$ are given by

$$\delta_r\left(\sum_{k=1}^{m} X_k\right) = \sum_{k=1}^{m} r^k X_k, \quad (X_k \in V_k).$$

Consequently, a Lie group is called *stratified* if it is a connected simply-connected Lie group whose Lie algebra is stratified.

Definition 1.1.6 (Homogeneous groups)**.** Let δ_r be dilations on $\mathbb{G}$. We say that a Lie group $\mathbb{G}$ is a *homogeneous group* if:

a. It is a connected and simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r\}$.
b. The maps $\exp \circ \delta_r \circ \exp^{-1}$ are group automorphism of $\mathbb{G}$.

Since the exponential mapping exp is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$ by Proposition 1.1.1, (i), it induces the corresponding family on $\mathbb{G}$ which we may still call the dilations on $\mathbb{G}$ and denote by δ_r. Thus, for $x \in \mathbb{G}$ we will write $\delta_r(x)$ or abbreviate it writing simply rx.

The origin of $\mathbb{G}$ will be usually denoted by 0.

Now let us give some well-known examples of homogeneous groups.

Example 1.1.7 (Abelian groups)**.** The Euclidean space $\mathbb{R}^n$ is a homogeneous group with dilation given by the scalar multiplication.

Example 1.1.8 (Heisenberg groups)**.** If n is a positive integer, the Heisenberg group $\mathbb{H}^n$ is the group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and whose multiplication is given by

$$(z_1, \ldots, z_n, t)(z_1', \ldots, z_n', t') = \left(z_1 + z_1', \ldots, z_n + z_n', t + t' + 2\mathrm{Im}\sum_{k=1}^{n} z_k \overline{z}_k'\right).$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(z_1, \ldots, z_n, t) = (rz_1, \ldots, rz_n, r^2 t).$$

Example 1.1.9 (Upper triangular groups)**.** Let $\mathbb{G}$ be the group of all $n \times n$ real matrices (a_{ij}) such that $a_{ii} = 1$ for $1 \leq i \leq n$ and $a_{ij} = 0$ when $i > j$. Then $\mathbb{G}$ is a homogeneous group with dilations

$$\delta_r(a_{ij}) = r^{j-i} a_{ij}.$$

These Examples 1.1.7, 1.1.8 and 1.1.9 are all examples of the stratified groups. It is also possible to define other families of dilations on these groups. For instance, on $\mathbb{R}^n$ we can define

$$\delta_r(x_1, \ldots, x_n) = (r^{d_1} x_1, \ldots, r^{d_n} x_n),$$

where $1 = d_1 \leq d_2 \leq \cdots \leq d_n$, and on $\mathbb{H}^n$ we can define

$$\delta_r(x_1 + iy_1, \ldots, x_n + iy_n, t) = (r^{a_1} x_1 + ir^{b_1} y_1, \ldots, r^{a_n} x_n + ir^{b_n} y_n, r^c t),$$

where $\min\{a_1, \ldots, a_n, b_1, \ldots, b_n\} = 1$ and $a_j + b_j = c$ for all j. In general, these dilations do not have to be stratified. However, when we refer to $\mathbb{R}^n$ or $\mathbb{H}^n$ we shall assume that they are equipped with the natural dilations defined in Examples 1.1.7, 1.1.8 unless we state otherwise.

Let $d_1, \ldots, d_n$ be the eigenvalues of A, enumerated in nondecreasing order according to their multiplicity, and let $\overline{d} = \max d_k$. The mappings $\{\delta_r = \exp(A \log r)\}$ give the dilation structure to an n-dimensional homogeneous group $\mathbb{G}$, with

$$1 = d_1 \leq d_2 \leq \cdots \leq d_n = \overline{d}. \tag{1.1}$$

Let us fix a basis $\{X_k\}_{k=1}^n$ of the Lie algebra $\mathfrak{g}$ of the Lie group $\mathbb{G}$ such that

$$AX_k = d_k X_k$$

for each k. Then one can define a standard Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$ by declaring the X_k's to be orthonormal. This norm can be also considered as a function on $\mathbb{G}$ by the formula

$$\|x\| = \|\exp^{-1} x\|. \tag{1.2}$$

The number

$$Q := \sum_{k=1}^{n} d_k = \mathrm{Tr}(A) \tag{1.3}$$

is called the *homogeneous dimension* of $\mathbb{G}$. From now on Q will always denote the homogeneous dimension of $\mathbb{G}$.

1.2 Properties of homogeneous groups

In this section we discuss properties of homogeneous groups that are important for their understanding and that will be also useful for our further analysis. For different further properties of homogeneous and graded groups we can refer the reader to the open access book [FR16, Chapter 3].

1.2.1 Homogeneous quasi-norms

We start by the definition of a quasi-norm.

Definition 1.2.1 (Quasi-norms)**.** Let us define a *homogeneous quasi-norm* on a homogeneous group $\mathbb{G}$ to be a continuous function $x \mapsto |x|$ from $\mathbb{G}$ to $[0, \infty)$ that satisfies

(a) for all $x \in \mathbb{G}$ and $r > 0$:

$$|x^{-1}| = |x| \quad \text{and} \quad |rx| = r|x|.$$

(b) The non-degeneracy:

$$|x| = 0 \quad \text{if and only if} \quad x = 0.$$

Here and elsewhere we denote by $rx = \delta_r x$ the dilation of x induced by the dilations on the Lie algebra through the exponential mapping.

There always exist homogeneous quasi-norms on homogeneous groups. Moreover, there always exist quasi-norms that are C^∞-smooth on $\mathbb{G}\backslash\{0\}$. Let us give such an example. Observe that

$$X = \sum_{k=1}^{n} c_k X_k \in \mathfrak{g} \quad \text{implies} \quad \|\delta_r X\| = \left(\sum_{k=1}^{n} c_k^2 r^{2d_k} \right)^{1/2},$$

where $\|\cdot\|$ is the Euclidean norm from (1.2). We can notice that for $X \neq 0$ the function $\|\delta_r X\|$ is a strictly increasing function of r, and it tends to 0 and ∞ as $r \to 0$ and $r \to \infty$, respectively. Now, for $x = \exp X$, we can define a homogeneous quasi-norm on $\mathbb{G}$ by setting

$$|0| := 0 \quad \text{and} \quad |x| := 1/r \quad \text{for} \quad x \neq 0,$$

where $r = r(X) > 0$ is the unique number such that

$$\|\delta_{r(X)} X\| = 1.$$

By the implicit function theorem and the fact that the Euclidean unit sphere is a C^∞ manifold we see that this function is C^∞ on $\mathbb{G}\backslash\{0\}$.

If $x \in \mathbb{G}$ and $r > 0$ we define the ball of radius r about x by

$$B(x,r) := \{y \in \mathbb{G} : |x^{-1}y| < r\}.$$

It can be noticed that $B(x,r)$ is the left translate by x of $B(0,r)$, which in turn is the image under δ_r of $B(0,1)$.

Lemma 1.2.2 (Closed quasi-balls are compact)**.** *$\overline{B(x,r)}$ is compact for any $x \in \mathbb{G}$ and $r > 0$.*

Proof. Let us define

$$\rho(x) := \sum_{k=1}^{n} \frac{|c_k|}{d_k} \quad \text{for} \quad x = \exp\left(\sum_{k=1}^{n} c_k X_k\right),$$

where d_k are as in (1.1). Then ρ satisfies all the properties of a homogeneous quasi-norm. Obviously $\{x : \rho(x) = 1\}$ is compact and does not contain 0, so the function $x \mapsto |x|$ attains a positive minimum η on it. Since $|rx| = r|x|$ and $\rho(rx) = r\rho(x)$, it follows that $|x| \geq \eta\rho(x)$ for all x and for some $\eta > 0$, and hence that

$$\overline{B(0,\eta)} \subset \{x : \rho(x) \leq 1\}.$$

Thus, $\overline{B(0,\eta)}$ is compact, and it follows by dilation and translation that $\overline{B(x,r)}$ is compact for all $r > 0$, $x \in \mathbb{G}$. $\square$

We can compare the quasi-norms with each other and with the Euclidean norm (1.2).

Proposition 1.2.3 (Quasi-norms and the Euclidean norm)**.** *We have the following properties:*

(1) *Any two homogeneous quasi-norms on a homogeneous group are equivalent.*

(2) *There are the constants $C_1, C_2 > 0$ such that*

$$C_1\|x\| \leq |x| \leq C_2\|x\|^{1/\overline{d}} \quad \textit{for all} \quad |x| \leq 1.$$

Proof. Proof of Part (2). When $y = \exp(\sum c_k X_k)$ we have $\|ry\| = (\sum c_k^2 r^{2d_k})^{1/2}$ and hence

$$r^{\overline{d}}\|y\| \leq \|ry\| \leq r\|y\|$$

for $r \leq 1$. A positive maximum C_1^{-1} and a positive minimum $C_2^{-\overline{d}}$ on $\{y : |y| = 1\}$ are attained by the Euclidean norm $\|y\|$ in view of the compactness in Lemma 1.2.2. Any $x \neq 0$ can be written as $x = |x|y$ where $|y| = 1$, so that for $|x| \leq 1$,

$$\|x\| \leq |x|\|y\| \leq C_1^{-1}|x|, \quad \|x\| \geq |x|^{\overline{d}}\|y\| \geq C_2^{-\overline{d}}|x|^{\overline{d}},$$

completing the proof of Part (2).

Proof of Part (1). Let $|\cdot|_1$ and $|\cdot|_2$ be two homogeneous quasi-norms. By a similar argument to Part (2), we observe that since the ball $\overline{B(0,1)}$ with respect to $|\cdot|_1$ is compact by Lemma 1.2.2, and $|\cdot|_2$ is continuous, we have

$$\left|\frac{x}{|x|_1}\right|_2 \leq C < \infty$$

for all $x \neq 0$. By homogeneity it follows that $|x|_2 \leq C|x|_1$ for all $x \in \mathbb{G}$. Switching the roles of $|\cdot|_1$ and $|\cdot|_2$ we obtain the statement. □

The reason why the homogeneous quasi-norms have the prefix 'quasi' becomes clear from the following proposition that shows that in general the triangle inequality is satisfied only with some constant:

Proposition 1.2.4 (Triangle inequality with constant)**.** *Let $\mathbb{G}$ be a homogeneous group. Then we have the following properties:*

(1) *If $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$, there exists $C > 0$ such that for every $x, y \in \mathbb{G}$, we have*
$$|xy| \leq C(|x| + |y|).$$

(2) *There always exists a homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ which satisfies the triangle inequality (with constant $C = 1$):*
$$|xy| \leq |x| + |y| \tag{1.4}$$
for all $x, y \in \mathbb{G}$.

Proof. Let us prove Part (1). The function $(x, y) \mapsto |xy|$ attains a finite maximum $C > 0$ on the set $\{(x, y) \in \mathbb{G} \times \mathbb{G} : |x| + |y| = 1\}$ which is compact by Lemma 1.2.2. Then, given any $x, y \in \mathbb{G}$, set $r = |x| + |y|$. It follows that

$$|xy| = r|r^{-1}(xy)| = r|(r^{-1}x)(r^{-1}y)| \leq Cr = C(|x| + |y|),$$

completing the proof.

We leave Part (2) without proof, referring to [FR16, Proposition 3.1.38 and Theorem 3.1.39] for the complete argument. □

Proposition 1.2.5. *There exists a constant $C > 0$ such that for every $x \in \mathbb{G}$ and $s \in [0, 1]$, we have*
$$|\exp(s \log(x))| \leq C|x|. \tag{1.5}$$

Proof. Let $x \neq 0$, otherwise (1.5) is trivial. Using the fact that $|\cdot|$ is homogeneous of degree 1, we have

$$\frac{|\exp(s\log(x))|}{|x|} = |\delta_{1/|x|}(\exp(s\log(x))| = |\exp(s\log(\delta_{1/|x|}(x))|.$$

With $|\delta_{1/|x|}(x)| = 1$, it follows that

$$\frac{|\exp(s\log(x))|}{|x|} \le \max_{\xi\in\mathbb{G}:|\xi|=1,s\in[0,1]} \rho(\exp(s\log(\xi))) =: C.$$

Note that C is finite, since the set $\{\xi : |\xi| = 1\}$ is compact (see Lemma 1.2.2) as well as $|\cdot|$, exp and log are all continuous functions. $\square$

The bi-invariant Haar measure on $\mathbb{G}$ comes from the Lebesgue measure on $\mathfrak{g}$ by Proposition 1.1.1. Fixing the normalisation of the Haar measure on $\mathbb{G}$ we require that the Haar measure of $B(0,1)$ is 1. (Thus, if $\mathbb{G} = \mathbb{R}^n$ with the usual Lebesgue measure, our Haar measure is $\Gamma((n+2)/2)/\pi^{n/2}$ times the Lebesgue measure.) The measure of any measurable set $E \subset \mathbb{G}$ will be denoted by $|E|$, and we shall denote the integral of a function f with respect to this measure by $\int_{\mathbb{G}} f dx$ or by $\int_{\mathbb{G}} f(x)dx$, or simply by $\int f$ or by $\int f(x)dx$.

Recalling (1.3), the homogeneous dimension of $\mathbb{G}$ is

$$Q = \sum_{k=1}^{n} d_k = \mathrm{Tr}(A),$$

and we have

$$|\delta_r(E)| = r^Q|E|, \quad d(rx) = r^Q dx. \tag{1.6}$$

In particular, we have $|B(x,r)| = r^Q$ for all $r > 0$ and $x \in \mathbb{G}$.

Definition 1.2.6 (Homogeneous functions and operators)**.** A function f on $\mathbb{G}\backslash\{0\}$ is said to be *homogeneous of degree* λ if it satisfies

$$f \circ \delta_r = r^\lambda f \quad \text{for all} \quad r > 0.$$

We note that for f and g, we have the formula

$$\int_{\mathbb{G}} f(x)\,(g\circ\delta_r)(x)dx = r^{-Q}\int_{\mathbb{G}} (f\circ\delta_{1/r})(x)g(x)dx,$$

given that the integrals exist. Hence we can extend the mapping $f \mapsto f\circ\delta_r$ to distributions by defining, for any distribution f and any test function ϕ, the distribution $f\circ\delta_r$ by

$$\langle f\circ\delta_r, \phi\rangle = r^{-Q}\langle f, \phi\circ\delta_{1/r}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality between functions and distributions. The distribution f is called *homogeneous of degree* λ if it satisfies

$$f \circ \delta_r = r^\lambda f \quad \text{for all} \quad r > 0.$$

Also, a linear operator D on $\mathbb{G}$ is called *homogeneous of degree* λ if it satisfies

$$D(f\circ\delta_r) = r^\lambda (Df)\circ\delta_r \quad \text{for all} \quad r > 0,$$

for any f. If D is a linear operator homogeneous of degree λ and f is a homogeneous function of degree μ, then Df is homogeneous of degree $\mu - \lambda$.

The following extension of the reverse triangle inequality is often useful:

Proposition 1.2.7 (Reverse triangle inequality)**.** *Let f be a homogeneous function of degree λ and of class C^1 on $\mathbb{G}\backslash\{0\}$. Then there is a constant $C>0$ such that we have*

$$|f(xy)-f(x)|\leq C|y||x|^{\lambda-1} \quad \textit{for all} \quad |y|\leq |x|/2.$$

Proof. Suppose that $|x|=1$ and $|y|\leq 1/2$, and we use the fact that both sides of the desired inequality are homogeneous of degree λ. In this case x and xy are bounded, and also bounded away from zero, and the map $y\mapsto xy$ is C^1, so by the usual mean value theorem and Proposition 1.2.3, we obtain

$$|f(xy)-f(x)|\leq C\|y\|\leq C'|y|=C'|y||x|^{\lambda-1},$$

using that both sides of the desired inequality are homogeneous functions of the same degree λ. □

In particular, this proposition can be applied to C^1 homogeneous quasi-norms. Specifically, the combination of Proposition 1.2.4 and Proposition 1.2.7 leads to a constant $\gamma>0$ such that we have

$$|xy|\leq\gamma(|x|+|y|) \quad \text{for all} \quad x,y\in\mathbb{G}, \tag{1.7}$$

$$||xy|-|x||\leq\gamma|y| \quad \text{for all} \quad x,y\in\mathbb{G} \quad \text{with} \quad |y|\leq|x|/2. \tag{1.8}$$

Henceforth, γ will always be called the minimal constant satisfying (1.7) and (1.8). Obviously, $\gamma\geq 1$. We will be using (1.7) and (1.8) without comment in the sequel. The following simple fact will also be useful later:

Lemma 1.2.8 (Peetre type inequality)**.** *For every $x,y\in\mathbb{G}$ and $s>0$, we have*

$$(1+|x|)^s(1+|y|)^{-s}\leq\gamma^s(1+|xy^{-1}|)^s.$$

Proof. Because of $|x|\leq\gamma(|xy^{-1}|+|y|)$ we have

$$1+|x|\leq\gamma(1+|xy^{-1}|)(1+|y|),$$

and we obtain the needed inequality by raising both sides to the sth power. □

Let us now fix the notation for some common function spaces on $\mathbb{G}$. Let $\Omega\subset\mathbb{G}$, and let $C(\Omega)$ ($C_0(\Omega)$) be the space of continuous functions on $\mathbb{G}$ (continuous functions with compact support, respectively). If Ω is open, then $C^{(k)}(\Omega)$ is called the class of k times continuously differentiable functions on Ω,

$$C^\infty(\Omega)=\bigcap_{k=1}^{\infty}C^{(k)}(\Omega) \;\text{ and }\; C_0^\infty(\Omega)=C^\infty(\Omega)\cap C_0(\Omega).$$

When $\Omega = \mathbb{G}$ we shall usually omit mentioning it. If $0 < p \leq \infty$, then L^p will denote the usual Lebesgue space on $\mathbb{G}$. For $0 < p < \infty$ we write

$$\|f\|_p := \left(\int_{\mathbb{G}} |f(x)|^p dx \right)^{1/p},$$

despite the fact that this is not a norm for $p < 1$. However, the map $(f, g) \mapsto \|f - g\|_p^p$ is a metric on L^p for $p < 1$. We recall that if f is a measurable function on $\mathbb{G}$, its *distribution function* $\lambda_f : [0, \infty] \to [0, \infty]$ is defined by

$$\lambda_f(\alpha) := |\{x : |f(x)| > \alpha\}|, \tag{1.9}$$

and its *nonincreasing rearrangement* $f^* : [0, \infty) \to [0, \infty)$ is defined by

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \leq t\}. \tag{1.10}$$

Moreover,

$$\int_{\mathbb{G}} |f(x)|^p dx = -\int_0^\infty \alpha^p d\lambda_f(\alpha) = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha = \int_0^\infty f^*(t)^p dt.$$

For $0 < p < \infty$, the weak-L^p is the space of functions f such that

$$[f]_p := \sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) = \sup_{t > 0} t^{1/p} f^*(t) < \infty.$$

This $[\cdot]_p$ is not a norm but it defines a topology on the weak-L^p space. A subadditive operator which is bounded from L^p to weak L^q is said to be *weak type* (p, q).

1.2.2 Polar coordinates

There is an analogue of polar coordinates on homogeneous groups. We start with the following observation:

Proposition 1.2.9 (Polar decomposition: a special case)**.** *Let f be a locally integrable function on $\mathbb{G}\backslash\{0\}$ and assume that it is homogeneous of degree $-Q$. Then there is a constant μ_f (the 'average value' of f) such that for every $g \in L^1((0, \infty), r^{-1}dr)$, we have*

$$\int_{\mathbb{G}} f(x) g(|x|) dx = \mu_f \int_0^\infty g(r) r^{-1} dr. \tag{1.11}$$

Proof. Define $L_f : (0, \infty) \to \mathbb{C}$ by

$$L_f(r) := \begin{cases} \int_{1 \leq |x| \leq r} f(x) dx & \text{if} \quad r \geq 1, \\ -\int_{r \leq |x| \leq 1} f(x) dx & \text{if} \quad r < 1. \end{cases}$$

By changing the variables $x \mapsto sx$ and using the homogeneity of f, it can be verified that

$$L_f(rs) = L_f(r) + L_f(s)$$

for all $r, s > 0$. From the continuity of L_f, it then follows that

$$L_f = L_f(e) \log r,$$

and we set $\mu_f := L_f(e)$. Then equality (1.11) is obvious when g is the characteristic function of an interval, and it follows in general by taking linear combinations and limits of such functions. $\square$

Proposition 1.2.10 (Polar decomposition)**.** *Let*

$$\wp := \{x \in \mathbb{G} : |x| = 1\} \tag{1.12}$$

be the unit sphere with respect to the homogeneous quasi-norm $|\cdot|$. Then there is a unique Radon measure σ on $\wp$ such that for all $f \in L^1(\mathbb{G})$,

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_\wp f(ry) r^{Q-1} d\sigma(y) dr. \tag{1.13}$$

Proof. Let $\widetilde{f} \in C(\mathbb{G}\backslash\{0\})$ be the homogeneous extension of $f \in C(\wp)$ defined by

$$\widetilde{f}(x) := |x|^{-Q} f(|x|^{-1}x).$$

Then $\widetilde{f}$ satisfies the hypotheses of Proposition 1.2.9. The map $f \mapsto \mu_{\widetilde{f}}$ is clearly a positive linear functional on $C(\wp)$, so it is given by the integration against some Radon measure σ on $\wp$. If $g \in C_0(0,\infty)$ then we have

$$\begin{aligned}\int_{\mathbb{G}} f(|x|^{-1}x) g(|x|) dx &= \int_{\mathbb{G}} \widetilde{f}(x)|x|^Q g(|x|) dx = \mu_{\widetilde{f}} \int_0^\infty r^{Q-1} g(r) dr \\ &= \int_0^\infty \int_\wp f(y) g(r) r^{Q-1} d\sigma(y) dr.\end{aligned}$$

Since linear combination of functions of the form $f(|x|^{-1}x)g(|x|)$ are dense in $L^1(\mathbb{G})$, this completes the existence proof, and from the decomposition it follows that such a measure is necessarily unique. $\square$

Corollary 1.2.11. *Let $C := \sigma(\wp)$. Then if $0 < a < b < \infty$ and $\alpha \in \mathbb{C}$, we have*

$$\int_{a<|x|<b} |x|^{\alpha - Q} dx = \begin{cases} C\alpha^{-1}(b^\alpha - a^\alpha) & \textit{if} \quad \alpha \neq 0, \\ C \log(b/a) & \textit{if} \quad \alpha = 0. \end{cases}$$

Corollary 1.2.12. *Let f be a measurable function on $\mathbb{G}$ such that*

$$f(x) = O(|x|^{\alpha - Q})$$

for some $\alpha \in \mathbb{R}$. If $\alpha > 0$ then f is integrable near 0, and if $\alpha < 0$ then f is integrable near ∞.

These two corollaries will be frequently used without comment in the sequel.

1.2.3 Convolutions

Let f and g be two integrable function on $\mathbb{G}$. Then their *convolution* $f * g$ is well defined by

$$(f * g)(x) := \int_{\mathbb{G}} f(y)g(y^{-1}x)dy = \int_{\mathbb{G}} f(xy^{-1})g(y)dy.$$

The basic facts about convolution of L^p and weak-L^p functions can be formulated in two propositions. For other properties of convolutions on groups we can refer to [FR16, Sections 1.5 and 3.1.10].

Proposition 1.2.13 (Young's inequality)**.** *Suppose*

$$1 \le p, q, r \le \infty \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and

$$\|f * g\|_{L^r(\mathbb{G})} \le \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})}.$$

Proof. First assuming $r = \infty$, in this case p and q are conjugate exponents and the result follows from Hölder's inequality.

Second assuming $r = q$, $p = 1$, let q' be the conjugate exponent to q. By Hölder's inequality,

$$\begin{aligned} |f * g(x)| &\le \int_{\mathbb{G}} |f(xy^{-1})|^{(1/q)+(1/q')} |g(y)| dy \\ &\le \left(\int_{\mathbb{G}} |f(xy^{-1})| dy \right)^{1/q'} \left(\int_{\mathbb{G}} |f(xy^{-1})| |g(y)|^q dy \right)^{1/q} \\ &= \|f\|_1^{1/q'} \left(\int_{\mathbb{G}} |f(xy^{-1})| |g(y)|^q dy \right)^{1/q}. \end{aligned}$$

Thus, by Fubini's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{G}} |f * g(x)|^q dx &\le \|f\|_1^{q/q'} \int_{\mathbb{G}} \int_{\mathbb{G}} |f(xy^{-1})| |g(y)|^q dy dx \\ &= \|f\|_1^{(q/q')+1} \|g\|_q^q, \end{aligned}$$

so that $\|f * g\|_q \le \|f\|_1 \|g\|_q$. The rest follows by interpolation. $\square$

Proposition 1.2.14 (Young's inequality for weak-L^p spaces)**.** *Suppose*

$$q \le p < \infty, \quad 1 < q, r < \infty, \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p$ and $g \in L^q$ then $f * g \in$ weak-L^r and there exists $C_1 = C_1(p, q)$ such that

$$[f * g]_r \le C_1 \|f\|_p [g]_q.$$

Moreover, if $p > 1$ then $f * g \in L^r$ and there exists $C_2 = C_2(p, q)$ such that

$$\|f * g\|_r \leq C_2 \|f\|_p [g]_q.$$

Proof. By the Marcinkiewicz interpolation theorem one can notice that the strong result for $p > 1$ follows from the weak result. Suppose then that $f \in L^p$ and $g \in L^q$, and (without loss of generality) that $\|f\|_p = [q]_q = 1$. Given $\alpha > 0$ set

$$M := (\alpha/2)^{r/q} (q/r)^{r/qp'},$$

where p' is the conjugate exponent to p. Define $g_1(x) := g(x)$ if $|g(x)| \leq M$ and $g_1(x) := 0$ otherwise, and set $g_2 := g - g_1$. Since

$$\lambda_{f*g}(\alpha) \leq \lambda_{f*g_1}(\alpha/2) + \lambda_{f*g_2}(\alpha/2),$$

it is enough to show that each term on the right side is bounded by $C\alpha^{-r}$, where C depends only on p and q. On the one hand, since $q^{-1} - (p')^{-1} = r^{-1} > 0$ we have $p'q > 0$ and therefore

$$\begin{aligned}
\int_{\mathbb{G}} |g_1(x)|^{p'} dx &= p' \int_0^\infty \alpha^{p'-1} \lambda_{g_1}(\alpha) d\alpha \leq p' \int_0^M \alpha^{p'-1} \lambda_g(\alpha) d\alpha \\
&\leq p' \int_0^M \alpha^{p'-1-q} d\alpha = \frac{p'}{p'-q} M^{p'-q} = \frac{r}{q} M^{qp'/r} = (\alpha/2)^{p'}.
\end{aligned}$$

Thus, for every $x \in \mathbb{G}$, by Hölder's inequality (or by Proposition 1.2.13) we have

$$|f * g_1(x)| \leq \|f\|_p \|g_1\|_{p'} \leq \alpha/2,$$

which implies that $\lambda_{f*g_1}(\alpha/2) = 0$. On the other hand, since $q > 1$, we have

$$\begin{aligned}
\int_{\mathbb{G}} |g_2(x)| dx = \int_0^\infty \lambda_{g_2}(\alpha) d\alpha &= \int_0^M \lambda_g(M) d\alpha + \int_M^\infty \lambda_g(\alpha) d\alpha \\
&\leq M \cdot M^{-q} + \int_M^\infty \alpha^{-q} d\alpha = \frac{q}{q-1} M^{1-q},
\end{aligned}$$

and therefore by Proposition 1.2.13,

$$\|f * g_2\|_p \leq \|f\|_p \|g_2\|_1 \leq q(q-1)^{-1} M^{1-q}.$$

But then

$$\begin{aligned}
\lambda_{f*g_2}(\alpha/2) &\leq [2\|f * g_2\|_p / \alpha]^p \\
&\leq \left(\frac{2}{\alpha}\right)^p \left(\frac{q}{q-1}\right)^p M^{(1-q)p} \\
&= C(p, q)\alpha^{-r}.
\end{aligned}$$

This completes the proof. □

Now let us summarize some properties of approximations to the identity in terms of the convolution. The following notation will be used throughout this monograph: if ϕ is a function on $\mathbb{G}$ and $t > 0$, we define ϕ_t by

$$\phi_t := t^{-Q}\phi \circ \delta_{1/t}, \quad \text{that is,} \quad \phi_t(x) := t^{-Q}\phi(x/t). \tag{1.14}$$

We notice that if $\phi \in L^1(\mathbb{G})$ then $\int_{\mathbb{G}} \phi_t(x)dx$ is independent of t.

Proposition 1.2.15 (Approximation of identity). *Let $\phi \in L^1(\mathbb{G})$ and let $a := \int_{\mathbb{G}} \phi(x)dx$. Then we have the following properties:*

(i) *If $f \in L^p(\mathbb{G})$ for $1 \le p < \infty$, then $\|f * \phi_t - af\|_p \to 0$ as $t \to 0$.*

(ii) *If f is bounded and right uniformly continuous, then $\|f * \phi_t - af\|_\infty \to 0$ as $t \to 0$.*

(iii) *If f is bounded on $\mathbb{G}$ and continuous on an open set $\Omega \subset \mathbb{G}$, then $f * \phi_t - af \to 0$ uniformly on compact subsets of Ω as $t \to 0$.*

Proof. For a function f on $\mathbb{G}$ and $y \in \mathbb{G}$, let us define

$$f^y(x) := f(xy^{-1}).$$

If $f \in L^p$ for $1 \le p < \infty$, then it can be shown that

$$\|f^y - f\|_p \to 0 \quad \text{as } y \to 0, \tag{1.15}$$

for example, using the fact that C_0 is dense in L^p. If $p = \infty$, property (1.15) holds if and only if f is (almost everywhere equal to) a right uniformly continuous function. We now observe that

$$\begin{aligned} f * \phi_t(x) - af(x) &= \int_{\mathbb{G}} f(xy^{-1})t^{-Q}\phi(y/t)dy - af(x) \\ &= \int_{\mathbb{G}} f(x(tz)^{-1})\phi(z)dz - af(x) \\ &= \int_{\mathbb{G}} [f(x(tz)^{-1}) - f(x)]\phi(z)dz. \end{aligned}$$

Hence by Minkowski's inequality,

$$\|f * \phi_t - af\|_p \le \int_{\mathbb{G}} \|f^{tz} - f\|_p |\phi(z)|dz.$$

Since $\|f^{tz} - f\|_p \le 2\|f\|_p$, under the hypothesis of (i) or (ii) it follows from (1.15) and the dominated convergence theorem that $\|f * \phi_t - af\| \to 0$. The routine modification of this argument (with $p = \infty$) needed to establish (iii) is left to the reader. $\square$

1.2.4 Polynomials

The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be understood from different prospectives:

- as tangent vectors at the origin,
- as left invariant (and right invariant) vector fields.

While in this book we will not make much use of the first interpretation, we will need the second. Consequently, let us denote by $\mathfrak{g}_L$ and $\mathfrak{g}_R$ the spaces of left invariant and right invariant vector fields on $\mathbb{G}$.

Let us fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ consisting of eigenvectors for the dilations δ_r with eigenvalues $r^{d_1}, \ldots, r^{d_n}$, i.e., such that

$$\delta_r X_k = r^{d_k} X_k.$$

In other words, first, we consider X_k as left invariant differential operators on $\mathbb{G}$ and we denote by $Y_1, \ldots, Y_n$ the corresponding basis for $\mathfrak{g}_R$: that is, Y_k is the element of $\mathfrak{g}_R$ such that $Y_k|_0 = X_k|_0$ and for $f \in C^1$ we have

$$X_k f(y) = \frac{d}{dt} f(y \cdot \exp(tX_k))|_{t=0},$$
$$Y_k f(y) = \frac{d}{dt} f(\exp(tX_k) \cdot y)|_{t=0}.$$

Then X_k and Y_k are the differential operators homogeneous of degree d_k since

$$\begin{aligned} X_k(f \circ \delta_r)(y) &= \frac{d}{dt} f((ry)\exp(r^{d_k} tX_k))|_{t=0} \\ &= r^{d_k} \frac{d}{dt} f((ry)\exp(tX_k))|_{t=0} \\ &= r^{d_k} (X_k f \circ \delta_r)(y), \end{aligned}$$

and similarly for Y_k. For $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we use the notation

$$X^I = X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}, \quad Y^I = Y_1^{i_1} Y_2^{i_2} \cdots Y_n^{i_n}.$$

According to the Poincaré–Birkhoff–Witt theorem, the operators X^I give a basis for the algebra of left invariant differential operators on the Lie group $\mathbb{G}$. In addition, we also use the notations

$$|I| := i_1 + i_2 + \cdots + i_n, \quad d(I) := d_1 i_1 + d_2 i_2 + \cdots + d_n i_n. \tag{1.16}$$

Here $|I|$ is called an order of the differential operators X^I and Y^I, and $d(I)$ is their degree of homogeneity or the homogeneous degree. If we denote by Δ the set of all numbers $d(I)$ as I ranges over $\mathbb{N}^n$ then we have $\mathbb{N} \subset \Delta$ as $d_1 = 1$.

There are two useful facts. On the one hand, left translations are isometries on $L^2(\mathbb{G})$, and the operators X_k and Y_k are formally skew-adjoint. Therefore,

$$\int_{\mathbb{G}} (X^I f) g = (-1)^{|I|} \int_{\mathbb{G}} f(X^I g), \quad \int_{\mathbb{G}} f(Y^I g) = (-1)^{|I|} \int_{\mathbb{G}} (Y^I f) g,$$

for all smooth functions f and g for which the integrands decay suitably at infinity. On the other hand, the operators X^I and Y^I interact with convolutions by the formulae

$$X^I(f*g) = f*(X^I g), \quad Y^I(f*g) = (Y^I f)*g, \quad (X^I f)*g = f*(Y^I g).$$

Except for the last one these equalities are direct consequences of differentiating, and the third can be obtained by integration by parts:

$$\begin{aligned}(X^I f)*g(x) &= \int_{\mathbb{G}} X^I f(xy)g(y^{-1})dy = (-1)^{|I|}\int_{\mathbb{G}} f(xy)X^I[g(y^{-1})]dy \\ &= \int_{\mathbb{G}} f(xy)(Y^I g)(y^{-1})dy = f*(Y^I g)(x).\end{aligned}$$

Definition 1.2.16 (Polynomials on the homogeneous group $\mathbb{G}$)**.** A function P on $\mathbb{G}$ will be called a *polynomial* if $P \circ \exp$ is a polynomial on $\mathfrak{g}$.

We can form a global coordinate system on $\mathbb{G}$ and generate the algebra of polynomials on $\mathbb{G}$ by setting

$$\eta_k = \xi_k \circ \exp^{-1},$$

where $\eta_1, \ldots, \eta_n$ are polynomials on $\mathbb{G}$, and $\xi_1, \ldots, \xi_n$ are the basis for the linear forms on $\mathfrak{g}$ dual to the basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Therefore, each polynomial on $\mathbb{G}$ can be defined uniquely in the form

$$P = \sum_I a_I \eta^I,$$

where

$$\eta^I = \eta_1^{i_1} \cdots \eta_n^{i_n},$$

$a_I \in \mathbb{C}$, and all but finitely many of the coefficients a_I vanish. Since η^I is homogeneous of degree $d(I)$, the set of possible homogeneous degrees for polynomials coincides with the set Δ. The *isotropic degree* of a polynomial P is

$$\max\{|I| : a_I \neq 0\}.$$

And the *homogeneous degree* of a polynomial P is

$$\max\{d(I) : a_I \neq 0\}.$$

By $\mathcal{P}_N^{\mathrm{iso}}$, for $N \in \mathbb{N}$, we denote the space of polynomials of isotropic degree $\leq N$, and by $\mathcal{P}_a$, for $a \in \Delta$, we denote the space of polynomials of homogeneous degree $\leq a$. Since $1 \leq d_k \leq \overline{d}$ for $k = 1, \ldots, n$, we observe that

$$\mathcal{P}_N \subset \mathcal{P}_N^{\mathrm{iso}} \subset \mathcal{P}_{\overline{d}N}$$

for $N \in \mathbb{N}$.

There is a more explicit description of the group law in terms of the coordinates η_k since the map

$$(x, y) \mapsto \eta_k(xy)$$

is a polynomial on $\mathbb{G} \times \mathbb{G}$. Thus, we have

$$\eta_k((rx)(ry)) = r^{d_k} \eta_k(xy),$$

that is, it is jointly homogeneous of degree d_k and, according to the Baker–Campbell–Hausdorff formula, we have $\eta_k(xy) = \eta_k(x) + \eta_k(y)$ modulo terms of isotropic degree ≥ 2. It follows that

$$\eta_k(xy) = \eta_k(x) + \eta_k(y) + \sum_{I \neq 0, J \neq 0, d(I)+d(J)=d_k} C_k^{IJ} \eta^I(x) \eta^J(y), \tag{1.17}$$

where C_k^{IJ} are constants. It is easy to see that the monomials η^I, η^J can only involve coordinates with homogeneous degree less than d_k, since the multi-indices I and J in (1.17) must satisfy $d(I) < d_k$ and $d(J) < d_k$. In particular, only the coordinates $\eta_1, \ldots, \eta_{j-1}$ can be involved, for instance:

$$\begin{aligned} d_k = 1: &\quad \eta_k(xy) = \eta_k(x) + \eta_k(y), \\ d_k = 2: &\quad \eta_k(xy) = \eta_k(x) + \eta_k(y) + \sum_{d_j = d_l = 1} C_k^{jl} \eta_j(x) \eta_l(y). \end{aligned}$$

Proposition 1.2.17 (Polynomials are translation invariant)**.** *For any $a \in \Delta$, $\mathcal{P}_a$ is left translation invariant.*

Proof. According the formula (1.17), it is easy to see that $\eta_k(xy)$ is in $\mathcal{P}_{d_k}$ (as a function of x for each y, and also as a function of y for each x). On the other hand, the η_k's generate all polynomials, therefore $\mathcal{P}_a$ is left translation invariant for all $a \in \Delta$. □

Definition 1.2.18 (Coordinate functions on the group)**.** For $x \in \mathbb{G}$ and $k = 1, \ldots, n$, we can think of

$$x_k := \eta_k(x)$$

as the *coordinates* of the variable x. Thus, each x_k becomes a polynomial of homogeneous degree k.

We now establish a link between left and right invariant differential operators and derivatives with respect to coordinate functions on the group.

Proposition 1.2.19 (Formulae for invariant derivatives)**.** *We have*

$$X_k = \sum P_{kj}(\partial/\partial x_j), \quad Y_k = \sum Q_{kj}(\partial/\partial x_j), \tag{1.18}$$

where $P_{kk} = Q_{kk} = 1$, $\quad P_{kj} = Q_{kj} = 0$ if $d_j < d_k$ or if $d_j = d_k$ and $j \neq k$, and P_{kj}, Q_{kj} are homogeneous polynomials of degree $d_k - d_j$ if $d_j > d_k$.

Proof. Let us define the operator $L_x : \mathbb{G} \to \mathbb{G}$ by $L_x(y) := xy$ for $x \in \mathbb{G}$. Then, using the fact that X_k agrees with $\partial/\partial x_k$ at 0, for each differentiable function f on $\mathbb{G}$ and $x \in \mathbb{G}$, we have

$$X_k f(x) = (X_k f) \circ L_x(0) = X_k(f \circ L_x)(0) = (\partial/\partial x_k)(f \circ L_x)(0).$$

Therefore, by the chain rule, we obtain

$$X_k f(x) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x) \frac{\partial [x_k \circ L_x]}{\partial x_k}(0).$$

But by formula (1.17) it follows that

$$\frac{\partial [x_j \circ L_x]}{\partial x_k}(0) = \delta_{kj} + \sum_{d(I)=d_j-d_k} C_j^{I[k]} \eta^I(x),$$

where $[k]$ is the multi-index with 1 in the kth place and zeros elsewhere. The desired result for X_k follows from this, and for Y_k it can be proved in a similar way. $\square$

There are also similar expressions for $\partial/\partial x_k$ in terms of X_j or Y_j:

$$\partial/\partial x_k = \sum P'_{kj} X_j = \sum Q'_{kj} Y_j,$$

where P'_{kj}, Q'_{kj} are of the same form as P_{kj}, Q_{kj} in (1.18). Above formulae can be directly obtained from (1.18) with $j = n$, that is, we have

$$\begin{aligned} X_n &= \partial/\partial x_n, \\ X_{n-1} &= \partial/\partial x_{n-1} + P_{(n-1)n} \partial/\partial x_n, \\ X_{n-2} &= \partial/\partial x_{n-2} + P_{(n-2)(n-1)} \partial/\partial x_{n-1} + \ P_{(n-2)n} \partial/\partial x_n, \end{aligned}$$

therefore, we obtain

$$\begin{aligned} \partial/\partial x_n &= X_n, \\ \partial/\partial x_{n-1} &= X_{n-1} - P_{(n-1)n} \partial/\partial x_n, \\ \partial/\partial x_{n-2} &= X_{n-2} - P_{(n-2)(n-1)} \partial/\partial x_{n-1} - \ P_{(n-2)n} \partial/\partial x_n, \end{aligned}$$

and so on. Similarly, one can obtain expressions for higher-order derivatives. For instance,

$$X^I = \sum_{|J| \le |I|, d(J) \ge d(I)} P_{IJ} (\partial/\partial x)^J, \tag{1.19}$$

where P_{IJ} is a homogeneous polynomial of degree $d(J) - d(I)$. Analogously, we obtain formulae for Y^I in terms of $(\partial/\partial x)^J$ and for $(\partial/\partial x)^I$ in terms of X^J or

Y^J, that is,

$$X^I = \sum_{|J|\le|I|,d(J)\le d(I)} P_{IJ}Y^J,$$
$$Y^I = \sum_{|J|\le|I|,d(J)\le d(I)} Q_{IJ}X^J,$$

where P_{IJ} and Q_{IJ} are homogeneous polynomials of degree $d(J) - d(I)$.

Proposition 1.2.20 (Determination of invariant differential operators)**.** *Let $a \in \Delta$ and let $\mu := \dim P_a$. Then the following maps are linear isomorphisms from P_a to $\mathbb{C}^\mu$:*

(i) $P \to ((\partial/\partial x)^I P(0))_{d(I)\le a}$,

(ii) $P \to (X^I P(0))_{d(I)\le a}$,

(iii) $P \to (Y^I P(0))_{d(I)\le a}$.

Proof. Note that Case (i) is a simple consequence of Taylor's theorem. Also, in view of (1.19), since P_{IJ} is a constant function when $d(I) = d(J)$ and $P_{IJ}(0) = 0$ when $d(J) > d(I)$, we have

$$X^I|_0 = \sum_{|J|\le|I|,d(J)=d(I)} P_{IJ}(\partial/\partial x)^J|_0,$$

and similarly for the other formulae relating X^I, Y^I and $(\partial/\partial\eta)^I$. Cases (ii) and (iii) follow easily from this observation together with Case (i). □

The properties above motivate the following:

Definition 1.2.21 (Taylor polynomials)**.** Let $x \in \mathbb{G}$, $a \in \Delta$, and let f be a function whose (distributional) derivatives $X^I f$ (resp. $Y^I f$) are continuous functions in a neighborhood of x for $d(I) \le a$. The left (resp. right) *Taylor polynomial* of f at x of homogeneous degree a is the unique $P \in P_a$ such that $X^I P(0) = X^I f(x)$ (resp. $Y^I P(0) = Y^I f(x)$) for all I such that $d(I) \le a$.

Now we provide simple proofs of an explicit expression of the Taylor formula and the Taylor inequality in the spirit of [Bon09].

Let $X \in \mathfrak{g}$ be given, and suppose $\gamma(t)$ is any integral curve of X, i.e.

$$\dot\gamma = X(\gamma(t))$$

for all $t \in \mathbb{R}$. If $m \in \mathbb{N} \cup \{0\}$ and $u \in C^{m+1}(\mathbb{G})$ is real-valued, then since

$$\frac{d^k}{dt^k}(u(\gamma(t))) = (X^k u)(\gamma(t)),$$

for each $k \in \mathbb{N}\cup\{0\}$, by applying the usual Taylor formula (with integral reminder) to $t \mapsto u(\gamma(t))$, we obtain

$$u(\gamma(t)) = \sum_{k=0}^{m} \frac{t^k}{k!}(X^k u)(\gamma(0)) + \frac{1}{m!}\int_0^t (t-s)^m (X^{m+1}u)(\gamma(s))ds. \tag{1.20}$$

Moreover, it is easy to see that the integral curve γ of $\log h$ starting at x is $s \mapsto \gamma(s) = x\exp(s\log h)$. With $\gamma(0) = x, \gamma(1) = xh$, we get

$$u(xh) = \sum_{k=0}^{m} \frac{1}{k!}((\log h)^k u)(x) + \frac{1}{m!}\int_0^1 (1-s)^m ((\log h)^{m+1}u)(x\exp(s\log h))ds. \tag{1.21}$$

On the other hand, there always exist (polynomial) functions $\mathbb{G} \ni h \mapsto \zeta_i(h) \in \mathbb{R}$ such that

$$\log h = \zeta_1(h)X_1 + \cdots + \zeta_n(h)X_n,$$

for all $h \in \mathbb{G}$, where $\{X_1, \ldots, X_n\}$ is a basis of the Lie algebra of $\mathbb{G}$. Thus, we have

$$(\log h)^k = \left(\sum_{i=1}^{n} \zeta_i(h)X_i\right)^k = \sum_{i_1,\ldots,i_k=1}^{n} \zeta_{i_1}(h)\cdots\zeta_{i_k}(h)X_{i_1}\cdots X_{i_k},$$

for every $k \in \mathbb{N}$.

Therefore, (1.21) implies that

$$\begin{aligned} u(xh) = u(x) + &\sum_{k=1}^{m} \sum_{I=(i_1,\ldots,i_k), i_1,\ldots,i_k \le n} \frac{X_I u(x)}{k!}\zeta_{i_1}(h)\cdots\zeta_{i_k}(h) \\ &+ \sum_{I=(i_1,\ldots,i_{m+1}), i_1,\ldots,i_{n+1}\le n} \zeta_{i_1}(h)\cdots\zeta_{i_{m+1}}(h) \\ &\times \int_0^1 (X_I u)\left(x\exp\left(\sum_{i\le n} s\zeta_i(h)X_i\right)\right)\frac{(1-s)^m}{m!}ds, \end{aligned} \tag{1.22}$$

where $X_I = X_{i_1}\cdots X_{i_k}$ and $I = (i_1, \ldots, i_k)$ with $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

For a multi-index α, we will be using the notations (1.16), i.e.,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad d(\alpha) = d_1\alpha_1 + \cdots + d_n\alpha_n,$$

where d_j is the homogeneous degree of X_k; these are called the Euclidean length and the homogeneous length of α, respectively. One also sets

$$\mathfrak{G} := \{d(\alpha) : \alpha \in (\mathbb{N}\cup\{0\})^n\}.$$

As usual, $[\beta]$ below is the integer part of the real number β. Now we are in a position to state the Taylor formula on homogeneous groups.

Theorem 1.2.22 (Taylor formula)**.** *Let $\mathbb{G}$ be a homogeneous group (identified with $\mathbb{R}^n$ as a topological space). Suppose $\{X_1, \ldots, X_n\}$ is the Jacobian basis for its Lie algebra, $m \in \mathfrak{G}$ and $u \in C^{[m]+1}(\mathbb{G})$. Let also $x_0 \in \mathbb{G}$ be fixed. Then, for every $x \in \mathbb{G}$ we have*

$$\begin{aligned} u(x) &= P_m(u, x_0)(x) + R_m(x, x_0) \\ &= u(x_0) + \sum_{k=1}^{[m]} \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \le n,\\ d(I)\le m}} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x) + R_m(x, x_0), \end{aligned} \tag{1.23}$$

where the reminder term $R_m(x, x_0)$ is given by

$$\begin{aligned} R_m(x, x_0) = \sum_{k=1}^{[m]} & \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \le n,\\ d(I)> m}} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x) \\ & + \sum_{\substack{I=(i_1,\ldots,i_{[n]+1}),\\ i_1,\ldots,i_{[m]+1} \le n}} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_{[m]+1}}(x_0^{-1}x) \\ & \times \int_0^1 (X_I u)\left(x_0 \exp\left(\sum_{i\le n} s\zeta_i(x_0^{-1}x) X_i \right)\right) \frac{(1-s)^{[m]}}{[m]!} ds. \end{aligned}$$

Proof of Theorem 1.2.22. If $x_0 \in \mathbb{G}$ is any fixed element, by replacing x and h in the formula (1.22) by respectively x_0 and $x_0^{-1}x$, we obtain the following:

$$\begin{aligned} u(x) = u(x_0) + \sum_{k=1}^{[m]} & \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \le n,\\ d(I)> m}} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x) \\ & + \sum_{\substack{I=(i_1,\ldots,i_{[m]+1}),\\ i_1,\ldots,i_{[m]+1} \le n}} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_{[m]+1}}(x_0^{-1}x) \\ & \times \int_0^1 (X_I u)\left(x_0 \exp\left(\sum_{i\le n} s\zeta_i(x_0^{-1}x) X_i \right)\right) \frac{(1-s)^{[m]}}{[m]!} ds. \end{aligned} \tag{1.24}$$

Since a polynomial $\zeta_i(x)$ is homogeneous of degree σ_i, there exists $C_1 > 0$ such that

$$C_1^{-1}|x|^{\sigma_i} \le |\zeta_i(x)| \le C_1|x|^{\sigma_i}, \quad \forall x \in \mathbb{G}, \quad i = 1, \ldots, n.$$

As a consequence, for every $k \in \mathbb{N}$, $\zeta_{i_1} \cdots \zeta_{i_k}$ is a homogeneous polynomial of degree $d_{i_1} + \cdots + d_{i_k}$. Similarly, $\zeta_{i_1} \cdots \zeta_{i_{[m]+1}}$ is homogeneous of degree $\ge [m] + 1$ (since the d_i's are ≥ 1) and, there appear only derivatives $X_I u$ with $d(I) \geq [m]+1 > m$ in the integral summands.

We restate (1.24) emphasizing out the polynomial of degree $\leq m$ in the right-hand side:

$$\begin{aligned} u(x) = u(x_0) &+ \sum_{k=1}^{[m]} \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \leq n,\\ d(I)\leq m}} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x) \\ &+ \sum_{k=1}^{[m]} \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \leq n,\\ d(I)> m}} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x) \\ &+ \sum_{\substack{I=(i_1,\ldots,i_{[m]+1}),\\ i_1,\ldots,i_{[m]+1} \leq n}} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_{[m]+1}}(x_0^{-1}x) \\ &\qquad \times \int_0^1 (X_I u) \left(x_0 \exp \left(\sum_{i\leq n} s\zeta_i(x_0^{-1}x) X_i \right) \right) \frac{(1-s)^{[m]}}{[m]!} ds \\ &=: P_m(u,x_0)(x) + R_m(x,x_0). \end{aligned}$$

By construction, $P_m(u,x_0)(x)$ is a polynomial of homogeneous degree $\leq m$. $\square$

Theorem 1.2.23 (Taylor inequality)**.** *Assume the hypotheses of Theorem 1.2.22. Then for every fixed homogeneous norm $|\cdot|$ on $\mathbb{G}$ and every $m \in \mathfrak{G}$, there exists $C > 0$ (depending on $\mathbb{G}$ and $|\cdot|$) such that*

$$|R_m(x,x_0)| \leq \sum_{k=1}^{[m]+1} \frac{C^k}{k!} \sum_{\substack{I=(i_1,\ldots,i_k),\\ i_1,\ldots,i_k \leq n,\\ d(I)> m}} |x_0^{-1}x|^{d(I)} \sup_{|y|\leq C|x_0^{-1}x|} |X_I u(x_0^{-1}y)|. \quad (1.25)$$

Moreover, an explicit formula for the Taylor polynomial $P_m(u,x_0)$ of degree $m \in \mathfrak{G}$ related to u about x_0 is

$$\begin{aligned} P_m(u,x_0)(x) &= u(x_0) \\ &+ \sum_{k=1}^{[m]} \sum_{I=(i_1,\ldots,i_k), i_1,\ldots,i_k\leq n, d(I)\leq m} \frac{X_I u(x_0)}{k!} \zeta_{i_1}(x_0^{-1}x) \cdots \zeta_{i_k}(x_0^{-1}x). \quad (1.26) \end{aligned}$$

Proof of Theorem 1.2.23. Since

$$\sum_{i=1}^{n} s\zeta_i(x)X_i = s\log(x)$$

by Proposition 1.2.5 we obtain

$$\left|(X_I u)\left(\exp\left(\sum_{i\le n} s\zeta_i(x)X_i\right)\right)\right| \le \sup_{|y|\le C_0|x|} |X_I u(y)|.$$

This implies that

$$\begin{aligned}|R_m(x_0^{-1}x)| &\le \sum_{k=1}^{[m]+1} \sum_{\substack{I=(i_1,\dots,i_k),\\ i_1,\dots,i_k\le n,\\ d(I)>m}} \sup_{|y|\le C_0|x_0^{-1}x|} |X_I u(x_0^{-1}y)| \frac{\zeta_{i_1}(x_0^{-1}x)\cdots\zeta_{i_k}(x_0^{-1}x)}{k!}\\ &\le \sum_{k=1}^{[m]+1} \frac{C_1^k}{k!} \sum_{\substack{I=(i_1,\dots,i_k),\\ i_1,\dots,i_k\le n,\\ d(I)>m}} |x_0^{-1}x|^{\sigma(I)} \sup_{|y|\le C_0|x_0^{-1}x|} |X_I u(x_0^{-1}y)|.\end{aligned}$$

Choosing $C := \max\{C_0, C_1\}$ we complete the proof of the Taylor inequality (1.25).

First, note that the above estimate of R_m gives $R_m(x) = \mathcal{O}(\rho^{m+\varepsilon}(x))$ as $x \to 0$, where

$$\varepsilon := \min_{k=1,\dots,[m]+1} \{d(I) - m : I = (i_1,\dots,i_k), i_1,\dots,i_k \le n, d(I) > m\}. \quad (1.27)$$

Thus, the Taylor formula (1.23) can be rewritten as $u(x)=P_m(x)+\mathcal{O}(|x|^{m+\varepsilon})$ as $x \to 0$, with $\varepsilon > 0$ as in (1.27).

Now let us see that there exists at most one polynomial function P on $\mathbb{G}$, with degree $\le m$, such that, for some $\varepsilon > 0$ (depending on P and m) it holds

$$u(x) = P(x) + \mathcal{O}_{x\to x_0}(|x_0^{-1}x|^{m+\varepsilon}). \quad (1.28)$$

Indeed, suppose there are two such polynomials, A and B (with related $\varepsilon_1, \varepsilon_2 > 0$). Then setting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ we have

$$Q(x) := B - A = \mathcal{O}_{x\to x_0}(|x_0^{-1}x|^{m+\varepsilon}).$$

Setting $\widetilde{Q}(z) := Q(x_0 z)$, this is equivalent to

$$\widetilde{Q}(z) = \mathcal{O}_{z\to 0}(|z|^{m+\varepsilon}). \quad (1.29)$$

The fact that Q is a polynomial of degree at most m and the ith component function of $x_0 z$ is a polynomial in z of degree at most d_i, it follows that $Q(x_0 z)$ is a polynomial in z of degree at most m.

Therefore, (1.29) is valid if and only if $\widetilde{Q} \equiv 0$, that is, $Q(x_0 z) = 0$ for all $z \in \mathbb{G}$. This is in turn equivalent to $Q \equiv 0$, i.e., $A \equiv B$. Note that the equivalence of all homogeneous norms (cf. Proposition 1.2.3) implies that a polynomial P as in (1.28) is independent of $|\cdot|$. Thus, P_m is the Taylor polynomial of degree m related to u, which has the explicit formula (1.26). $\square$

1.3 Radial and Euler operators

An important tool for working on homogeneous groups will be an extensive use of radial and Euler operators. We now discuss them in some detail and establish a number of properties used throughout the book.

1.3.1 Radial derivative

First we introduce a radial derivative (acting on a differentiable function f) on a homogeneous group $\mathbb{G}$ by

$$\mathcal{R}f(x) := \frac{df(x)}{d|x|}, \tag{1.30}$$

where $|x|$ is a homogeneous quasi-norm of $\mathbb{G}$. Note that the homogeneous quasi-norm $|x|$ in the formula (1.30) can be arbitrary, that is, in general the radial operator $\mathcal{R}$ depends on a chosen homogeneous quasi-norm.

Let $\{X_1, \ldots, X_n\}$ be a basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ such that we have

$$AX_k = \nu_k X_k \quad \text{for every} \;\; k = 1, \ldots, n.$$

Then the matrix A can be taken to be $A = \operatorname{diag}(\nu_1, \ldots, \nu_n)$ and each X_k is homogeneous of degree ν_k. By decomposing the vector $\exp_{\mathbb{G}}^{-1}(x)$ in $\mathfrak{g}$ with respect to the basis $\{X_1, \ldots, X_n\}$, we get the vector

$$e(x) = (e_1(x), \ldots, e_n(x))$$

given by the formula

$$\exp_{\mathbb{G}}^{-1}(x) = e(x) \cdot \nabla \equiv \sum_{j=1}^{n} e_j(x) X_j,$$

where

$$\nabla = (X_1, \ldots, X_n)$$

is the full gradient. It gives the equality

$$x = \exp_{\mathbb{G}} \left(e_1(x) X_1 + \cdots + e_n(x) X_n \right). \tag{1.31}$$

By homogeneity and denoting $x = ry$, with $y \in \wp$ being on the quasi-sphere (1.12), we get

$$e(x) = e(ry) = (r^{\nu_1} e_1(y), \ldots, r^{\nu_n} e_n(y)).$$

Indeed, since each X_k is homogeneous of degree ν_k, from (1.31) we get that

$$rx = \exp_{\mathbb{G}} \left(r^{\nu_1} e_1(x) X_1 + \cdots + r^{\nu_n} e_n(x) X_n \right),$$

and hence

$$e(rx) = (r^{\nu_1} e_1(x), \ldots, r^{\nu_n} e_n(x)).$$

Thus, since $r > 0$ is arbitrary, without loss of generality taking $|x| = 1$, we can write

$$\frac{d}{d|rx|} f(rx) = \frac{d}{dr} f(\exp_{\mathbb{G}}(r^{\nu_1} e_1(x) X_1 + \cdots + r^{\nu_n} e_n(x) X_n)). \tag{1.32}$$

So, summarizing, one obtains

$$\frac{d}{d|x|}(f(x)) = \frac{d}{dr}(f(ry)) = \frac{d}{dr}(f(\exp_{\mathbb{G}}(r^{\nu_1} e_1(y) X_1 + \cdots + r^{\nu_n} e_n(y) X_n))). \tag{1.33}$$

Throughout this book we will be often abbreviate the notation by writing

$$\mathcal{R} := \frac{d}{dr}, \tag{1.34}$$

meaning that the derivative is taken with respect to the radial direction with respect to the quasi-norm $|\cdot|$.

We can also observe that for any differentiable function f we have

$$\frac{d}{d|x|} f(x) = \frac{d}{d|x|} f\left(\frac{x}{|x|}|x|\right) = \frac{x}{|x|}\frac{d}{dx} f(x) = \frac{x \cdot \nabla_E}{|x|} f(x), \tag{1.35}$$

since for $x \in \mathbb{G}$, we have that $\frac{x}{|x|}$ does not depend on $|x|$, and where

$$\nabla_E = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

is an anisotropic (Euclidean) gradient on $\mathbb{G}$ consisting of partial derivatives with respect to coordinate functions.

Although x_j and $\frac{\partial}{\partial x_j}$ may have degrees of homogeneity depending on j, the operator

$$\mathcal{R} = \frac{x \cdot \nabla_E}{|x|} = \frac{d}{d|x|} \tag{1.36}$$

is homogeneous of degree -1.

1.3.2 Euler operator

Given the radial derivative operator $\mathcal{R}$, we define the *Euler operator* on $\mathbb{G}$ by

$$\mathbb{E} := |x|\mathcal{R}. \tag{1.37}$$

Since $\mathcal{R}$ is homogeneous of degree -1, the operator $\mathbb{E}$ is homogeneous of degree 0.

We can note the following useful property shedding some more light on the link between the radial derivative and the Euler operators, also clarifying how to take derivatives with respect to points that are not on the quasi-sphere $\wp$. Thus, for $x \in \mathbb{G}$, we can write $x = ry$ with $y \in \wp$. Then, denoting $\rho := e^t r$ for $t \in \mathbb{R}$, we have

$$\frac{d}{dt}(f(e^t x)) = \frac{d}{dt}(f(e^t ry)) = \frac{d}{dt}(f(\rho y)) = \rho \frac{d}{d\rho}(f(\rho y)) = \mathbb{E}f(\rho y) = \mathbb{E}f(e^t x),$$

that is,

$$\frac{d}{dt} f(e^t x) = \mathbb{E}f(e^t x). \tag{1.38}$$

The Euler operator has the following useful properties, also justifying the name of Euler associated to this operator.

Proposition 1.3.1 (Properties of the Euler operator)**.** *We have the following properties:*

(i) *Let $\nu \in \mathbb{R}$. If $f : \mathbb{G}\backslash\{0\} \to \mathbb{R}$ is differentiable, then*

$$\mathbb{E}(f) = \nu f \quad \textit{if and only if} \quad f(rx) = r^\nu f(x) \ \ (\forall r > 0, x \neq 0).$$

(ii) *The formal adjoint operator of $\mathbb{E}$ has the form*

$$\mathbb{E}^* = -Q\mathbb{I} - \mathbb{E}, \tag{1.39}$$

where $\mathbb{I}$ is the identity operator.

(iii) *For all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ we have*

$$\|\mathbb{E}f\|_{L^2(\mathbb{G})} = \|\mathbb{E}^* f\|_{L^2(\mathbb{G})}. \tag{1.40}$$

Proof. Part (i). If a function f is positively homogeneous of order ν, that is, if

$$f(rx) = r^\nu f(x)$$

holds for all $r > 0$ and $x := \rho y \neq 0$, $y \in \wp$, then using (1.30) for such an f, it follows that

$$\mathbb{E}f = \nu f(x).$$

Conversely, let us fix $x \neq 0$ and define

$$g(r) := f(rx).$$

Using (1.30), the equality $\mathbb{E}f(rx) = \nu f(rx)$ means that

$$g'(r) = \frac{d}{dr} f(rx) = \frac{1}{r}\mathbb{E}f(rx) = \frac{\nu}{r} f(rx) = \frac{\nu}{r} g(r).$$

Consequently, $g(r) = g(1)r^\nu$, i.e., $f(rx) = r^\nu f(x)$ and thus f is positively homogeneous of order ν.

Part (ii). We can calculate the formal adjoint operator of $\mathbb{E}$ on $C_0^\infty(\mathbb{G}\backslash\{0\})$ as follows:

$$\begin{aligned}\int_{\mathbb{G}} \mathbb{E}f(x)\overline{g(x)}dx &= \int_0^\infty \int_{\wp} \frac{d}{dr} f(ry)\overline{g(ry)}r^Q d\sigma(y)dr \\ &= -\int_0^\infty \int_{\wp} f(ry)\left(Qr^{Q-1}\overline{g(ry)} + r^Q \frac{d}{dr}\overline{g(ry)}\right) d\sigma(y)dr \\ &= -\int_{\mathbb{G}} f(x)(Q+\mathbb{E})\overline{g(x)}dx,\end{aligned}$$

by the polar decomposition in Proposition 1.2.10 and the integration by parts using formula (1.30).

Part (iii). By using the representation of $\mathbb{E}^*$ in (1.39), we get

$$\begin{aligned}\|\mathbb{E}^* f\|^2_{L^2(\mathbb{G})} &= \|(-Q\mathbb{I}-\mathbb{E})f\|^2_{L^2(\mathbb{G})} \\ &= Q^2\,\|f\|^2_{L^2(\mathbb{G})} + 2Q\mathrm{Re}\int_{\mathbb{G}} f(x)\overline{\mathbb{E}f(x)}dx + \|\mathbb{E}f\|^2_{L^2(\mathbb{G})}.\end{aligned} \tag{1.41}$$

Then we have

$$\begin{aligned}2Q\mathrm{Re}\int_{\mathbb{G}} f(x)\overline{\mathbb{E}f(x)}dx &= 2Q\mathrm{Re}\int_0^\infty \int_{\wp} f(ry)\frac{d}{dr}\overline{f(ry)}r^Q d\sigma(y)dr \\ &= Q\int_0^\infty r^Q \int_{\wp} \frac{d}{dr}(|f(ry)|^2)d\sigma(y)dr \\ &= -Q^2\int_0^\infty \int_{\wp} |f(ry)|^2 r^{Q-1} d\sigma(y)dr \\ &= -Q^2\,\|f\|^2_{L^2(\mathbb{G})}.\end{aligned} \tag{1.42}$$

Combining this with (1.41) we obtain (1.40). □

Let us introduce the following operator that will be of importance in the sequel,

$$\mathbb{A} := \mathbb{E}\mathbb{E}^*.$$

It is easy to see that this operator is formally self-adjoint, that is,

$$\mathbb{A} = \mathbb{E}\mathbb{E}^* = \mathbb{E}^*\mathbb{E} = \mathbb{A}^*,$$

where we can use Proposition 1.3.1, Part (ii), to also write

$$\mathbb{E}\mathbb{E}^* = \mathbb{E}^*\mathbb{E} = -Q\mathbb{E} - \mathbb{E}^2.$$

Then by replacing f by $\mathbb{E}f$ in (1.40), we obtain the equality

$$\|\mathbb{A}f\|_{L^2(\mathbb{G})} = \left\|\mathbb{E}^2 f\right\|_{L^2(\mathbb{G})} \tag{1.43}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. Moreover, the operator $\mathbb{A}$ is *Komatsu-non-negative* in $L^2(\mathbb{G})$, which means that $(-\infty,0)$ is included in the resolvent set $\rho(\mathbb{A})$ of $\mathbb{A}$ and we have the property

$$\exists M>0,\ \forall\lambda>0,\quad \|(\lambda+\mathbb{A})^{-1}\|_{L^2(\mathbb{G})\to L^2(\mathbb{G})}\leq M\lambda^{-1}.$$

Indeed, and more precisely, we have the following:

Lemma 1.3.2 ($\mathbb{A}=\mathbb{E}\mathbb{E}^*$ is Komatsu-non-negative)**.** *The operator* $\mathbb{A}=\mathbb{E}\mathbb{E}^*$ *is Komatsu-non-negative in* $L^2(\mathbb{G})$*:*

$$\|(\lambda+\mathbb{A})^{-1}\|_{L^2(\mathbb{G})\to L^2(\mathbb{G})}\leq \lambda^{-1}\ \ \textit{for all } \lambda>0. \tag{1.44}$$

Proof. We start with $f\in C_0^\infty(\mathbb{G}\backslash\{0\})$. Using Proposition 1.3.1, Part (ii), a direct calculation shows that we have the equality

$$\begin{aligned}\|(\lambda\mathbb{I}+\mathbb{A})f\|^2_{L^2(\mathbb{G})}&=\|(\lambda\mathbb{I}-\mathbb{E}(Q\mathbb{I}+\mathbb{E}))f\|^2_{L^2(\mathbb{G})}\\&=\lambda^2\|f\|^2_{L^2(\mathbb{G})}+\|\mathbb{E}(Q\mathbb{I}+\mathbb{E})f\|^2_{L^2(\mathbb{G})}-2\lambda\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{Q\mathbb{E}f+\mathbb{E}^2f}dx.\end{aligned}\tag{1.45}$$

Since

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{\mathbb{E}^2f}dx&=\mathrm{Re}\int_0^\infty\int_{\wp}f(ry)\overline{\frac{d}{dr}(\mathbb{E}f(ry))}r^Q d\sigma(y)dr\\&=-\mathrm{Re}\int_0^\infty\int_{\wp}\overline{(\mathbb{E}f(ry))}\left(r^Q\frac{d}{dr}f(ry)+Qr^{Q-1}f(ry)\right)d\sigma(y)dr\\&=-\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}-Q\mathrm{Re}\int_{\mathbb{G}}\overline{\mathbb{E}f(x)}f(x)dx,\end{aligned}$$

we have

$$\begin{aligned}&-2\lambda\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{Q\mathbb{E}f(x)+\mathbb{E}^2f(x)}dx\\&=-2\lambda Q\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{\mathbb{E}f(x)}dx-2\lambda\mathrm{Re}\int_{\mathbb{G}}f(x)\overline{\mathbb{E}^2f(x)}dx=2\lambda\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}.\end{aligned}\tag{1.46}$$

Combining (1.45) with (1.46), we obtain the equality

$$\|(\lambda\mathbb{I}-\mathbb{E}(Q\mathbb{I}+\mathbb{E}))f\|^2_{L^2(\mathbb{G})}=\lambda^2\|f\|^2_{L^2(\mathbb{G})}+2\lambda\|\mathbb{E}f\|^2_{L^2(\mathbb{G})}+\|\mathbb{E}(Q\mathbb{I}+\mathbb{E})f\|^2_{L^2(\mathbb{G})}.$$

By dropping positive terms, it follows that

$$\|(\lambda\mathbb{I}-\mathbb{E}(Q\mathbb{I}+\mathbb{E}))f\|^2_{L^2(\mathbb{G})}\geq\lambda^2\|f\|^2_{L^2(\mathbb{G})},$$

which implies (1.44). $\square$

We can refer to [FR16, Section A.3] for more details on general further properties of Komatsu-non-negative operators and their use in the theory of fractional powers of operators.

1.3.3 From radial to non-radial inequalities

Now we show that the Euler operator and, consequently, also the radial derivative operator, have a very useful property that in order to prove certain inequalities on $\mathbb{G}$ is may be enough to prove them only for radially symmetric functions. We summarize it in the following proposition. Such ideas will be of use, for example, in the analysis of remainder estimates in Theorem 2.3.1 or in Theorem 3.2.6.

As usual, we will say that a function $f = f(x)$ on $\mathbb{G}$ is *radial*, or *radially symmetric*, if it depends only on $|x|$; clearly, this notion depends on the seminorm that we are using.

Proposition 1.3.3 (Radialisation of functions)**.** *Let $\phi_1, \phi_2, \phi_3 \in L^1_{\mathrm{loc}}(\mathbb{G})$ be arbitrary radially symmetric functions. For $f \in L^p_{\mathrm{loc}}(\mathbb{G})$, define its radial average by*

$$\widetilde{f}(|x|) := \left(\frac{1}{|\wp|} \int_{\wp} |f(|x|y)|^p d\sigma(y) \right)^{1/p}. \tag{1.47}$$

Then for any $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ and $1 < p < \infty$ we have the equality

$$\int_{\mathbb{G}} \phi_1(x) \left| \widetilde{f}(|x|) \right|^p dx = \int_{\mathbb{G}} \phi_1(x) \, |f(x)|^p \, dx. \tag{1.48}$$

Moreover, if $\phi_2, \phi_3 \geq 0$, we have the inequalities

$$\int_{\mathbb{G}} \phi_2(x) \left| \mathbb{E}^k \widetilde{f}(|x|) \right|^p dx \leq \int_{\mathbb{G}} \phi_2(x) \left| \mathbb{E}^k f(x) \right|^p dx \tag{1.49}$$

and

$$\int_{\mathbb{G}} \phi_3(x) \left| \mathcal{R}^k \widetilde{f}(|x|) \right|^p dx \leq \int_{\mathbb{G}} \phi_3(x) \left| \mathcal{R}^k f(x) \right|^p dx, \tag{1.50}$$

for all $1 < p < \infty$, any $k \in \mathbb{N}$ and all $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ such that $\mathbb{E}^k f \in L^p_{\mathrm{loc}}(\mathbb{G})$ or $\mathcal{R}^k f \in L^p_{\mathrm{loc}}(\mathbb{G})$, respectively. The constants in these inequalities are sharp, and are attained when $f = \widetilde{f}$.

Proof. Using definition (1.47) and the polar decomposition formula in Proposition 1.2.10, we have

$$\begin{aligned} \int_{\mathbb{G}} |\widetilde{f}(|x|)|^p \phi_1(x) dx &= |\wp| \int_0^\infty |\widetilde{f}(r)|^p \phi_1(r) r^{Q-1} dr \\ &= |\wp| \int_0^\infty \frac{1}{|\wp|} \int_{\wp} |f(ry)|^p d\sigma(y) \phi_1(r) r^{Q-1} dr = \int_{\mathbb{G}} |f(x)|^p \phi_1(x) dx, \end{aligned} \tag{1.51}$$

which proves the identity (1.48).

To prove (1.49) let us show first that

$$|\mathbb{E}^k \widetilde{f}| \leq \left(\frac{1}{|\wp|} \int_{\wp} \left| \mathbb{E}^k f(ry) \right|^p d\sigma(y) \right)^{\frac{1}{p}}, \; r = |x|, \tag{1.52}$$

holds for any $k \in \mathbb{N}$. We use the induction. For $k = 1$, by the Hölder inequality we obtain

$$
\begin{aligned}
|\mathbb{E}\widetilde{f}| &= r\left(\frac{1}{|\wp|}\int_{\wp}|f(ry)|^p d\sigma(y)\right)^{\frac{1}{p}-1}\frac{1}{|\wp|}\left|\int_{\wp}|f(ry)|^{p-2}f(ry)\overline{\frac{d}{dr}f(ry)}d\sigma(y)\right| \\
&\le \left(\frac{1}{|\wp|}\int_{\wp}|f(ry)|^p d\sigma(y)\right)^{\frac{1-p}{p}}\frac{1}{|\wp|}\int_{\wp}|f(ry)|^{p-1}\left|r\frac{d}{dr}f(ry)\right|d\sigma(y) \\
&\le \left(\frac{1}{|\wp|}\int_{\wp}|f(ry)|^p d\sigma(y)\right)^{\frac{1-p}{p}}\frac{1}{|\wp|}\left(\int_{\wp}\left|r\frac{d}{dr}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}}\left(\int_{\wp}|f(ry)|^p d\sigma(y)\right)^{\frac{p-1}{p}} \\
&= \left(\frac{1}{|\wp|}\int_{\wp}|\mathbb{E}f(ry)|^p\, d\sigma(y)\right)^{\frac{1}{p}}.
\end{aligned}
$$

For the induction step, we assume that for some $\ell \in \mathbb{N}$ we have

$$
|\mathbb{E}^{\ell}\widetilde{f}| \le \left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^p d\sigma(y)\right)^{1/p}, \tag{1.53}
$$

and we want to prove that it then follows that

$$
|\mathbb{E}^{\ell+1}\widetilde{f}| \le \left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell+1}f(ry)\right|^p d\sigma(y)\right)^{1/p}.
$$

So, using (1.53), similarly to the case $\ell = 1$ above, we calculate

$$
\begin{aligned}
|\mathbb{E}^{\ell+1}\widetilde{f}(r)| &\le \left|\mathbb{E}\left(\left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}}\right)\right| \\
&= r\left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}-1}\left|\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^{p-2}\mathbb{E}f(ry)\overline{\frac{d}{dr}(\mathbb{E}^{\ell}f(ry))}d\sigma(y)\right| \\
&\le \left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}-1}\left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell}f(ry)\right|^{p-1}\left|\mathbb{E}^{\ell+1}f(ry)\right|d\sigma(y)\right) \\
&\le \left(\frac{1}{|\wp|}\int_{\wp}|\mathbb{E}^{\ell}f(ry)|^p d\sigma(y)\right)^{\frac{1}{p}-1}\frac{1}{|\wp|}\left(\int_{\wp}\left|\mathbb{E}^{\ell+1}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_{\wp}|\mathbb{E}^{\ell}f(ry)|^p d\sigma(y)\right)^{\frac{p-1}{p}} \\
&= \left(\frac{1}{|\wp|}\int_{\wp}\left|\mathbb{E}^{\ell+1}f(ry)\right|^p d\sigma(y)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Here in the last line we have used Hölder's inequality. It proves (1.52).

Now (1.52) yields

$$\begin{aligned}\int_{\mathbb{G}} \left|\mathbb{E}^k \widetilde{f}(x)\right|^p \phi_2(x)dx &= |\wp| \int_0^\infty \left|\mathbb{E}^k \widetilde{f}(r)\right|^p \phi_2(r) r^{Q-1} dr \\ &\le |\wp| \int_0^\infty \frac{1}{|\wp|} \int_{\wp} \left|\mathbb{E}^k f(ry)\right|^p \phi_2(r) r^{Q-1} d\sigma(y) dr \\ &= \int_{\mathbb{G}} \left|\mathbb{E}^k f(x)\right|^p \phi_2(x) dx.\end{aligned}$$

This completes the proof of (1.49). The proof of (1.50) is similar. □

1.3.4 Euler semigroup $e^{-t\mathbb{E}^*\mathbb{E}}$

Here we will describe the operator semigroup $\{e^{-t\mathbb{E}^*\mathbb{E}}\}_{t>0}$ associated with the Euler operator on homogeneous groups.

Theorem 1.3.4 (Euler semigroup)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $x \in \mathbb{G}$, $x \neq 0$, and let $y := \frac{x}{|x|}$, and $t > 0$. Then the semigroup $e^{-t\mathbb{E}^*\mathbb{E}}$ is given by*

$$(e^{-t\mathbb{E}^*\mathbb{E}} f)(x) = \frac{e^{-tQ^2/4}}{\sqrt{4\pi t}} |x|^{-Q/2} \int_0^\infty e^{-\frac{(\ln|x| - \ln s)^2}{4t}} s^{-Q/2} f(sy) s^{Q-1} ds. \tag{1.54}$$

Before we prove formula (1.54), let us introduce some notation that will be useful in the sequel. Thus, let us define the map $F : L^2(\mathbb{G}) \to L^2(\mathbb{R} \times \wp)$ by the formula

$$(Ff)(s, y) := e^{sQ/2} f(e^s y), \tag{1.55}$$

for $y \in \wp$ and $s \in \mathbb{R}$. Its inverse map $F^{-1} : L^2(\mathbb{R} \times \wp) \to L^2(\mathbb{G})$ can be given by the formula

$$(F^{-1}g)(x) := r^{-Q/2} g(\ln r, y), \tag{1.56}$$

and one can readily check that F preserves the L^2 norm. The map F can be also described as

$$(Ff)(s, y) = (U(s)f)(y)$$

for all $y \in \wp$ and $s \in \mathbb{R}$, with the dilation mapping $U(t)$ defined by

$$U(t)f(x) := e^{tQ/2} f(e^t x). \tag{1.57}$$

We then immediately have

$$(F(U(t)f))(s, y) = (U(s)(U(t)f))(y) = (U(s + t)f)(y) = (Ff)(s + t, y). \tag{1.58}$$

The dilations $U(t)$ can be linked to the Euler operator through the relation

$$\frac{d}{dt} f(e^t x) = \mathbb{E} f(e^t x), \quad x \in \mathbb{G},$$

see (1.38). It then follows that these dilations can be also seen as a group of unitary operators $U(t) = e^{iAt}$ with the generator

$$Af = \frac{1}{i}\frac{d}{dt}U(t)f|_{t=0} = \frac{1}{i}\left(\mathbb{E} + \frac{Q}{2}\right)f = -i\mathbb{E}f - i\frac{Q}{2}f. \tag{1.59}$$

Since $\mathbb{E}^* = -Q\mathbb{I} - \mathbb{E}$ by Proposition 1.3.1, Part (ii), the formula (1.59) implies that

$$A = A^* = -i\mathbb{E} - i\frac{Q}{2}, \tag{1.60}$$

which yields the relation

$$\mathbb{A} = \mathbb{E}^*\mathbb{E} = \left(-iA - \frac{Q}{2}\right)\left(iA - \frac{Q}{2}\right) = A^2 + \frac{Q^2}{4}. \tag{1.61}$$

The family $U(t)$ is mapped to the multiplication by exponents $e^{it\tau}$ through the Mellin transformation $M : L^2(\mathbb{G}) \to L^2(\mathbb{R}\times\wp)$ defined by the formula $M = \mathcal{F}\circ F$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$, that is,

$$(Mf)(\tau, y) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-is\tau}(Ff)(s, y)ds. \tag{1.62}$$

Indeed, using (1.58) and changing variables, we have

$$\begin{aligned}(MU(t)f)(\tau, y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-is\tau}(Ff)(s+t, y)ds \\ &= \frac{e^{it\tau}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-is\tau}(Ff)(s, y)ds = e^{it\tau}(Mf)(\tau, y).\end{aligned} \tag{1.63}$$

Before finally proving Theorem 1.3.4, let us point out that it implies the following representation of the semigroup e^{-tA^2}.

Corollary 1.3.5 (Semigroup e^{-tA^2})**.** *Let F and F^{-1} be mappings as in (1.55) and (1.56), respectively. Then we have*

$$Fe^{-tA^2}F^{-1}f(r, y) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \exp\left(-\frac{(r-s)^2}{4t}\right)f(sy)ds. \tag{1.64}$$

Proof of Corollary 1.3.5. Setting $e^{-tA^2} = e^{tQ^2/4}e^{-t\mathbb{E}^*\mathbb{E}}$ as well as combining (1.55) and (1.56), and using (1.54) we get

$$\begin{aligned}&Fe^{-tA^2}F^{-1}f(r, y) \\ &= F\left(e^{tQ^2/4}\frac{e^{-tQ^2/4}r^{-Q/2}}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}}s^{Q/2-1}(s^{-Q/2}f(\ln s, y))ds\right) \\ &= e^{rQ/2}\frac{e^{-rQ/2}}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(r-\ln s)^2}{4t}}\frac{f(\ln s, y)}{s}ds = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^\infty e^{-\frac{(r-s_1)^2}{4t}}f(s_1y)ds_1,\end{aligned}$$

which is (1.64), where we have used the new variable $s = e^{s_1}$ in the last line. $\square$

Let us finally prove Theorem 1.3.4.

Proof of Theorem 1.3.4. Noting that from the definition we have $iAe^{itA} = \partial_t U(t)$, we can calculate

$$(MiAe^{iAt}f)(\tau, y) = (M\partial_t U(t)f)(\tau, y) = \partial_t (MU(t)f)(\tau, y).$$

Now using (1.63) we get from above that

$$(MiAe^{iAt}f)(\tau, y) = \partial_t e^{it\tau}(Mf)(\tau, y) = i\tau e^{it\tau}(Mf)(\tau, y),$$

which (after setting $t = 0$) implies

$$(MAf)(\tau, y) = \tau(Mf)(\tau, y) \tag{1.65}$$

for f in the domain $\mathcal{D}(A)$. It follows that the condition $f \in \mathcal{D}(A)$ can be described by the property that the function $(\tau, y) \mapsto \tau(Mf)(\tau, y) \in L^2(\mathbb{R} \times \wp)$.

So, first we prove that

$$(Me^{-tA^2}f)(\tau, y) = e^{-t\tau^2}(Mf)(\tau, y). \tag{1.66}$$

We have

$$(Me^{-tA^2}f)(\tau, y) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}(MA^{2k}f)(\tau, y). \tag{1.67}$$

Moreover, by iterating (1.65), it follows that

$$(MA^{2k}f)(\tau, y) = \tau^{2k}(Mf)(\tau, y), \quad k = 0, 1, 2, \ldots.$$

Combining this with (1.67), we get

$$(Me^{-tA^2}f)(\tau, y) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\tau^{2k}(Mf)(\tau, y) = e^{-t\tau^2}(Mf)(\tau, y).$$

That is, we have showed that (1.66) holds. Thus, it follows that

$$e^{-tA^2} = M^{-1}e^{-t\tau^2}M.$$

Here by using that $M = \mathcal{F} \circ F$, we have

$$e^{-tA^2} = F^{-1} \circ \mathcal{F}^{-1}(e^{-t\tau^2}\mathcal{F} \circ F). \tag{1.68}$$

Furthermore, we calculate

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-t\tau^2}Mf)(\lambda, y) &= \mathcal{F}^{-1}(e^{-t\tau^2}\mathcal{F} \circ F)(\lambda, y) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\tau}e^{-t\tau^2}e^{-is\tau}(Ff)(s, y)dsd\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-t\tau^2 + i(\lambda - s)\tau}d\tau\right)(Ff)(s, y)ds \\
&= \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(\lambda - s)^2}{4t}}(Ff)(s, y)ds =: \varphi_t(\lambda, y),
\end{aligned}$$

since

$$\int_{\mathbb{R}} e^{-t\tau^2+i(\lambda-s)\tau}d\tau = \sqrt{\frac{\pi}{t}}e^{-\frac{(\lambda-s)^2}{4t}}.$$

From this and (1.68), using (1.55), (1.56) and $M = \mathcal{F}\circ F$ with $x = |x|y$, we compute

$$\begin{aligned}(e^{-tA^2}f)(|x|y) &= (F^{-1}\varphi_t)(|x|y) = r^{-Q/2}\varphi_t(\ln|x|, y)\\ &= \frac{1}{\sqrt{4\pi t}}|x|^{-Q/2}\int_{\mathbb{R}} e^{-\frac{(\ln|x|-s)^2}{4t}}(Ff)(s,y)ds\\ &= \frac{1}{\sqrt{4\pi t}}|x|^{-Q/2}\int_0^{\infty} e^{-\frac{(\ln|x|-\ln z)^2}{4t}}z^{\frac{Q}{2}-1}f(zy)dz,\end{aligned}$$

where we have used the change of variables $z = e^s$ in the last line.

Since we have $e^{-t\mathbb{E}^*\mathbb{E}} = e^{-tQ^2/4}e^{-tA^2}$ by (1.61), we arrive at

$$\begin{aligned}(e^{-t\mathbb{E}^*\mathbb{E}}f)(|x|y) &= e^{-tQ^2/4}(e^{-tA^2}f)(|x|y)\\ &= \frac{1}{\sqrt{4\pi t}}|x|^{-Q/2}e^{-tQ^2/4}\int_0^{\infty} e^{-\frac{(\ln|x|-\ln z)^2}{4t}}z^{\frac{Q}{2}-1}f(zy)dz\\ &= \frac{1}{\sqrt{4\pi t}}|x|^{-Q/2}e^{-tQ^2/4}\int_0^{\infty} e^{-\frac{(\ln|x|-\ln z)^2}{4t}}z^{-\frac{Q}{2}}f(zy)z^{Q-1}dz,\end{aligned}$$

completing the proof of (1.54). □

Remark 1.3.6.

1. The representation of the Euler semigroup in Theorem 1.3.4 becomes instrumental in deriving several forms of the Hardy–Sobolev and Gagliardo–Nirenberg type inequalities for the Euler operator, as we will show in the sequel, see, e.g., Section 10.4.
2. In the Euclidean case $\mathbb{R}^n$, the results of this section have been obtained in [BEHL08]. For general homogeneous groups, our presentation followed the results obtained in [RSY18a].

1.4 Stratified groups

An important special case of homogeneous groups is that of stratified groups introduced in Definition 1.1.5. Because this is an important class that will be analysed in Chapter 6 from the point of view of Hardy and other inequalities, here we will provide more details on it and fix the corresponding notation.

1.4.1 Stratified Lie groups

We recall the definition of stratified groups.

Definition 1.4.1 (Stratified groups)**.** A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a *stratified group* (or a *homogeneous Carnot group*) if it satisfies the following conditions:

(a) For some natural numbers $N + N_2 + \cdots + N_r = n$, that is $N = N_1$, the decomposition $\mathbb{R}^n = \mathbb{R}^N \times \cdots \times \mathbb{R}^{N_r}$ is valid, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda(x', x^{(2)}, \ldots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.

(b) Let N be as in (a) and let $X_1, \ldots, X_N$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N$. Then

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_N\}) = n,$$

for every $x \in \mathbb{R}^n$, i.e., the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of $\mathbb{G}$.

The number r is called the *step of* $\mathbb{G}$ and the left invariant vector fields $X_1, \ldots, X_N$ are called the (Jacobian) generators of $\mathbb{G}$. The homogeneous dimension of a stratified Lie group $\mathbb{G}$ is given by

$$Q = \sum_{k=1}^{r} k N_k, \quad N_1 = N.$$

The second-order differential operator

$$\mathcal{L} = \sum_{k=1}^{N} X_k^2 \tag{1.69}$$

is called the (*canonical*) *sub-Laplacian* on $\mathbb{G}$. The sub-Laplacian $\mathcal{L}$ is a left invariant homogeneous hypoelliptic differential operator and it is elliptic if and only if the step of $\mathbb{G}$ is equal to 1.

The *hypoellipticity* of $\mathcal{L}$ means that for a distribution $f \in \mathcal{D}'(\Omega)$ in any open set Ω, if $\mathcal{L}f \in C^\infty(\Omega)$ then $f \in C^\infty(\Omega)$. It is a special case of Hörmander's sum of squares theorem [Hör67].

The left invariant vector field X_k has an explicit form given in Proposition 1.2.19, namely,

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \ldots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.70}$$

where $a_{k,m}^{(l)}$ is a homogeneous (with respect to δ_λ) polynomial function of degree $l-1$. We will also use the following notation for the *horizontal gradient*

$$\nabla_H := (X_1, \ldots, X_N),$$

for the *horizontal divergence*

$$\mathrm{div}_H v := \nabla_H \cdot v,$$

and for the *horizontal p-Laplacian* (*or p-sub-Laplacian*)

$$\mathcal{L}_p f := \mathrm{div}_H(|\nabla_H f|^{p-2} \nabla_H f), \quad 1 < p < \infty. \tag{1.71}$$

Denoting the Euclidean distance by

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$, the representation (1.70) for derivatives leads to the identities

$$|\nabla_H |x'|^\gamma| = \gamma |x'|^{\gamma-1}, \tag{1.72}$$

and

$$\mathrm{div}_H \left(\frac{x'}{|x'|^\gamma} \right) = \frac{\sum_{j=1}^N |x'|^\gamma X_j x'_j - \sum_{j=1}^N x'_j \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N-\gamma}{|x'|^\gamma} \tag{1.73}$$

for all $\gamma \in \mathbb{R}$, $|x'| \neq 0$.

It was shown by Folland [Fol75] that the sub-Laplacian $\mathcal{L}$ in (1.69) on a general stratified group $\mathbb{G}$ has a unique *fundamental solution* ε, that is,

$$\mathcal{L}\varepsilon = \delta, \tag{1.74}$$

where δ is the delta-distribution at the unit element of $\mathbb{G}$. Moreover, the function ε is homogeneous of degree $2-Q$.

The function

$$d(x) := \begin{cases} \varepsilon(x)^{\frac{1}{2-Q}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases} \tag{1.75}$$

is called the *$\mathcal{L}$-gauge* on $\mathbb{G}$. It is a *homogeneous quasi-norm* on $\mathbb{G}$, that is, it is a continuous function $d : \mathbb{G} \to [0, \infty)$, smooth away from the origin, which satisfies the conditions

$$d(\lambda x) = \lambda d(x),\ d(x^{-1}) = d(x) \text{ and } d(x) = 0 \text{ if only if } x = 0. \tag{1.76}$$

We refer to the original paper [Fol75] by Folland as well as to a recent presentation in [FR16, Section 3.2.7] for further details and properties of these fundamental solutions.

For future use, we record the action of $\mathcal{L}$ on d and its powers. Since $\mathcal{L}d^{2-Q} = 0$ in $\mathbb{G}\backslash\{0\}$, a straightforward calculation shows that for $Q \geq 3$ we have

$$\mathcal{L}d = (Q-1)\frac{|\nabla_H d|^2}{d} \text{ in } \mathbb{G}\backslash\{0\}, \tag{1.77}$$

as well as, consequently, for all $\alpha \in \mathbb{R}$,

$$\mathcal{L}d^\alpha = \alpha(\alpha + Q - 2)d^{\alpha-2}|\nabla_H d|^2 \text{ in } \mathbb{G}\backslash\{0\}. \tag{1.78}$$

1.4.2 Extended sub-Laplacians

In general, most of the results described in this book in the setting of stratified groups can be extended to any second-order hypoelliptic differential operators which are "equivalent" to the sub-Laplacian $\mathcal{L}$. Let us very briefly discuss this matter in the sprit of [BLU07].

Let $A = (a_{k,j})_{1\leq k,j\leq N_1}$ be a positive-definite symmetric matrix. Consider the following second-order hypoelliptic differential operator based on the matrix A and the vector fields $\{X_1, \ldots, X_{N_1}\}$ from the first stratum, given by

$$\mathcal{L}_A = \sum_{k,j=1}^{N_1} a_{k,j} X_k X_j. \tag{1.79}$$

For instance, in the Euclidean case, that is, for $\mathbb{G} = (\mathbb{R}^N, +)$ and $N_1 = N$, the constant coefficients second-order elliptic operator

$$\Delta_A = \sum_{k,j=1}^{N} a_{k,j} \frac{\partial^2}{\partial x_k \partial x_j}$$

is transformed into the Laplacian

$$\Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}$$

under a linear change of coordinates in $\mathbb{R}^N$. Thus, the operator Δ_A is "equivalent" to the operator Δ by a linear change of the coordinate system.

In general, to apply the above argument to transform $\mathcal{L}_A$ to the sub-Laplacian $\mathcal{L}$ it is not enough to change the basis by a linear transformation. However, it is enough in the setting of free stratified groups. We say that a stratified group $\mathbb{G}$ is a *free stratified group* if its Lie algebra is (isomorphic to) a free Lie algebra. For instance, the Heisenberg group $\mathbb{H}^1$ is a free stratified group. In this case we have the following result.

Theorem 1.4.2 ([BLU07]). *Let $\mathbb{G}$ be a free stratified group and let A be a given positive-definite symmetric matrix. Let $X = \{X_1, \ldots, X_{N_1}\}$ be left invariant vector fields in the first stratum of the Lie algebra of $\mathbb{G}$. Let*

$$Y_k := \sum_{j=1}^{N_1} \left(A^{\frac{1}{2}}\right)_{k,j} X_j, \quad k = 1, \ldots, N_1.$$

Consider the related second-order differential operator

$$\mathcal{L}_A = \sum_{k=1}^{N_1} Y_k^2 = \sum_{k,j=1}^{N_1} a_{k,j} X_k X_j.$$

Then there exists a Lie group automorphism T_A of $\mathbb{G}$ such that

$$Y_k(u \circ T_A) = (X_k u) \circ T_A, \quad k = 1, \ldots, N_1,$$
$$\mathcal{L}_A(u \circ T_A) = (\mathcal{L} u) \circ T_A,$$

for every smooth function $u : \mathbb{G} \to \mathbb{R}$. Moreover, T_A has polynomial component functions and commutes with the dilations of $\mathbb{G}$.

Remark 1.4.3. The automorphism T_A may not exist when $\mathbb{G}$ is not a free stratified group. However, for any stratified group $\mathbb{G}$ one can find a different stratified group $\mathbb{G}_* = (\mathbb{R}^N, *, \delta_\lambda)$, that is, the stratified group with the same underlying manifold $\mathbb{R}^N$ and the same group of dilations δ_λ as $\mathbb{G}$, and a Lie-group isomorphism from $\mathbb{G}$ to $\mathbb{G}_*$ turning the extended sub-Laplacian $\mathcal{L}_A$ on $\mathbb{G}$ into the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}_*$, see [BLU07, Chapter 16.3].

1.4.3 Divergence theorem

Here we discuss the divergence theorem on stratified Lie groups that will be useful for our analysis at different places of the book.

Let $d\nu$ denote the volume element on $\mathbb{G}$ corresponding to the first stratum on $\mathbb{G}$:

$$d\nu := d\nu(x) = \bigwedge_{j=1}^{N} dx_j. \tag{1.80}$$

However, for simplicity of the exposition, we will mainly use the notation $dx := d\nu(x)$. Regarding it as a differential form, let $\langle X_k, d\nu \rangle$ denote the natural pairing between vector fields and differential forms. As it will follow from the proof of Theorem 1.4.5, using formula (1.70) for the left invariant operators X_k expressing them in terms of the Euclidean derivatives, the pairing $\langle X_k, d\nu\rangle$ can be also expressed in terms of the differential forms corresponding to the Euclidean coordinates in the form

$$\langle X_k, d\nu(x)\rangle = \bigwedge_{j=1, j\neq k}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r} \bigwedge_{m=1}^{N_l} \theta_{l,m}, \tag{1.81}$$

with

$$\theta_{l,m} = -\sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \ldots, x^{(l-1)}) dx_k^{(1)} + dx_m^{(l)}, \tag{1.82}$$

for $l = 2, \ldots, r$ and $m = 1, \ldots, N_l$, where $a_{k,m}^{(l)}$ is a homogeneous polynomial of degree $l-1$ from (1.70).

Definition 1.4.4 (Admissible domains)**.** A bounded open set $\Omega \subset \mathbb{G}$ will be called an *admissible domain* if its boundary $\partial\Omega$ is piecewise smooth and simple, that is, it has no self-intersections. The condition for the boundary to be simple amounts to $\partial\Omega$ being orientable.

The following divergence theorem can be regarded as a consequence of the abstract Stokes formula. However, we give a detailed local proof which will also lead to the explicit representation formula (1.82) that will be of use in the sequel.

Theorem 1.4.5 (Divergence formula)**.** *Let $\Omega \subset \mathbb{G}$ be an admissible domain. Let $f_k \in C^1(\Omega) \bigcap C(\overline{\Omega})$, $k = 1, \ldots, N_1$. Then for each $k = 1, \ldots, N_1$, we have*

$$\int_{\Omega} X_k f_k d\nu = \int_{\partial\Omega} f_k \langle X_k, d\nu\rangle. \tag{1.83}$$

Consequently, we also have

$$\int_{\Omega} \sum_{k=1}^{N_1} X_k f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^{N_1} f_k \langle X_k, d\nu\rangle. \tag{1.84}$$

Proof of Theorem 1.4.5. Using (1.70), for any function f we obtain the following differentiation formula

$$\begin{aligned}
df &= \sum_{k=1}^{N_1} \frac{\partial f}{\partial x_k^{(1)}} dx_k^{(1)} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} \frac{\partial f}{\partial x_m^{(l)}} dx_m^{(l)} \\
&= \sum_{k=1}^{N_1} X_k f dx_k^{(1)} - \sum_{k=1}^{N_1} \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x^{(1)}, \ldots, x^{(l-1)}) \frac{\partial f}{\partial x_m^{(l)}} dx_k^{(l)} \sum_{l=2}^{r} \sum_{m=1}^{N_l} \frac{\partial f}{\partial x_m^{(l)}} dx_m^{(l)} \\
&= \sum_{k=1}^{N_1} X_k f dx_k^{(1)} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} \frac{\partial f}{\partial x_m^{(l)}} \left(-\sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \ldots, x^{(l-1)}) dx_k^{(1)} + dx_m^{(l)} \right) \\
&= \sum_{k=1}^{N_1} X_k f dx_k^{(1)} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} \frac{\partial f}{\partial x_m^{(l)}} \theta_{l,m},
\end{aligned}$$

where for each $l = 2, \dots, r$ and $m = 1, \dots, N_l$,

$$\theta_{l,m} = -\sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) dx_k^{(1)} + dx_m^{(l)}.$$

That is, we can write

$$df = \sum_{k=1}^{N_1} X_k f dx_k^{(1)} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} \frac{\partial f}{\partial x_m^{(l)}} \theta_{l,m}. \tag{1.85}$$

It is simple to see that

$$\langle X_s, dx_j^{(1)} \rangle = \frac{\partial}{\partial x_s^{(1)}} dx_j^{(1)} = \delta_{sj},$$

where δ_{sj} is the Kronecker delta. Moreover, we have

$$\begin{aligned}
&\langle X_s, \theta_{l,m} \rangle \\
&\quad = \left(\frac{\partial}{\partial x_s^{(1)}} + \sum_{h=2}^{r} \sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \frac{\partial}{\partial x_g^{(h)}} \right) \\
&\qquad \times \left(-\sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) dx_k^{(1)} + dx_m^{(l)} \right) \\
&\quad = -\sum_{k=1}^{N_1} \left(\frac{\partial}{\partial x_s^{(1)}} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \right) dx_k^{(1)} \\
&\qquad - \sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \frac{\partial}{\partial x_s^{(1)}} dx_k^{(1)} + \frac{\partial}{\partial x_s^{(1)}} dx_m^{(l)} \\
&\qquad - \sum_{k=1}^{N_1} \sum_{h=2}^{r} \sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \left(\frac{\partial}{\partial x_g^{(h)}} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \right) dx_k^{(1)} \\
&\qquad - \sum_{k=1}^{N_1} \sum_{h=2}^{r} \sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \frac{\partial}{\partial x_g^{(h)}} dx_k^{(1)} \\
&\qquad + \sum_{h=2}^{r} \sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \frac{\partial}{\partial x_g^{(h)}} dx_m^{(l)} \\
&\quad = -\sum_{k=1}^{N_1} \left(\frac{\partial}{\partial x_s^{(1)}} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \right) dx_k^{(1)} - \sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \delta_{sk} \\
&\qquad - \sum_{k=1}^{N_1} \sum_{h=2}^{r} \sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)}, \dots, x^{(h-1)}) \left(\frac{\partial}{\partial x_g^{(h)}} a_{k,m}^{(l)}(x^{(1)}, \dots, x^{(l-1)}) \right) dx_k^{(1)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{h=2}^{r}\sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)},\ldots,x^{(h-1)})\delta_{gm}\delta_{hl}\\
=&-\sum_{k=1}^{N_1}\sum_{h=2}^{r}\sum_{g=1}^{N_h} a_{s,g}^{(h)}(x^{(1)},\ldots,x^{(h-1)})\left(\frac{\partial}{\partial x_g^{(h)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)})\right)dx_k^{(1)}\\
&-\sum_{k=1}^{N_1}\left(\frac{\partial}{\partial x_s^{(1)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)})\right)dx_k^{(1)}\\
=&-\sum_{k=1}^{N_1}\left[\sum_{h=2}^{r}\sum_{g=1}^{N_l} a_{s,g}^{(h)}(x^{(1)},\ldots,x^{(h-1)})\left(\frac{\partial}{\partial x_g^{(h)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)})\right)\right.\\
&\left.+\frac{\partial}{\partial x_s^{(1)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)})\right]dx_k^{(1)}.
\end{aligned}
$$

That is, we have

$$\langle X_s, dx_j^{(1)}\rangle = \delta_{sj},$$

for $s, j = 1, \ldots, N_1$, and

$$\langle X_s, \theta_{l,m}\rangle = \sum_{k=1}^{N_1}\mathcal{C}_k dx_k^{(1)},$$

for $s = 1, \ldots, N_1,\ l = 2, \ldots, r,\ m = 1, \ldots, N_l$. Here we used the notation

$$
\begin{aligned}
\mathcal{C}_k =&-\sum_{h=2}^{r}\sum_{g=1}^{N_l} a_{s,g}^{(h)}(x^{(1)},\ldots,x^{(h-1)})\frac{\partial}{\partial x_g^{(h)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)})\\
&-\frac{\partial}{\partial x_s^{(1)}}a_{k,m}^{(l)}(x^{(1)},\ldots,x^{(l-1)}).
\end{aligned}
$$

By (1.80) we have

$$d\nu = d\nu(x) = \bigwedge_{j=1}^{N} dx_j = \bigwedge_{j=1}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r}\bigwedge_{m=1}^{N_l} dx_m^{(l)} = \bigwedge_{j=1}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r}\bigwedge_{m=1}^{N_l}\theta_{l,m},$$

so that we obtain

$$\langle X_k, d\nu(x)\rangle = \bigwedge_{j=1, j\neq k}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r}\bigwedge_{m=1}^{N_l}\theta_{l,m}.$$

Therefore, using (1.85) we get

$$
\begin{aligned}
d(f_s\langle X_s, d\nu(x)\rangle) &= df_s \wedge \langle X_s, d\nu(x)\rangle\\
&= \sum_{k=1}^{N_1} X_k f_s dx_k^{(1)} \wedge \langle X_s, d\nu(x)\rangle + \sum_{l=2}^{r}\sum_{m=1}^{N_l}\frac{\partial f_s}{\partial x_m^{(l)}}\theta_{l,m}\wedge\langle X_s, d\nu(x)\rangle\\
&= X_s f_s d\nu(x),
\end{aligned}
$$

that is, we have

$$d(\langle f_k X_k, d\nu(x)\rangle) = X_k f_k d\nu(x), \quad k = 1, \dots, N_1.$$

Now using the classical Stokes theorem (see, e.g., [DFN84, Theorem 26.3.1]) we obtain (1.83). Taking a sum over k we also obtain (1.84). □

1.4.4 Green's identities for sub-Laplacians

In this section we prove Green's first and second formulae for the sub-Laplacian on stratified groups. These formulae will be useful throughout the book when we will be dealing with inequalities and with the potential theory on stratified groups. We will formulate them in admissible domains in the sense of Definition 1.4.4.

Theorem 1.4.6 (Green's first and second identities)**.** *Let $\mathbb{G}$ be a stratified group and let $\Omega \subset \mathbb{G}$ be an admissible domain.*

(1) *Green's first identity: Let $v \in C^1(\Omega) \bigcap C(\overline{\Omega})$ and $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. Then*

$$\int_\Omega \left((\widetilde{\nabla} v) u + v \mathcal{L} u \right) d\nu = \int_{\partial\Omega} v \langle \widetilde{\nabla} u, d\nu \rangle, \tag{1.86}$$

where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{G}$ and where the vector field $\widetilde{\nabla} u$ is defined by

$$\widetilde{\nabla} u := \sum_{k=1}^{N_1} (X_k u)\, X_k. \tag{1.87}$$

(2) *Green's second identity: Let $u, v \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. Then*

$$\int_\Omega (u \mathcal{L} v - v \mathcal{L} u) d\nu = \int_{\partial\Omega} (u \langle \widetilde{\nabla} v, d\nu \rangle - v \langle \widetilde{\nabla} u, d\nu \rangle). \tag{1.88}$$

Remark 1.4.7.

1. The definition (1.87) means that $\widetilde{\nabla} u$ is a vector field. Consequently, the expression $(\widetilde{\nabla} v) u$ is a scalar, given by

$$\left(\widetilde{\nabla} v \right) u = \widetilde{\nabla} v u = \sum_{k=1}^{N_1} (X_k v)\, (X_k u) = \sum_{k=1}^{N_1} X_k v X_k u.$$

   At the same time the expression $\widetilde{\nabla}(vu)$ is a vector field, also understood as an operator.

2. Although we formulate Green's identities in bounded domains, they are still applicable in unbounded domains for functions with necessary decay rates at infinity. It can be readily shown by the standard argument using quasi-balls with radii $R \to \infty$.

3. The version (1.86) of Green's first identity was proved for the ball in [Gav77] and for any smooth domain of the complex Heisenberg group in [Rom91]. Other analogues have been also obtained in [BLU07] and [CGN08] but using different terminologies. Also, the group structure is not needed for it, see Proposition 12.2.1, with other versions also known, see, e.g., [CGL93]. The version given in Theorem 1.4.6 was obtained in [RS17c].

Proof of Theorem 1.4.6. Part (1). Let $f_k := vX_k u$, so that

$$\sum_{k=1}^{N_1} X_k f_k = (\widetilde{\nabla} v)u + v\mathcal{L}u.$$

By using the divergence formula in Theorem 1.4.5 we obtain

$$\begin{aligned}\int_\Omega \left(\widetilde{\nabla} vu + v\mathcal{L}u\right) d\nu &= \int_\Omega \sum_{k=1}^{N_1} X_k f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^{N_1} \langle f_k X_k, d\nu\rangle \\ &= \int_{\partial\Omega} \sum_{k=1}^{N_1} \langle vX_k u X_k, d\nu\rangle = \int_{\partial\Omega} v\langle \widetilde{\nabla} u, d\nu\rangle,\end{aligned}$$

yielding (1.86).

Part (2). Rewriting (1.86) we have

$$\int_\Omega \left((\widetilde{\nabla} u)v + u\mathcal{L}v\right) d\nu = \int_{\partial\Omega} u\langle \widetilde{\nabla} v, d\nu\rangle,$$

$$\int_\Omega \left((\widetilde{\nabla} v)u + v\mathcal{L}u\right) d\nu = \int_{\partial\Omega} v\langle \widetilde{\nabla} u, d\nu\rangle.$$

By subtracting the second identity from the first one and using

$$(\widetilde{\nabla} u)v = (\widetilde{\nabla} v)u,$$

we obtain (1.88). □

Taking $v = 1$ in Theorem 1.4.6 we obtain the following analogue of Gauss' mean value formula for harmonic functions:

Corollary 1.4.8 (Gauss' mean value formula)**.** *If $\mathcal{L}u = 0$ in an admissible domain $\Omega \subset \mathbb{G}$, then*

$$\int_{\partial\Omega} \langle \widetilde{\nabla} u, d\nu\rangle = 0.$$

As in the classical theory, we can approximate functions with (weak) singularities such as smooth functions because the Green formulae are still valid for them. In this sense, without further justification and using these Green formulae, in particular, we apply them to the fundamental solution ε of the sub-Laplacian $\mathcal{L}$ as in (1.74). We define the function

$$\varepsilon(x,y) := \varepsilon(x^{-1}y). \tag{1.89}$$

The properties of the $\mathcal{L}$-gauge imply that $\varepsilon(x,y) = \varepsilon(y,x)$.

Thus, for $x \in \Omega$, taking $v = 1$ and $u(y) = \varepsilon(x,y)$ we can record the following consequence of Theorem 1.4.6, (1):

Corollary 1.4.9. *If $\Omega \subset \mathbb{G}$ is an admissible domain, and $x \in \Omega$, then*

$$\int_{\partial\Omega} \langle \widetilde{\nabla}\varepsilon(x,y), d\nu(y)\rangle = 1,$$

where ε is the fundamental solution of the sub-Laplacian $\mathcal{L}$.

Putting the fundamental solution ε instead of v in (1.88) we obtain the following representation formulae.

Corollary 1.4.10 (Representation formulae for functions on stratified groups)**.** *Let $\mathbb{G}$ be a stratified group and let $\Omega \subset \mathbb{G}$ be an admissible domain.*

(1) *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. Then for $x \in \Omega$ we have*

$$u(x) = \int_{\Omega} \varepsilon(x,y)\mathcal{L}u(y)d\nu(y) + \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(x,y), d\nu(y)\rangle - \int_{\partial\Omega} \varepsilon(x,y)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle.$$

(2) *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ and $\mathcal{L}u = 0$ on Ω, then for $x \in \Omega$ we have*

$$u(x) = \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(x,y), d\nu(y)\rangle - \int_{\partial\Omega} \varepsilon(x,y)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle.$$

(3) *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ and $u(x) = 0,\ x \in \partial\Omega$, then*

$$u(x) = \int_{\Omega} \varepsilon(x,y)\mathcal{L}u(y)d\nu(y) - \int_{\partial\Omega} \varepsilon(x,y)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle.$$

(4) *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ and $\sum_{j=1}^{N_1} X_j u\langle X_j, d\nu\rangle = 0$ on $\partial\Omega$, then*

$$u(x) = \int_{\Omega} \varepsilon(x,y)\mathcal{L}u(y)d\nu(y) + \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(x,y), d\nu(y)\rangle.$$

1.4.5 Green's identities for p-sub-Laplacians

In this section we show how Green's first and second formulae for the sub-Laplacian from Section 1.4.4 on stratified groups can be extended to the p-sub-Laplacian for all $1 < p < \infty$. As before, we will formulate them in admissible domains in the sense of Definition 1.4.4. We recall the definition of the p-sub-Laplacian from (1.71) as

$$\mathcal{L}_p f := \operatorname{div}_H(|\nabla_H f|^{p-2}\nabla_H f), \quad 1 < p < \infty.$$

Theorem 1.4.11 (Green's first and second identities for p-sub-Laplacian)**.** *Let $\mathbb{G}$ be a stratified group and let $\Omega \subset \mathbb{G}$ be an admissible domain. Let $1 < p < \infty$.*

(1) *Green's first identity: Let $v \in C^1(\Omega)\bigcap C(\overline{\Omega})$ and $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$. Then*

$$\int_\Omega \left((|\nabla_{\mathbb{G}} u|^{p-2}\widetilde{\nabla} v)u + v\mathcal{L}_p u\right) d\nu = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2} v\langle \widetilde{\nabla} u, d\nu\rangle, \tag{1.90}$$

where

$$\widetilde{\nabla} u = \sum_{k=1}^{N_1} (X_k u)\, X_k.$$

(2) *Green's second identity: Let $u, v \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$. Then*

$$\begin{aligned}\int_\Omega &\left(u\mathcal{L}_p v - v\mathcal{L}_p u + (|\nabla_{\mathbb{G}} v|^{p-2} - |\nabla_{\mathbb{G}} u|^{p-2})(\widetilde{\nabla} v)u\right) d\nu \\ &= \int_{\partial\Omega} (|\nabla_{\mathbb{G}} v|^{p-2} u\langle \widetilde{\nabla} v, d\nu\rangle - |\nabla_{\mathbb{G}} u|^{p-2} v\langle \widetilde{\nabla} u, d\nu\rangle).\end{aligned} \tag{1.91}$$

Proof of Theorem 1.4.11. Part (1). Let $f_k := v|\nabla_{\mathbb{G}} u|^{p-2} X_k u$, then

$$\sum_{k=1}^{N_1} X_k f_k = (|\nabla_{\mathbb{G}} u|^{p-2}\widetilde{\nabla} v)u + v\mathcal{L}_p u.$$

By integrating both sides of this equality over Ω and using Proposition 1.4.5 we obtain

$$\begin{aligned}\int_\Omega \left((|\nabla_{\mathbb{G}} u|^{p-2}\widetilde{\nabla} v)u + v\mathcal{L}_p u\right) d\nu &= \int_\Omega \sum_{k=1}^{N_1} X_k f_k d\nu = \int_{\partial\Omega} \sum_{k=1}^{N_1} \langle f_k X_k, d\nu\rangle \\ = \int_{\partial\Omega} \sum_{k=1}^{N_1} \langle v|\nabla_{\mathbb{G}} u|^{p-2} X_k u X_k, d\nu\rangle &= \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2} v\langle \widetilde{\nabla} u, d\nu\rangle,\end{aligned}$$

showing (1.90).

Part (2). Using (1.90) we have

$$\int_{\Omega}\left((|\nabla_{\mathbb{G}} v|^{p-2}\widetilde{\nabla} u)v + u\mathcal{L}_p v\right) d\nu = \int_{\partial\Omega} |\nabla_{\mathbb{G}} v|^{p-2} u\langle \widetilde{\nabla} v, d\nu\rangle,$$
$$\int_{\Omega}\left((|\nabla_{\mathbb{G}} u|^{p-2}\widetilde{\nabla} v)u + v\mathcal{L}_p u\right) d\nu = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2} v\langle \widetilde{\nabla} u, d\nu\rangle.$$

By subtracting the second identity from the first one, the equality

$$(\widetilde{\nabla} u)v = (\widetilde{\nabla} v)u$$

implies (1.91). □

Taking $v = 1$ in Theorem 1.4.11 we get the following analogue of Gauss' mean value formula for p-harmonic functions:

Corollary 1.4.12 (Gauss' mean value formula for p-harmonic functions)**.** *If* $1 < p < \infty$ *and* $\mathcal{L}_p u = 0$ *in an admissible domain* $\Omega \subset \mathbb{G}$*, then*

$$\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2}\langle \widetilde{\nabla} u, d\nu\rangle = 0.$$

1.4.6 Sub-Laplacians with drift

In this section we briefly describe the so-called sub-Laplacians with drift. While such operators can be analysed on more general groups, we restrict our presentation to stratified groups $\mathbb{G}$ only since this will be the setting where we will be using these operators.

Definition 1.4.13 (Sub-Laplacian with drift)**.** The (extended) positive sub-Laplacian with drift is defined on $C_0^\infty(\mathbb{G})$ as the operator

$$\mathcal{L}_X := -\sum_{i,j=1}^{N} a_{i,j} X_i X_j - \gamma X, \tag{1.92}$$

where $\gamma \in \mathbb{R}$, the matrix $(a_{i,j})_{i,j=1}^N$ is real, symmetric, positive definite, and $X \in \mathfrak{g}$ is a left invariant vector field on $\mathbb{G}$.

Similar to Section 1.4.2 the operator (1.92) can be transformed to the (*positive*) *sub-Laplacian with drift* of the form

$$\mathcal{L}_X = -\sum_{j=1}^{N} X_j^2 - \gamma X := \mathcal{L}_0 - \gamma X, \tag{1.93}$$

where $\mathcal{L}_0$ is the positive sub-Laplacian on $\mathbb{G}$ defined by

$$\mathcal{L}_0 = -\sum_{j=1}^{N} X_j^2. \tag{1.94}$$

The details of such a transformation can be found in [HMM05] or [MOV17].

If $X = \sum_{j=1}^{N} a_j X_j$, then we denote $\|X\| := \left(\sum_{j=1}^{N} a_j^2\right)^{1/2}$ and

$$b_X := \frac{\|X\|}{2}, \tag{1.95}$$

where $a_j \in \mathbb{R}$ for $j = 1, \ldots, N$.

Let us collect the following spectral properties of (positive) sub-Laplacians with drift (1.93) which are true on more general groups than the stratified ones. In the case $\gamma = 1$ this was shown in [HMM05, Proposition 3.1] while here we follow [RY18b] for general $\gamma \in \mathbb{R}$:

Proposition 1.4.14 (Spectral properties of sub-Laplacians with drift)**.** *Let $\mathbb{G}$ be a connected Lie group with unit e, $X_1, \ldots, X_N$ an algebraic basis of $\mathfrak{g}$ and let $X \in \mathfrak{g}\backslash\{0\}$. Let $\gamma \in \mathbb{R}$. Then we have for the operator $\mathcal{L}_X$, with domain $C_0^\infty(\mathbb{G})$, the following properties:*

(i) *the operator $\mathcal{L}_X$ is symmetric on $L^2(\mathbb{G}, \mu)$ for some positive measure μ on $\mathbb{G}$ if and only if there exists a positive character χ of $\mathbb{G}$ and a constant C such that $\mu = C\mu_X$ and $\nabla_H \chi|_e = \gamma X|_e$, where μ_X is the measure absolutely continuous with respect to the Haar measure μ with density χ;*

(ii) *assume that $\nabla_H \chi|_e = \gamma X|_e$ for some positive character χ of $\mathbb{G}$. Then the operator $\mathcal{L}_X$ is essentially self-adjoint on $L^2(\mathbb{G}, \mu_X)$ and its spectrum is contained in the interval $[\gamma^2 b_X^2, \infty)$.*

Proof of Proposition 1.4.14. Let μ be a positive measure on $\mathbb{G}$. Then for all test functions $\phi, \psi \in C_0^\infty(\mathbb{G})$ we can calculate

$$\begin{aligned}
\int_{\mathbb{G}} (\mathcal{L}_X \phi)\psi d\mu &= -\sum_{j=1}^{N} \left(\int_{\mathbb{G}} (X_j^2 \phi)\psi d\mu\right) - \gamma \int_{\mathbb{G}} \psi X \phi d\mu \\
&= \sum_{j=1}^{N} \left(\int_{\mathbb{G}} X_j \phi X_j \psi d\mu + \int_{\mathbb{G}} \psi X_j \phi X_j \mu\right) + \gamma \int_{\mathbb{G}} \phi X \psi d\mu + \gamma \int_{\mathbb{G}} \phi \psi X \mu \\
&= \sum_{j=1}^{N} \left(-\int_{\mathbb{G}} \phi X_j^2 \psi d\mu - 2\int_{\mathbb{G}} \phi X_j \psi X_j \mu - \int_{\mathbb{G}} \phi \psi X_j^2 \mu\right) \\
&\quad + \gamma \int_{\mathbb{G}} \phi X \psi d\mu + \gamma \int_{\mathbb{G}} \phi \psi X \mu \\
&= \int_{\mathbb{G}} \phi (\mathcal{L}_X \psi) d\mu + 2\gamma \int_{\mathbb{G}} \phi X \psi d\mu - 2\int_{\mathbb{G}} \phi \nabla_H \psi \nabla_H \mu + \int_{\mathbb{G}} \phi \psi (\mathcal{L}_0 + \gamma X)\mu \\
&= \int_{\mathbb{G}} \phi (\mathcal{L}_X \psi) d\mu + \langle \phi, 2\gamma (X\psi)\mu - 2\nabla_H \psi \nabla_H \mu + \psi(\mathcal{L}_0 + \gamma X)\mu \rangle \\
&=: \int_{\mathbb{G}} \phi (\mathcal{L}_X \psi) d\mu + \mathfrak{I}(\phi, \psi, \mu),
\end{aligned} \tag{1.96}$$

where $\mathcal{L}_0$ is defined in (1.94), $\nabla_H = (X_1, \ldots, X_N)$ and $\langle\cdot,\cdot\rangle$ is the pairing between distributions and test functions on $\mathbb{G}$. From this we see that $\mathcal{L}_X$ is symmetric on $L^2(\mathbb{G},\mu)$ if and only if $\mathfrak{I}(\phi,\psi,\mu) = 0$ for all functions ϕ and ψ, that is,

$$(\mathcal{L}_0 + \gamma X)\mu = 0, \;\; \gamma(X\psi)\mu - \nabla_H\psi\nabla_H\mu = 0, \;\; \forall\psi \in C_0^\infty(\mathbb{G}). \tag{1.97}$$

The vector fields $X_1, \ldots, X_N$ satisfy Hörmander's condition, so that $\mathcal{L}_0 + \gamma X$ is hypoelliptic, which implies with the condition $(\mathcal{L}_0 + \gamma X)\mu = 0$ that μ has a smooth density ω with respect to the Haar measure. Then, as in [HMM05, Proof of Proposition 3.1], we show that

$$X = \sum_{j=1}^{N} a_j X_j, \tag{1.98}$$

for some coefficients $a_1, \ldots, a_N$. Using the fact that $X_1, \ldots, X_N$ are linearly independent and the second equation of (1.97), we obtain that

$$X_k\omega = \gamma a_k \omega, \tag{1.99}$$

where $k = 1, \ldots, N$. The solution of (1.99) is given by

$$\omega(x) = \omega(e)\exp\left(\gamma \int_0^1 \sum_{k=1}^{N} a_k \vartheta_k(t)dt\right),$$

which is a positive and uniquely determined by its value at the identity, where $\vartheta_k(t)$ is the piecewise C^1 path. By normalizing ω, we get that $\omega(e) = 1$, and that it is a character of $\mathbb{G}$. Then, we see that the function $x \mapsto \omega(xy)/\omega(y)$ is a solution of (1.99) for any y in $\mathbb{G}$. Since the value of this function at the identity is 1, we have $\omega(xy) = \omega(x)\omega(y)$ for any $x, y \in \mathbb{G}$, and ω is a character of $\mathbb{G}$. From (1.98) and (1.99), we get $\nabla_H\chi|_e = \gamma X|_e$ with $\chi = \omega$. This proves Part (i) of Proposition 1.4.14.

As in the case $\gamma = 1$ (see [HMM05, Proposition 3.1]), by considering the isometry $\mathcal{U}_2 f = \chi^{-1/2} f$ of $L^2(\mathbb{G},\mu)$ onto $L^2(\mathbb{G},\mu_X)$, we have

$$\chi^{\frac{1}{2}}\mathcal{L}_X(\chi^{-\frac{1}{2}}f) = (\mathcal{L}_0 + \gamma^2 b_X^2)f, \tag{1.100}$$

which is an essentially self-adjoint operator on $L^2(\mathbb{G},\mu)$, where b_X is defined in (1.95). Since the spectrum of this operator is contained in $[\gamma^2 b_X^2, \infty)$, we obtain that $\mathcal{L}_X$ is essentially self-adjoint on $L^2(\mathbb{G},\mu_X)$ and its spectrum is contained in $[\gamma^2 b_X^2, \infty)$.

This completes the proof of Proposition 1.4.14. □

As a corollary of Proposition 1.4.14 let us collect the properties that will be important for us in the sequel.

Corollary 1.4.15 (Transformation of sub-Laplacian with drift). *Let $\mathbb{G}$ be a stratified group and assume conditions of Proposition 1.4.14. Assume that there exists a positive character χ of $\mathbb{G}$ such that*

$$\nabla_H \chi|_e = \gamma X|_e.$$

Then the operator $\mathcal{L}_X$ is formally self-adjoint with respect to the positive measure $\mu_X = \chi\mu$, where μ is the Haar measure of $\mathbb{G}$. The operator $\mathcal{L}_X$ is self-adjoint on $L^2(\mathbb{G}, \mu_X)$ and the mapping

$$L^2(\mathbb{G}, \mu) \ni f \mapsto \chi^{-1/2} f \in L^2(\mathbb{G}, \mu_X) \tag{1.101}$$

is an isometric isomorphism.

For a detailed discussion about more properties of the sub-Laplacians with drift we refer to [HMM04], [HMM05] and [MOV17].

1.4.7 Polarizable Carnot groups

In (1.74) we recalled the result of Folland that the sub-Laplacian $\mathcal{L}$ on general stratified groups always has a unique fundamental solution ε. The explicit formula (1.75) relating the fundamental solution to the $\mathcal{L}$-gauge turns out to be useful in many explicit calculations.

In applications to nonlinear partial differential equations, a natural question arises to express the fundamental solution of the p-sub-Laplacian (1.71) in terms of the fundamental solution of the sub-Laplacian or, equivalently, in terms of the $\mathcal{L}$-gauge. One of the largest classes of stratified Lie groups, for which the fundamental solution of the p-sub-Laplacian is known to be expressed explicitly in terms of the $\mathcal{L}$-gauge are the so-called polarizable Carnot groups which we now briefly discuss.

A Lie group $\mathbb{G}$ is called a *polarizable Carnot group* if the $\mathcal{L}$-gauge d satisfies the following ∞-sub-Laplacian equality

$$\mathcal{L}_\infty d := \frac{1}{2}\langle \nabla_H(|\nabla_H d|^2), \nabla_H d\rangle = 0 \quad \text{in } \mathbb{G}\backslash\{0\}. \tag{1.102}$$

It is known that the Euclidean space, the Heisenberg group $\mathbb{H}^n$ and Kaplan's H-type groups are polarizable Carnot groups.

It was shown by Balogh and Tyson in [BT02b] that if $\mathbb{G}$ is a polarizable Carnot group, then the fundamental solutions of the p-sub-Laplacian (1.71) are given by the explicit formulae

$$\varepsilon_p := \begin{cases} c_p d^{\frac{p-Q}{p-1}}, & \text{if } p \neq Q, \\ -c_Q \log d, & \text{if } p = Q. \end{cases} \tag{1.103}$$

This class of groups also admits an advantageous version of the polar coordinates decomposition. In particular, it can be shown (see [BT02b, Proposition

2.20]) that (1.103) implies a useful identity

$$\mathcal{L}_\infty u = \frac{Q-1}{Q-2}\frac{|\nabla_H u|^4}{u},$$

which can be also written as

$$\sum_{j=1}^{N} \frac{uX_j uX_j |\nabla_H u|}{|\nabla_H u|^3} = \frac{Q-1}{Q-2}. \tag{1.104}$$

It can be shown that the $\mathcal{L}$-gauge d on polarizable Carnot groups satisfies a number of further useful relations. For example, the following formula established in [BT02b] will be useful for some calculations in the sequel:

$$\nabla_H \left(\frac{d}{|\nabla_H d|^2} \nabla_H d \right) = Q \text{ in } \mathbb{G}\backslash\mathcal{Z}, \tag{1.105}$$

where the set

$$\mathcal{Z} := \{0\} \bigcup \{x \in \mathbb{G}\backslash\{0\} : \nabla_H d = 0\}$$

has Haar measure zero, and we have $\nabla_H d \neq 0$ for a.e. $x \in \mathbb{G}$.

As usual, the Green identities are still valid for functions with (weak) singularities provided we can approximate them by smooth functions. Thus, for example, for $x \in \Omega$ in a polarizable Carnot group, taking $v = 1$ and $u(y) = \varepsilon_p(x,y)$ we have the following corollary of Theorem 1.4.11 as an extension of Corollary 1.4.9:

Corollary 1.4.16. *Let Ω be an admissible domain in a polarizable Carnot group $\mathbb{G}$ and let $x \in \Omega$. Then we have*

$$\int_{\partial\Omega} |\nabla_{\mathbb{G}} \varepsilon_p|^{p-2} \langle \widetilde{\nabla} \varepsilon_p(x,y), d\nu(y)\rangle = 1.$$

Note that there are stratified Lie groups other than polarizable Carnot groups where the fundamental solution of the sub-Laplacian can be expressed explicitly (see, e.g., [BT02b, Section 6]).

In particular, since on the polarizable Carnot groups we have the fundamental solution ε_p, putting it instead of v in (1.91) we get the following representation type formulae extending those for $p = 2$ from Corollary 1.4.10:

Corollary 1.4.17 (Representation formulae for functions on polarizable Carnot groups)**.** *Let Ω be an admissible domain in a polarizable Carnot group $\mathbb{G}$.*

1. *Let $u \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$. Then for $x \in \Omega$ we have*

$$\begin{aligned} u(x) = &\int_\Omega \varepsilon_p \mathcal{L}_p u - (|\nabla_{\mathbb{G}} \varepsilon_p|^{p-2} - |\nabla_{\mathbb{G}} u|^{p-2})(\widetilde{\nabla} \varepsilon_p) u d\nu \\ &+ \int_{\partial\Omega} (|\nabla_{\mathbb{G}} \varepsilon_p|^{p-2} u \langle \widetilde{\nabla} \varepsilon_p, d\nu\rangle - |\nabla_{\mathbb{G}} u|^{p-2} \varepsilon_p \langle \widetilde{\nabla} u, d\nu\rangle). \end{aligned}$$

2. *Let* $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *and* $\mathcal{L}_p u = 0$ *on* Ω*, then for* $x \in \Omega$ *we have*

$$\begin{aligned} u(x) = &\int_\Omega (|\nabla_{\mathbb{G}} u|^{p-2} - |\nabla_{\mathbb{G}}\varepsilon_p|^{p-2})(\widetilde{\nabla}\varepsilon_p) u d\nu \\ &+ \int_{\partial\Omega} (|\nabla_{\mathbb{G}}\varepsilon_p|^{p-2} u\langle \widetilde{\nabla}\varepsilon_p, d\nu\rangle - |\nabla_{\mathbb{G}} u|^{p-2}\varepsilon_p \langle \widetilde{\nabla} u, d\nu\rangle). \end{aligned}$$

3. *Let* $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *and*

$$u(x) = 0,\ x \in \partial\Omega,$$

then

$$\begin{aligned} u(x) = &\int_\Omega \varepsilon_p \mathcal{L}_p u - (|\nabla_{\mathbb{G}}\varepsilon_p|^{p-2} - |\nabla_{\mathbb{G}} u|^{p-2})(\widetilde{\nabla}\varepsilon_p) u d\nu \\ &- \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^{p-2}\varepsilon_p \langle \widetilde{\nabla} u, d\nu\rangle. \end{aligned}$$

4. *Let* $u \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *and* $\sum_{j=1}^{N_1} X_j u\langle X_j, d\nu\rangle = 0$ *on* $\partial\Omega$*, then*

$$\begin{aligned} u(x) = &\int_\Omega \varepsilon_p \mathcal{L}_p u - (|\nabla_{\mathbb{G}}\varepsilon_p|^{p-2} - |\nabla_{\mathbb{G}} u|^{p-2})(\widetilde{\nabla}\varepsilon_p) u d\nu \\ &+ \int_{\partial\Omega} |\nabla_{\mathbb{G}}\varepsilon_p|^{p-2} u \langle \widetilde{\nabla}\varepsilon_p, d\nu\rangle. \end{aligned}$$

1.4.8 Heisenberg group

One of the important examples of the stratified groups is the Heisenberg group that was introduced in Example 1.1.8. Here we collect several of its basic properties that will be of use later in the book. We will give both real and complex descriptions of the Heisenberg group as both will be of use to us in the sequel.

Real description of the Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is the manifold $\mathbb{R}^{2n+1}$ but with the group law given by

$$\begin{aligned} &(x^{(1)}, y^{(1)}, t^{(1)})(x^{(2)}, y^{(2)}, t^{(2)}) \\ &\quad := \left(x^{(1)} + x^{(2)}, y^{(1)} + y^{(2)}, t^{(1)} + t^{(2)} + \frac{1}{2}(x^{(1)} \cdot y^{(2)} - x^{(2)} \cdot y^{(1)})\right), \end{aligned} \tag{1.106}$$

for $(x^{(1)}, y^{(1)}, t^{(1)}), (x^{(2)}, y^{(2)}, t^{(2)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}^n$, where $x^{(1)} \cdot y^{(2)}$ and $x^{(2)} \cdot y^{(1)}$ are the usual scalar products on $\mathbb{R}^n$. The canonical basis of the Lie algebra $\mathfrak{h}^n$ of the Heisenberg group $\mathbb{H}^n$ is given by the left invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t,\ \ Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t,\ \ j = 1, \ldots, n,\ \ T = \partial_t. \tag{1.107}$$

It follows that the basis elements X_j, Y_j, T, $j = 1, \ldots, n$, have the following commutator relations,

$$[X_j, Y_j] = T, \quad j = 1, \ldots, n,$$

with all the other commutators being zero. The Heisenberg (Lie) algebra $\mathfrak{h}^n$ is stratified via the decomposition

$$\mathfrak{h}^n = V_1 \oplus V_2,$$

where V_1 is linearly spanned by the X_j's and Y_j's, and $V_2 = \mathbb{R}T$. Therefore, the natural dilations on $\mathfrak{h}^n$ are given by

$$\delta_r(X_j) = rX_j, \;\; \delta_r(Y_j) = rY_j, \;\; \delta_r(T) = r^2 T.$$

On the level of the Heisenberg group $\mathbb{H}^n$ this can be expressed as

$$\delta_r(x, y, t) = r(x, y, t) = (rx, ry, r^2 t), \;\; (x, y, t) \in \mathbb{H}^n, \; r > 0.$$

Consequently, $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$. The (negative) sub-Laplacian on $\mathbb{H}^n$ is given by

$$\mathcal{L} := \sum_{j=1}^{n} (X_j^2 + Y_j^2) = \sum_{j=1}^{n} \left(\partial_{x_j} - \frac{y_j}{2} \partial_t \right)^2 + \left(\partial_{y_j} + \frac{x_j}{2} \partial_t \right)^2,$$

corresponding to the horizontal gradient

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n).$$

We can also write

$$\mathcal{L} = \Delta_{x,y} + \frac{|x|^2 + |y|^2}{4} \partial_t^2 + Z\partial_t, \qquad \text{with } Z = \sum_{j=1}^{n} (x_j \partial_{y_j} - y_j \partial_{x_j}),$$

where $\Delta_{x,y}$ is the Euclidean Laplacian with respect to x, y, and Z is the tangential derivative in the (x, y)-variables.

Complex description of the Heisenberg group. There is an alternative description of the Heisenberg group using complex rather than real variables. It is easy to see that both descriptions are equivalent.

The Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{C}^n \times \mathbb{R}$ with the group operation given by

$$(\zeta, t) \circ (\eta, \tau) = (\zeta + \eta, t + \tau + 2 \operatorname{Im} \zeta\eta), \tag{1.108}$$

for $(\zeta, t), (\eta, \tau) \in \mathbb{C}^n \times \mathbb{R}$.

Comparing (1.106) with (1.108) we can note the change of the constant from $\frac{1}{2}$ to 2. As a result, we are getting different constants in the group law and in the formulae for the left invariant vector fields. We chose to give two descriptions with different constants since the adaptation of such constants in real and complex descriptions of the Heisenberg group seems to be happening in most of the literature. As a result, it will make it more convenient to refer to the relevant literature when needed. This should lead to no confusion since we will never be using these two descriptions at the same time. We note that in general, one can put any constant instead of $\frac{1}{2}$ or 2, the appearing objects are all isomorphic. We can refer the reader to [FR16, Section 6.1.1] for a detailed discussion on the choice of constants in the descriptions of the Heisenberg group.

Writing $\zeta = x + iy$ with x_j, y_j, $j = 1, \ldots, n$, the real coordinates on $\mathbb{H}^n$, the left invariant vector fields

$$\begin{aligned} \tilde{X}_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \\ \tilde{Y}_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n, \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

form a basis for the Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$; again, this can be compared to (1.107).

At the same time, $\mathbb{H}^n$ can be seen as the boundary of the Siegel upper half-space in $\mathbb{C}^{n+1}$,

$$\mathbb{H}^n = \{(\zeta, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = |\zeta|^2, \zeta = (z_1, \ldots, z_n)\}.$$

Again, we can refer the reader, e.g., to [FR16, Section 6.1.1] for more details on different descriptions of the Heisenberg group.

Parametrising $\mathbb{H}_n$ by $z = (\zeta, t)$ where $t = \operatorname{Re} z_{n+1}$, a basis for the complex tangent space of $\mathbb{H}_n$ at the point z is given by the left invariant vector fields

$$X_j = \frac{\partial}{\partial z_j} + i\overline{z}\frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

We denote their conjugates by

$$X_{\overline{j}} \equiv \overline{X}_j = \frac{\partial}{\partial \overline{z}_j} - iz\frac{\partial}{\partial t}, \quad j = 1, \ldots, n.$$

The operator

$$\mathcal{L}_{a,b} = \sum_{j=1}^{n} (aX_j X_{\overline{j}} + bX_{\overline{j}} X_j), \quad a + b = n, \tag{1.109}$$

is a left invariant, rotation invariant differential operator that is homogeneous of degree two. We can refer to the book of Folland and Kohn [FK72] for further properties of such operators. However, we can note that this operator is a slight generalization of the standard sub-Laplacian or Kohn-Laplacian $\mathcal{L}_b$ on the

Heisenberg group $\mathbb{H}^n$ which, when acting on the coefficients of a $(0,q)$-form can be written as

$$\mathcal{L}_b = -\frac{1}{n}\sum_{j=1}^{n}\big((n-q)X_jX_{\overline{j}} + qX_{\overline{j}}X_j\big).$$

Folland and Stein [FS74] obtained the fundamental solution of the operator $\mathcal{L}_{a,b}$ as a constant multiple of the function

$$\varepsilon_{a,b}(z) = \varepsilon(z) = \varepsilon(\zeta,t) = \frac{1}{(t+i|\zeta|^2)^a(t-i|\zeta|^2)^b}. \tag{1.110}$$

More precisely, the distribution $\frac{1}{c_{a,b}}\varepsilon$ is the fundamental solution of $\mathcal{L}_{a,b}$ since ε from (1.110) satisfies the equation

$$\mathcal{L}_{a,b}\varepsilon = c_{a,b}\delta. \tag{1.111}$$

The constant $c_{a,b}$ is zero if a and $b = -1,-2,\ldots,n,n+1,\ldots$, and $c_{a,b}\neq 0$ if a or $b\neq -1,-2,\ldots,n,n+1,\ldots$. In fact, then we can take

$$c_{a,b} = \frac{2(a^2+b^2)\mathrm{Vol}(B_1)}{(2i)^{n+1}}\,\frac{n!}{a(a-1)\cdots(a-n)}(1-\exp(-2ia\pi)) \tag{1.112}$$

for $a\notin\mathbb{Z}$, see Romero [Rom91, Proof of Theorem 1.6]. We will use the above description of the Heisenberg group and of the (rescaled) fundamental solution (1.110) to $\mathcal{L}_{a,b}$ in Section 11.3.3.

1.4.9 Quaternionic Heisenberg group

In this section we describe the basics of the quaternionic version of the Heisenberg group. We start by recalling the notion of quaternions and summarizing their main properties. As the space of quaternions is usually denoted by $\mathbb{H}$, we keep this notation here as well. There should be no notational confusion with the Heisenberg group since the quaternionic notation will be mostly localized to this section only.

Let $\mathbb{H}$ be the set of quaternions

$$x := x_0 + x_1i_1 + x_2i_2 + x_3i_3,$$

where $(x_0,x_1,x_2,x_3)\in\mathbb{R}^4$, and $1,i_1,i_2,i_3$ are the basis elements of $\mathbb{H}$ with the following rules of multiplication:

$$i_1^2 = i_2^2 = i_3^2 = i_1i_2i_3 = -1,$$
$$i_1i_2 = -i_2i_1 = i_3,\ i_2i_3 = -i_3i_2 = i_1,\ i_3i_1 = -i_1i_3 = i_2.$$

The usual convention is that the real part of $x\in\mathbb{H}$ is the real number x_0 and its imaginary part is the point $(x_1,x_2,x_3)\in\mathbb{R}^3$. And so, the real and imaginary parts of x can be denoted by $\Re x$ and $\Im x$, respectively. In addition, we use more precise notations for the imaginary parts as

$$\Im_1 x := x_1, \quad \Im_2 x := x_2, \quad \Im_3 x := x_3.$$

The conjugate of x is denoted by

$$\overline{x} := x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3,$$

and the modulus $|x|$ is defined by

$$|x|^2 := x\overline{x} = \sum_{j=0}^{3} x_j^2.$$

The Grassmanian product (or the quaternion product) of x and y is defined by

$$xy := (x_0 y_0 - \Im x \cdot \Im y) + (x_0 \Im y + y_0 \Im x + \Im x \times \Im y),$$

where

$$\Im x \times \Im y := \det \begin{pmatrix} i_1 & i_2 & i_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Let us denote $\mathbb{H}_q := \mathbb{H} \times \mathbb{R}^3$, it is called the *quaternion Heisenberg group*. Then $\mathbb{H}_q$ becomes a non-commutative group with the group law

$$\begin{aligned}
&(x, t_1, t_2, t_3) \circ (y, \tau_1, \tau_2, \tau_3) \\
&\quad := (x + y, t_1 + \tau_1 - 2\Im_1(\overline{y}x), t_2 + y_2 - 2\Im_2(\overline{y}x), t_3 + \tau_3 - 2\Im_3(\overline{y}x)),
\end{aligned}$$

for all $(x, t), (y, \tau) \in \mathbb{H}_q$. We note that $e = (0, 0, 0, 0)$ is the identity element of $\mathbb{H}_q$ and the inverse of every element $(x, t_1, t_2, t_3) \in \mathbb{H}_q$ is $(-x, -t_1, -t_2, -t_3)$.

The Haar measure on $\mathbb{H}_q$ coincides with the Lebesgue measure on $\mathbb{H} \times \mathbb{R}^3$ which is denoted by $d\nu = dxdt$. Let $\mathfrak{h}_q$ be the Lie algebra of left invariant vector fields on $\mathbb{H}_q$. A basis of $\mathfrak{h}_q$ is given by $\{X_0, X_1, X_2, X_3\}$ and $\{T_1, T_2, T_3\}$, where

$$\begin{aligned}
X_0 &= \frac{\partial}{\partial x_0} - 2x_1 \frac{\partial}{\partial t_1} - 2x_2 \frac{\partial}{\partial t_2} - 2x_3 \frac{\partial}{\partial t_3}, \\
X_1 &= \frac{\partial}{\partial x_1} + 2x_0 \frac{\partial}{\partial t_1} - 2x_3 \frac{\partial}{\partial t_2} + 2x_2 \frac{\partial}{\partial t_3}, \\
X_2 &= \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial t_1} + 2x_0 \frac{\partial}{\partial t_2} - 2x_1 \frac{\partial}{\partial t_3}, \\
X_3 &= \frac{\partial}{\partial x_3} - 2x_2 \frac{\partial}{\partial t_1} + 2x_1 \frac{\partial}{\partial t_2} + 2x_0 \frac{\partial}{\partial t_3},
\end{aligned}$$

and

$$T_k = \frac{\partial}{\partial t_k}, \quad k = 1, 2, 3.$$

The Lie brackets of these vector fields are given by

$$[X_0, X_1] = [X_3, X_2] = 4T_1,$$
$$[X_0, X_2] = [X_1, X_3] = 4T_2,$$
$$[X_0, X_3] = [X_2, X_1] = 4T_3.$$

Thus, the sub-Laplacian on $\mathbb{H}_q$ is given by

$$\mathcal{L} = \sum_{j=0}^{3} X_j^2 = \Delta_x - 4|x|^2\Delta_t - 4\sum_{k=1}^{3}(i_k x \cdot \nabla_x)\frac{\partial}{\partial t_k}, \tag{1.113}$$

where

$$\Delta_x = \sum_{k=0}^{3} \frac{\partial^2}{\partial x_k^2}, \quad \text{and} \quad \Delta_t = \sum_{k=1}^{3} \frac{\partial^2}{\partial t_k^2}.$$

Note that the fundamental solution of the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_q$ was found by Tie and Wong in [TW09]. We restate their results in the following theorem.

Theorem 1.4.18 (Fundamental solutions for sub-Laplacian on quaternion Heisenberg groups)**.** *The fundamental solution $\Gamma(\xi)$ of the sub-Laplacian $\mathcal{L}$ on the quaternion Heisenberg group $\mathbb{H}_q$ is given by*

$$\Gamma(\xi) := \Gamma(|x|, t) = \frac{2}{(2\pi)^{7/2}|x|^2} \int_{\mathbb{S}^2} \frac{1}{(|x|^2 - i(t \cdot n))^3} d\sigma, \tag{1.114}$$

where $\xi = (x, t) \in \mathbb{H}_q$, $n = (n_1, n_2, n_3)$ is a point on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ with centre at the origin, and $d\sigma$ is the surface measure on $\mathbb{S}^2$. That is,

$$\mathcal{L}\Gamma_\zeta = -\delta_\zeta, \tag{1.115}$$

where $\Gamma_\zeta(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_ζ is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{H}_q$.

The quaternion Heisenberg group is a special case of the model step two nilpotent Lie group. It is a homogeneous group with respect to the dilation

$$\delta_\lambda : \mathbb{R}^7 \to \mathbb{R}^7, \quad \delta_\lambda = (\lambda x, \lambda^2 t).$$

Thus,

$$d(\xi) = \frac{1}{\Gamma^{1/8}(\xi)}, \quad \xi = (x, t) \in \mathbb{H}_q, \tag{1.116}$$

is a homogeneous quasi-norm on $\mathbb{H}_q$ with respect to the dilation δ_λ (see, e.g., [Cyg81]).

1.4.10 H-type groups

The H-type groups are a special family of stratified groups with a similar structure to that of the Heisenberg group; one of their important features is that the fundamental solutions to the sub-Laplacian are known explicitly.

We briefly recall the main notions related to this family of groups adopting the notation from [BLU07]; we refer to it for further details.

Definition 1.4.19 (Prototype H-type groups)**.** The space $\mathbb{R}^{m+n}$ equipped with the group law

$$(x,t)\circ(y,\tau)=\begin{pmatrix} x_k+y_k, \quad k=1,\dots,m \\ t_k+\tau_k+\frac{1}{2}\langle A^{(k)}x,y\rangle, \quad k=1,\dots,n\end{pmatrix} \tag{1.117}$$

and with the dilations

$$\delta_\lambda(x,t)=(\lambda x,\lambda^2 t)$$

is called a *prototype H-type group*. Here $A^{(k)}$ is an $m\times m$ skew-symmetric orthogonal matrix, such that,

$$A^{(k)}A^{(l)}+A^{(l)}A^{(k)}=0$$

for all $k,l\in\{1,\dots,n\}$ with $k\neq l$.

Clearly, the Euclidean (Abelian) group and the Heisenberg group are examples of prototype H-type groups.

We leave aside the general H-type groups since it can be shown that any (abstract) H-type group is naturally isomorphic to a prototype H-group (see [BLU07, Theorem 18.2.1]).

It can be directly checked that prototype H-groups are two step nilpotent Lie groups in which the identity of the group is the origin $(0,0)$ and the inverse of (x,t) is

$$(x,t)^{-1}=(-x,-t).$$

It can be also verified that the vector field in the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ that agrees at the origin with $\frac{\partial}{\partial x_j}$, $j=1,\dots,m$, is given by

$$X_j=\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m}a^k_{j,i}x_i\right)\frac{\partial}{\partial t_k}, \tag{1.118}$$

where $a^k_{j,i}$ is the (j,i)th element of the matrix $A^{(k)}$.

The prototype H-type groups are stratified with a basis of the first stratum given by these vector fields $X_1,\dots,X_m$. Thus, the (negative) sub-Laplacian on a prototype H-type group $\mathbb{G}$ is given by

$$\mathcal{L}=\sum_{j=1}^{m}X_j^2=\Delta_x+\frac{1}{4}|x|^2\Delta_t+\sum_{k=1}^{n}\langle A^{(k)}x,\nabla_x\rangle\frac{\partial}{\partial t_k}, \tag{1.119}$$

where Δ and ∇ are the Euclidean Laplacian and the Euclidean gradient, respectively. There is no restriction to suppose that, if ϱ is the centre of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, $\varrho^{\perp}$ is the orthogonal complement of ϱ and

$$m = \dim(\varrho^{\perp}), \quad n = \dim(\varrho).$$

So, the $\mathbb{R}^n$-component of the prototype H-type group $\mathbb{G} \simeq \mathbb{R}^{m+n}$ can be thought of as of its centre.

We have that the homogeneous dimension of the group is

$$Q = m + 2n.$$

We note that since for H-type groups we have $m \geq 2$ and $n \geq 1$, we actually always have $Q \geq 4$.

Now using a generic coordinate $\xi \equiv (x, t)$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, let us introduce the following functions on $\mathbb{G}$:

$$v : \mathbb{G} \to \varrho^{\perp}, \quad v(\xi) := \sum_{j=1}^{m} \langle \exp_{\mathbb{G}}^{-1}(\xi),\, X_j \rangle X_j,$$

where $\{X_1, \ldots, X_m\}$ is an orthogonal basis of $\varrho^{\perp}$,

$$z : \mathbb{G} \to \varrho, \quad z(\xi) := \sum_{j=1}^{n} \langle \exp_{\mathbb{G}}^{-1}(\xi),\, Z_j \rangle Z_j,$$

where $\{Z_1, \ldots, Z_n\}$ is an orthogonal basis of ϱ. Thus, by the definition of v and z, for any $\xi \in \mathbb{G}$, one has

$$\xi = \exp(v(\xi) + z(\xi)), \quad v(\xi) \in \varrho^{\perp}, \quad z(\xi) \in \varrho,$$

and by a direct calculation we have (see, e.g., [BLU07, Proof of Remark 18.3.3]) that

$$|v(\xi)| = |x|, \quad |z(\xi)| = |t|.$$

The fundamental solutions for the sub-Laplacian on abstract H-type groups were found by A. Kaplan in [Kap80]. Such results boil down to the following statement.

Theorem 1.4.20 (Fundamental solutions for sub-Laplacian on H-type groups)**.** *There exists a positive constant c such that*

$$\Gamma(\xi) := c \left(|x|^4 + 16|t|^2 \right)^{(2-Q)/4}$$

is the fundamental solution of the sub-Laplacian $\mathcal{L}$, that is,

$$\mathcal{L}\Gamma_{\zeta} = -\delta_{\zeta}, \tag{1.120}$$

where $\Gamma_{\zeta}(\xi) = \Gamma(\zeta^{-1} \circ \xi)$ and δ_{ζ} is the Dirac distribution at $\zeta \equiv (y, \tau) \in \mathbb{G}$.

For future use in Section 11.5, we will prefer to have the appearing function Γ positive, which leads to the appearance of the minus sign in (1.120).

For further details and analysis on H-type and related groups we may refer the reader to Kohn–Nirenberg [KN65], Folland [Fol75], Kaplan [Kap80], as well as to a more detailed exposition in [BLU07].