Chapter 10

Function Spaces on Homogeneous Groups

In this chapter, we describe several function spaces on homogeneous groups. The origins of the extensive use of homogeneous groups in analysis go back to the book [FS82] of Folland and Stein where Hardy spaces on homogeneous groups have been thoroughly analysed. It turns out that several other function spaces can be defined on homogeneous groups since their main structural properties essentially depend only on the group and dilation structures. Thus, in this chapter we carry out such a construction for Morrey and Campanato spaces and analyse their main properties. Moreover, we describe a version of Sobolev spaces associated to the Euler operator. We call such spaces the Euler–Hilbert–Sobolev spaces.

The constructions of this chapter are based on the analysis in [RSY18d] and [RSY18c]. Since on general homogeneous groups we may not have a (hypoelliptic) differential operator to start with for the usual construction of Sobolev spaces, we develop a version of Sobolev spaces associated to the Euler operator. The development of such spaces is linked to relevant Hardy and Sobolev inequalities from Chapter 2 and Chapter 3. Consequently, we can also analyse properties of maximal operators and fractional integral operators in the constructed Morrey and Campanato spaces on homogeneous groups.

In Definition 6.5.4 and the subsequent analysis we have already considered a collection of (horizontal) weighted Sobolev type on stratified groups, but here we will concentrate on general homogeneous groups. Since horizontal gradients are not available in such a setting, its action will be replaced by that of the radial derivative combined with the corresponding Euler operator.

Thus, throughout this chapter $\mathbb{G}$ will denote a general homogeneous group of homogeneous dimension denoted by Q.

10.1 Euler–Hilbert–Sobolev spaces

In this section, we introduce an Euler–Hilbert–Sobolev space on a homogeneous group $\mathbb{G}$ of homogeneous dimension Q. We start with more general Euler–Sobolev

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function spaces. The definitions and further analysis are based on the Euler operator $\mathbb{E}$ discussed in Section 1.3.

Definition 10.1.1 (Euler-Sobolev function spaces)**.** We define the *Euler–Sobolev function space* on $\mathbb{G}$ by

$$\mathfrak{L}^{k,p}(\mathbb{G}) := \overline{C_0^\infty(\mathbb{G}\backslash\{0\})}^{\|\cdot\|_{\mathfrak{L}^{k,p}(\mathbb{G})}}, \; k \in \mathbb{Z}, \tag{10.1}$$

i.e., as the completion of $C_0^\infty(\mathbb{G}\backslash\{0\})$ with respect to the semi-norm

$$\|f\|_{\mathfrak{L}^{k,p}(\mathbb{G})} := \|\mathbb{E}^k f\|_{L^p(\mathbb{G})}.$$

Let us recall a special case of inequality (3.87) with $\alpha = 0$, that is,

$$\|f\|_{L^p(\mathbb{G})} \leq \left(\frac{p}{Q}\right)^k \|\mathbb{E}^k f\|_{L^p(\mathbb{G})}, \; 1 < p < \infty, \; k \in \mathbb{N}. \tag{10.2}$$

From the definition of the Euler–Sobolev space it follows that this inequality extends to all functions $f \in \mathfrak{L}^{k,p}(\mathbb{G})$:

Corollary 10.1.2 (Embeddings of Euler–Sobolev spaces)**.** *The semi-normed spaces* $(\mathfrak{L}^{k,p}, \|\cdot\|_{\mathfrak{L}^{k,p}})$*,* $k \in \mathbb{Z}$*, are complete spaces for any* $1 < p < \infty$*. The norm of the embedding operator* $\iota : (\mathfrak{L}^{k,p}, \|\cdot\|_{\mathfrak{L}^{k,p}}) \hookrightarrow (L^p, \|\cdot\|_{L^p})$ *satisfies*

$$\|\iota\|_{\mathfrak{L}^{k,p}\to L^p} \leq \left(\frac{p}{Q}\right)^k, \quad k \in \mathbb{N}, \; 1 < p < \infty, \tag{10.3}$$

where the embedding ι *is an embedding of a semi-normed subspace of* L^p*.*

Based on Lemma 1.3.2 we can use the general theory of fractional powers of operators as in [MS01, Chapter 5], to define fractional powers of the operator $\mathbb{A} = \mathbb{E}\mathbb{E}^*$, and we denote

$$|\mathbb{E}|^\beta := \mathbb{A}^{\frac{\beta}{2}}, \quad \beta \in \mathbb{C}.$$

For a brief and specific account of the relevant theory of fractional powers that is required for this construction we can refer the reader to the open access presentation in [FR16, Appendix A]. Consequently, we obtain the following fractional Hardy inequalities in $L^2(\mathbb{G})$.

Theorem 10.1.3 (Fractional Hardy inequalities in $L^2(\mathbb{G})$)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q \geq 1$*. Let* $\beta \in \mathbb{C}_+$ *and let* $k > \frac{\mathrm{Re}\beta}{2}$ *be a positive integer. Then for all complex-valued functions* $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$ *we have*

$$\|f\|_{L^2(\mathbb{G})} \leq C(k - \frac{\beta}{2}, k) \left(\frac{2}{Q}\right)^{\mathrm{Re}\beta} \left\| |\mathbb{E}|^\beta f \right\|_{L^2(\mathbb{G})}, \tag{10.4}$$

where

$$C(\beta, k) = \frac{\Gamma(k+1)}{|\Gamma(\beta)\Gamma(k-\beta)|} \frac{2^{k-\mathrm{Re}\beta}}{\mathrm{Re}\beta(k-\mathrm{Re}\beta)}. \tag{10.5}$$

Proof of Theorem 10.1.3. By using [MS01, Proposition 7.2.1, p. 176] we have the interpolation inequality

$$\left\| |\mathbb{E}|^{-\beta} f \right\|_{L^2(\mathbb{G})} \leq C(k - \frac{\beta}{2}, k) \, \|f\|_{L^2(\mathbb{G})}^{1-\frac{\mathrm{Re}\beta}{2k}} \, \left\| \mathbb{A}^{-k} f \right\|_{L^2(\mathbb{G})}^{\frac{\mathrm{Re}\beta}{2k}}. \tag{10.6}$$

By the equality (1.43) and (3.119) with $\alpha = 0$ it follows that

$$\begin{aligned} C(k - \frac{\beta}{2}, k) \, \|f\|_{L^2(\mathbb{G})}^{1-\frac{\mathrm{Re}\beta}{2k}} \, \left\| \mathbb{A}^{-k} f \right\|_{L^2(\mathbb{G})}^{\frac{\mathrm{Re}\beta}{2k}} &\leq C(k - \frac{\beta}{2}, k) \, \|f\|_{L^2(\mathbb{G})}^{1-\frac{\mathrm{Re}\beta}{2k}} \left(\frac{4}{Q^2} \right)^{\frac{\mathrm{Re}\beta}{2}} \|f\|_{L^2(\mathbb{G})}^{\frac{\mathrm{Re}\beta}{2k}}, \\ &= C(k - \frac{\beta}{2}, k) \left(\frac{4}{Q^2} \right)^{\frac{\mathrm{Re}\beta}{2}} \|f\|_{L^2(\mathbb{G})}, \end{aligned}$$

which combined with (10.6) implies (10.4). □

Definition 10.1.4 (Euler–Hilbert–Sobolev function spaces)**.** For $\beta \in \mathbb{C}_+$, we define the *Euler–Hilbert–Sobolev function space* on $\mathbb{G}$ by

$$\mathbb{H}^{\beta}(\mathbb{G}) := \overline{C_0^{\infty}(\mathbb{G} \backslash \{0\})}^{\|\cdot\|_{\mathbb{H}^{\beta}(\mathbb{G})}}, \tag{10.7}$$

that is, as the completion of $C_0^{\infty}(\mathbb{G} \backslash \{0\})$ with respect to the semi-norm

$$\|f\|_{\mathbb{H}^{\beta}(\mathbb{G})} := \| |\mathbb{E}|^{\beta} f \|_{L^2(\mathbb{G})}.$$

In view of this definition we have inequality (10.4) for all $f \in \mathbb{H}^{\beta}(\mathbb{G})$:

$$\|f\|_{L^2(\mathbb{G})} \leq C(k - \frac{\beta}{2}, k) \left(\frac{2}{Q} \right)^{\mathrm{Re}\beta} \left\| |\mathbb{E}|^{\beta} f \right\|_{L^2(\mathbb{G})}, \tag{10.8}$$

where $\beta \in \mathbb{C}_+$, $k > \frac{\mathrm{Re}\beta}{2}$, $k \in \mathbb{N}$, and $C(k - \frac{\beta}{2}, k)$ is given by (10.5). We can summarize these facts as follows:

Proposition 10.1.5 (Embeddings of Euler–Hilbert–Sobolev spaces)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 1$. For any $\beta \in \mathbb{C}$ the semi-normed space $(\mathbb{H}^{\beta}, \|\cdot\|_{\mathbb{H}^{\beta}})$ is a complete space. Moreover, the norm of the embedding operator $\iota : (\mathbb{H}^{\beta}, \|\cdot\|_{\mathbb{H}^{\beta}}) \hookrightarrow (L^2, \|\cdot\|_{L^2})$ satisfies*

$$\|\iota\|_{\mathbb{H}^{\beta} \to L^2} \leq C\left(k - \frac{\beta}{2}, k\right) \left(\frac{2}{Q} \right)^{\mathrm{Re}\beta}, \quad \beta \in \mathbb{C}_+, \ k > \frac{\mathrm{Re}\beta}{2}, \ k \in \mathbb{N}, \tag{10.9}$$

where we understand the embedding ι as an embedding of a semi-normed subspace of L^2.

10.1.1 Poincaré type inequality

Let $\Omega \subset \mathbb{G}$ be an open set and let $\widehat{\mathfrak{L}}^{1,p}_0(\Omega)$ be the completion of $C_0^\infty(\Omega\backslash\{0\})$ with respect to

$$\|f\|_{\widehat{\mathfrak{L}}^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\mathbb{E}f\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

Then we have the following completion of Hardy inequalities:

Theorem 10.1.6 (Poincaré type inequality on homogeneous groups)**.** *Let Ω be a bounded open subset of a homogeneous group $\mathbb{G}$ of homogeneous dimension Q. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}^{1,p}_0(\Omega)$ and $\mathcal{R}f \equiv \frac{1}{|x|}\mathbb{E}f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{Rp}{Q}\|\mathcal{R}f\|_{L^p(\Omega)} = \frac{Rp}{Q}\left\|\frac{1}{|x|}\mathbb{E}f\right\|_{L^p(\Omega)}, \tag{10.10}$$

where $R = \sup\limits_{x\in\Omega}|x|$.

In order to prove Theorem 10.1.6, we first show the following Hardy inequality on open sets.

Lemma 10.1.7 (Hardy inequality on open sets)**.** *Let $\Omega \subset \mathbb{G}$ be an open set. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}^{1,p}_0(\Omega)$ and $\mathbb{E}f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\Omega)}. \tag{10.11}$$

Proof of Lemma 10.1.7. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be an even smooth function satisfying

- $0 \leq \zeta \leq 1$,
- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

For $\lambda > 0$, we set

$$\zeta_\lambda(x) := \zeta(\lambda|x|).$$

By (3.70) we already have inequality (10.11) for $f \in C_0^\infty(\mathbb{G}\backslash\{0\})$. There exists some $\{f_\ell\}_{\ell=1}^\infty \in C_0^\infty(\Omega\backslash\{0\})$ such that $f_\ell \to f$ in $\widehat{\mathfrak{L}}^{1,p}_0(\Omega)$ as $\ell \to \infty$. Let $\lambda > 0$. From (3.70) we obtain

$$\|\zeta_\lambda f_\ell\|_{L^p(\Omega)} \leq \frac{p}{Q}\left(\|(\mathbb{E}\zeta_\lambda)f_\ell\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E}f_\ell)\|_{L^p(\Omega)}\right)$$

for all $\ell \geq 1$. It is easy to see that

$$\lim_{\ell\to\infty} \zeta_\lambda f_\ell = \zeta_\lambda f,$$

$$\lim_{\ell\to\infty} (\mathbb{E}\zeta_\lambda) f_\ell = (\mathbb{E}\zeta_\lambda) f,$$

$$\lim_{\ell\to\infty} \zeta_\lambda(\mathbb{E}f_\ell) = \zeta_\lambda(\mathbb{E}f)$$

in $L^p(\Omega)$. These properties imply that

$$\|\zeta_\lambda f\|_{L^p(\Omega)} \le \frac{p}{Q}\left\{\|(\mathbb{E}\zeta_\lambda)f\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E}f)\|_{L^p(\Omega)}\right\}.$$

Since

$$|(\mathbb{E}\zeta_\lambda)(x)| \le \begin{cases} \sup|\mathbb{E}\zeta|, & \text{if } \lambda^{-1} < |x| < 2\lambda^{-1}, \\ 0, & \text{otherwise,} \end{cases}$$

we obtain (10.11) in the limit as $\lambda \to 0$. $\square$

Proof of Theorem 10.1.6. Since $R = \sup_{x\in\Omega}|x|$ using Proposition 10.1.7 we obtain

$$\|f\|_{L^p(\Omega)} \le \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\Omega)} \le \frac{Rp}{Q}\|\mathcal{R}f\|_{L^p(\Omega)} = \frac{Rp}{Q}\left\|\frac{1}{|x|}\mathbb{E}f\right\|_{L^p(\Omega)},$$

which gives (10.10). $\square$

10.2 Sobolev–Lorentz–Zygmund spaces

In this section, we define several families of Lorentz type spaces and analyse their basic properties.

Definition 10.2.1 (Lorentz, Lorentz–Zygmund, and Sobolev–Lorentz–Zygmund spaces)**.** Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. We define the *Lorentz type spaces* on $\mathbb{G}$ by

$$L_{|\cdot|,Q,p,q}(\mathbb{G}) := \{f \in L^1_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{L_{|\cdot|,Q,p,q}(\mathbb{G})} < \infty\}, \quad 0 \le p, q \le \infty,$$

where

$$\|f\|_{L_{|\cdot|,Q,p,q}(\mathbb{G})} := \left(\int_{\mathbb{G}} (|x|^{\frac{Q}{p}}|f(x)|)^q \frac{1}{|x|^Q}dx\right)^{1/q}.$$

In the sequel, if the quasi-norm $|\cdot|$ on $\mathbb{G}$ is fixed we will often abbreviate the notation by writing

$$L_{p,q}(\mathbb{G}) := L_{|\cdot|,Q,p,q}(\mathbb{G}).$$

Moreover, we define the *Lorentz–Zygmund spaces* on $\mathbb{G}$ by

$$L_{p,q,\lambda}(\mathbb{G}) := \{f \in L^1_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{L_{p,q,\lambda}(\mathbb{G})} < \infty\}, \quad 0 \le p, q \le \infty, \lambda \in \mathbb{R},$$

where

$$\|f\|_{L_{p,q,\lambda}(\mathbb{G})} := \sup_{R>0}\left(\int_{\mathbb{G}}\left(|x|^{\frac{Q}{p}}\left|\log\frac{R}{|x|}\right|^{\lambda}|f(x)|\right)^q \frac{1}{|x|^Q}dx\right)^{1/q}.$$

Furthermore, we define the *Sobolev–Lorentz–Zygmund spaces* by

$$W^1L_{p,q,\lambda}(\mathbb{G}) := \left\{ f \in L_{p,q,\lambda}(\mathbb{G}) : \frac{1}{|x|}\mathbb{E}f \in L_{p,q,\lambda}(\mathbb{G}) \right\},$$

endowed with the norm

$$\|\cdot\|_{W^1L_{p,q,\lambda}(\mathbb{G})} := \|\cdot\|_{L_{p,q,\lambda}(\mathbb{G})} + \left\| \frac{1}{|x|}\mathbb{E}\cdot \right\|_{L_{p,q,\lambda}(\mathbb{G})}.$$

We also define

$$W_0^1L_{p,q,\lambda}(\mathbb{G}) := \overline{C_0^\infty(\mathbb{G})}^{\|\cdot\|_{W^1L_{p,q,\lambda}(\mathbb{G})}}$$

as the completion of $C_0^\infty(\mathbb{G})$ with respect to $\|\cdot\|_{W^1L_{p,q,\lambda}(\mathbb{G})}$.

In addition, for $\lambda_1, \lambda_2 \in \mathbb{R}$ we introduce the Lorentz–Zygmund spaces involving the double logarithmic weights by

$$L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \{ f \in L^1_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} < \infty \},$$

where

$$\|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} := \sup_{R>0} \left(\int_{\mathbb{G}} \left(|x|^{\frac{Q}{p}} \left| \log \frac{R}{|x|} \right|^{\lambda_1} \left| \log \left| \log \frac{R}{|x|} \right| \right|^{\lambda_2} |f(x)| \right)^q \frac{dx}{|x|^Q} \right)^{1/q}.$$

Remark 10.2.2. The space $L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})$ extends the scale of the spaces $L_{p,q,\lambda}(\mathbb{G})$ and $L_{p,q}(\mathbb{G})$ in the sense that $L_{p,q,\lambda,0}(\mathbb{G}) = L_{p,q,\lambda}(\mathbb{G})$ and $L_{p,q,0,0}(\mathbb{G}) = L_{p,q}(\mathbb{G})$.

Similarly, we define the Sobolev–Lorentz–Zygmund spaces $W^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})$ by

$$W^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \left\{ f \in L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) : \frac{1}{|x|}\mathbb{E}f \in L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) \right\}, \tag{10.12}$$

endowed with the norm

$$\|\cdot\|_{W^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} := \|\cdot\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} + \left\| \frac{1}{|x|}\mathbb{E}\cdot \right\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})},$$

and

$$W_0^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \overline{C_0^\infty(\mathbb{G})}^{\|\cdot\|_{W^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})}}. \tag{10.13}$$

Taking into account the special behaviour of functions

$$f_R(x) := f\left(R\frac{x}{|x|} \right),$$

we introduce the Lorentz–Zygmund type spaces $\mathfrak{L}_{p,q,\lambda}$ by

$$\mathfrak{L}_{p,q,\lambda}(\mathbb{G}) := \{ f \in L^1_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{\mathfrak{L}_{p,q,\lambda}(\mathbb{G})} < \infty \}, \quad \lambda \in \mathbb{R},$$

where

$$\|f\|_{\mathfrak{L}_{p,q,\lambda}(\mathbb{G})} := \sup_{R>0}\left(\int_{\mathbb{G}}\left(|x|^{\frac{Q}{p}}\left|\log\frac{R}{|x|}\right|^{\lambda}|f-f_R|\right)^{q}\frac{dx}{|x|^Q}\right)^{1/q}.$$

For $p=\infty$ we define

$$\|f\|_{\mathfrak{L}_{\infty,q,\lambda}(\mathbb{G})} := \sup_{R>0}\left(\int_{\mathbb{G}}\left(\left|\log\frac{R}{|x|}\right|^{\lambda}|f-f_R|\right)^{q}\frac{dx}{|x|^Q}\right)^{1/q}.$$

Moreover, we define the Lorentz–Zygmund type spaces $\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})$ by

$$\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \{f\in L^1_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})}<\infty\}, \tag{10.14}$$

where

$$\begin{aligned}\|f\|_{\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} &\\ := \sup_{R>0}\Bigg(\int_{\mathbb{G}}\Big(|x|^{\frac{Q}{p}}&\left|\log\frac{eR}{|x|}\right|^{\lambda_1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\lambda_2}\\ &\times\big(\chi_{B(0,eR)}(x)|f-f_R|+\chi_{B^c(0,eR)}(x)|f-f_{e^2R}|\big)\Big)^{q}\frac{dx}{|x|^Q}\Bigg)^{1/q},\end{aligned}$$

$$\chi_{B(0,eR)}(x)=\begin{cases}1, & x\in B(0,eR);\\ 0, & x\notin B(0,eR).\end{cases}$$

For $p=\infty$ we define

$$\begin{aligned}\|f\|_{\mathfrak{L}_{\infty,q,\lambda_1,\lambda_2}(\mathbb{G})} := \sup_{R>0}\Bigg(\int_{\mathbb{G}}\Bigg(&\left|\log\frac{eR}{|x|}\right|^{\lambda_1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\lambda_2}\\ &\times\big(\chi_{B(0,eR)}(x)|f-f_R|+\chi_{B^c(0,eR)}(x)|f-f_{e^2R}|\big)\Bigg)^{q}\frac{dx}{|x|^Q}\Bigg)^{1/q}.\end{aligned}$$

We now show several embeddings between the Sobolev–Lorentz–Zygmund spaces.

Theorem 10.2.3 (Embeddings of Sobolev–Lorentz–Zygmund spaces)**.** *For all* $1<\gamma<\infty$ *and* $\max\{1,\gamma-1\}<q<\infty$ *we have the continuous embedding*

$$W^1_0L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})\hookrightarrow\mathfrak{L}_{\infty,q,-\frac{1}{q},-\frac{\gamma}{q}}(\mathbb{G}).$$

In particular, for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$ *and for any* $R > 0$ *the following inequality holds*

$$\begin{aligned} &\left(\int_{\mathbb{G}} \frac{\chi_{B(0,eR)}(x)|f-f_R|^q + \chi_{B^c(0,eR)}(x)|f-f_{e^2R}|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \\ &\leq \frac{q}{\gamma-1}\left(\int_{\mathbb{G}} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx\right)^{1/q}, \end{aligned} \tag{10.15}$$

where the embedding constant $\frac{q}{\gamma-1}$ *is sharp and where we denote* $f_R(x) := f(R\frac{x}{|x|})$.

Remark 10.2.4.

1. The function spaces extend with respect to indices some known results for spaces in the Abelian case $\mathbb{R}^n$ analysed in [MOW15b]. Embeddings of such Euclidean spaces for some indices were considered in [Wad14], and logarithmic type Hardy inequalities in Euclidean Sobolev–Zygmund spaces were investigated in [MOW13a].
2. Despite the fact that the integrand on the right-hand side of (10.15) has singularities for $|x| = R$, $|x| = eR$ and $|x| = e^2R$ as it will be clear from the proof we do not need to subtract the boundary value of functions on $|x| = eR$ on the left-hand side.

In order to prove Theorem 10.2.3, let us first establish the following estimate.

Proposition 10.2.5. *Let* $Q \in \mathbb{N}$, $1 < \gamma < \infty$, *and assume that* $\max\{1, \gamma - 1\} < q < \infty$. *Then for all* $f \in C_0^\infty(\mathbb{G})$ *and any* $R > 0$ *we have the inequality*

$$\begin{aligned} &\left(\int_{B(0,eR)} \frac{|f-f_R|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \\ &\leq \frac{q}{\gamma-1}\left(\int_{B(0,eR)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx\right)^{1/q}. \end{aligned} \tag{10.16}$$

Proof of Proposition 10.2.5. Using the polar coordinates as in Proposition 1.2.10 and integrating by parts in a quasi-ball $B(0,R)$ we have

$$\begin{aligned} &\int_{B(0,R)} \frac{|f-f_R|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q} \\ &= \int_0^R \frac{1}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^{\gamma}}\int_{\wp}|f(ry)-f(Ry)|^q d\sigma(y)dr \end{aligned}$$

$$= \frac{1}{\gamma - 1}\left[\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma+1}\int_{\wp}|f(ry) - f(Ry)|^q d\sigma(y)\right]_{r=0}^{r=R}$$
$$- \frac{1}{\gamma-1}\int_0^R\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma+1}\frac{d}{dr}\int_{\wp}|f(ry) - f(Ry)|^q d\sigma(y)dr$$
$$= -\frac{q}{\gamma-1}\int_0^R\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma+1}\mathrm{Re}\int_{\wp}|f(ry) - f(Ry)|^{q-2}$$
$$\times (f(ry) - f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)dr.$$

In the above calculation, since $q - \gamma + 1 > 0$, the boundary term at $r = R$ vanishes due to inequalities

$$\log\left(\log\frac{eR}{r}\right) = \int_1^{\log\frac{eR}{r}}\frac{dt}{t} \geq \frac{\log\frac{eR}{r} - 1}{\log\frac{eR}{r}} = \frac{\log\frac{R}{r}}{\log\frac{eR}{r}}$$
$$= \frac{1}{\log\frac{eR}{r}}\int_1^{\frac{R}{r}}\frac{dt}{t} \geq \frac{1}{\log\frac{eR}{r}}\frac{\frac{R}{r} - 1}{\frac{R}{r}} = \frac{R - r}{R\log\frac{eR}{r}}$$

and

$$|f(ry) - f(Ry)| \leq C(R - r),$$

for $0 < r \leq R$. It follows that

$$\int_0^R\frac{1}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^{\gamma}}\int_{\wp}|f(ry) - f(Ry)|^q d\sigma(y)dr$$
$$\leq \frac{q}{\gamma-1}\int_0^R\frac{1}{(\log(\log\frac{eR}{r}))^{\gamma-1}}\int_{\wp}|f(ry) - f(Ry)|^{q-1}\left|\frac{df(ry)}{dr}\right|d\sigma(y)dr$$
$$= \frac{q}{\gamma-1}$$
$$\times\int_0^R\frac{1}{r^{\frac{q-1}{q}}(\log\frac{eR}{r})^{\frac{q-1}{q}}r^{-\frac{q-1}{q}}(\log\frac{eR}{r})^{-\frac{q-1}{q}}}\frac{1}{(\log(\log\frac{eR}{r}))^{\frac{(q-1)\gamma}{q}}(\log(\log\frac{eR}{r}))^{\frac{\gamma-q}{q}}}$$
$$\times\int_{\wp}|f(ry) - f(Ry)|^{q-1}\left|\frac{df(ry)}{dr}\right|d\sigma(y)dr.$$

By the Hölder inequality, we get

$$\int_0^R\frac{1}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^{\gamma}}\int_{\wp}|f(ry) - f(Ry)|^q d\sigma(y)dr$$
$$\leq \frac{q}{\gamma-1}\left(\int_0^R\int_{\wp}\frac{|f(ry) - f(Ry)|^q}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^{\gamma}}d\sigma(y)dr\right)^{(q-1)/q}$$
$$\times\left(\int_0^R\int_{\wp}r^{q-1}\left(\log\frac{eR}{r}\right)^{q-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{q-\gamma}\left|\frac{df(ry)}{dr}\right|^q d\sigma(y)dr\right)^{1/q}.$$

This gives

$$\left(\int_{B(0,R)} \frac{|f-f_R|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \tag{10.17}$$
$$\leq \frac{q}{\gamma-1}\left(\int_{B(0,R)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^{q}dx\right)^{1/q}.$$

Now we calculate the integrals in (10.16) restricted on $B(0,eR)\backslash B(0,R)$:

$$\int_{B(0,eR)\backslash B(0,R)} \frac{|f-f_R|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}$$
$$= \int_R^{eR}\frac{1}{r(\log\frac{eR}{r})(\log((\log\frac{eR}{r})^{-1}))^{\gamma}}\int_{\wp}|f(ry)-f(Ry)|^q d\sigma(y)dr$$
$$= -\frac{1}{\gamma-1}\left[\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\gamma+1}\int_{\wp}|f(ry)-f(Ry)|^q d\sigma(y)\right]_{r=R}^{r=eR}$$
$$+\frac{1}{\gamma-1}\int_R^{eR}\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\gamma+1}\frac{d}{dr}\int_{\wp}|f(ry)-f(Ry)|^q d\sigma(y)dr$$
$$= \frac{q}{\gamma-1}\int_R^{eR}\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\gamma+1}\mathrm{Re}\int_{\wp}|f(ry)-f(Ry)|^{q-2}$$
$$\times (f(ry)-f(Ry))\overline{\frac{df(ry)}{dr}}d\sigma(y)dr.$$

Again, in the calculation above, since $q-\gamma+1>0$, the boundary term at $r=R$ vanishes due to inequalities

$$\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right) = \int_1^{\left(\log\frac{eR}{r}\right)^{-1}}\frac{dt}{t}$$
$$\geq \left(\log\frac{eR}{r}\right)\left(\left(\log\frac{eR}{r}\right)^{-1}-1\right)$$
$$= 1-\log\frac{eR}{r} \geq \frac{r-R}{R}$$

and

$$|f(ry)-f(Ry)| \leq C(R-r),$$

for $R \leq r \leq eR$. This implies

$$\int_R^{eR} \frac{1}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^{\gamma}} \int_{\wp} |f(ry) - f(Ry)|^q d\sigma(y) dr$$

$$\leq \frac{q}{\gamma - 1} \int_R^{eR} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{-\gamma+1}$$

$$\times \int_{\wp} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr = \frac{q}{\gamma - 1}$$

$$\times \int_R^{eR} \frac{1}{r^{\frac{q-1}{q}} (\log \frac{eR}{r})^{\frac{q-1}{q}} r^{-\frac{q-1}{q}} (\log \frac{eR}{r})^{-\frac{q-1}{q}}}$$

$$\times \frac{1}{\left(\log\left((\log \frac{eR}{r})^{-1}\right)\right)^{\frac{(q-1)\gamma}{q}} \left(\log\left((\log \frac{eR}{r})^{-1}\right)\right)^{\frac{\gamma-q}{q}}}$$

$$\times \int_{\wp} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr.$$

By using the Hölder inequality, we get

$$\int_R^{eR} \frac{1}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^{\gamma}} \int_{\wp} |f(ry) - f(Ry)|^q d\sigma(y) dr$$

$$\leq \frac{q}{\gamma - 1} \left(\int_R^{eR} \int_{\wp} \frac{|f(ry) - f(Ry)|^q}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^{\gamma}} d\sigma(y) dr \right)^{(q-1)/q}$$

$$\times \left(\int_R^{eR} \int_{\wp} r^{q-1} \left(\log \frac{eR}{r} \right)^{q-1} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{q-\gamma}\right.$$

$$\left. \times \left| \frac{df(ry)}{dr} \right|^q d\sigma(y) dr \right)^{1/q}.$$

Thus, we arrive at

$$\left(\int_{B(0,eR) \setminus B(0,R)} \frac{|f - f_R|^q}{\left|\log\left|\log \frac{eR}{|x|}\right|\right|^{\gamma} \left|\log \frac{eR}{|x|}\right|} \frac{dx}{|x|^Q} \right)^{1/q}$$

$$\leq \frac{q}{\gamma - 1} \left(\int_{B(0,eR) \setminus B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E} f \right|^q dx \right)^{1/q}.$$

This and (10.17) imply (10.16). $\square$

By a similar argument one can prove a dual inequality to (10.16), that is, we have

Proposition 10.2.6. *Let* $1 < \gamma < \infty$ *and* $\max\{1, \gamma - 1\} < q < \infty$. *Then for all* $f \in C_0^\infty(\mathbb{G})$ *and for any* $R > 0$ *we have*

$$\left(\int_{B^c(0,R)} \frac{|f - f_{eR}|^q}{\left|\log\left|\log\frac{R}{|x|}\right|\right|^{\gamma}\left|\log\frac{R}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \tag{10.18}$$
$$\leq \frac{q}{\gamma - 1}\left(\int_{B^c(0,R)} |x|^{q-Q}\left|\log\frac{R}{|x|}\right|^{q-1}\left|\log\left|\log\frac{R}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx\right)^{1/q}.$$

Now we are ready to prove Theorem 10.2.3.

Proof of Theorem 10.2.3. By using (10.18) with R replaced by eR we have

$$\left(\int_{B^c(0,eR)} \frac{|f - f_{e^2R}|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \tag{10.19}$$
$$\leq \frac{q}{\gamma - 1}\left(\int_{B^c(0,eR)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx\right)^{1/q}.$$

Then from (10.16) and (10.19) we get (10.15) for functions $f \in C_0^\infty(\mathbb{G})$.

Let us now show (10.15) for general functions $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. First let us verify that (10.16) holds for $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. Let $\{f_m\} \subset C_0^\infty(\mathbb{G})$ be a sequence such that $f_m \to f$ in $W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$ as $m \to \infty$ and almost everywhere by the definition (10.13). If we define

$$f_{R,m}(x) := \frac{f_m(x) - f_m(R\frac{x}{|x|})}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\frac{\gamma}{q}}\left|\log\frac{eR}{|x|}\right|^{\frac{1}{q}}},$$

then $\{f_{R,m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{G}; \frac{dx}{|x|^Q})$, which is a weighted Lebesgue space, since the inequality (10.16) holds for $f_m - f_k \in C_0^\infty(\mathbb{G})$. Consequently, there exists $g_R \in L^q(\mathbb{G}; \frac{dx}{|x|^Q})$ such that $f_{R,m} \to g_R$ in $L^q(G; \frac{dx}{|x|^Q})$ as $m \to \infty$. From the inclusion

$$\left\{x \in \mathbb{G}\backslash\{0\} : f_m\left(R\frac{x}{|x|}\right) \nrightarrow f\left(R\frac{x}{|x|}\right)\right\}$$
$$\subset \bigcup_{r>0}\left\{x \in \mathbb{G}\backslash\{0\} : f_m\left(r\frac{x}{|x|}\right) \nrightarrow f\left(r\frac{x}{|x|}\right)\right\} = \{x \in \mathbb{G}\backslash\{0\} : f_m(x) \nrightarrow f(x)\},$$

it follows that $f_m\left(R\frac{x}{|x|}\right) \to f\left(R\frac{x}{|x|}\right)$, that is, we obtain the equality

$$\frac{f(x) - f\left(R\frac{x}{|x|}\right)}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma/q}\left|\log\frac{eR}{|x|}\right|^{1/q}} = g_R(x)$$

almost everywhere.

That is, inequality (10.16) holds for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. In the same way we establish the inequality (10.19) for any $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. Since inequalities (10.16) and (10.19) hold for any $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$, we get (10.15) for $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$.

Now it remains to show the sharpness of the constant $\frac{q}{\gamma-1}$ in (10.15). By (10.15) for each $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(B(0,R))$, we have

$$\left(\int_{B(0,R)} \frac{|f(x)|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{1/q} \tag{10.20}$$
$$\leq \frac{q}{\gamma-1}\left(\int_{B(0,R)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx\right)^{1/q}.$$

Therefore, it is sufficient to show the sharpness of the constant $\frac{q}{\gamma-1}$ in (10.20). As in the Abelian case (see [MOW15b, Section 3]), we consider a sequence of functions $\{f_\ell\}$ for large $\ell \in \mathbb{N}$ defined by

$$f_\ell(x) := \begin{cases} (\log(\log(\ell eR)))^{\frac{\gamma-1}{q}}, & \text{when} \quad |x| \leq \frac{1}{\ell}, \\ (\log(\log\frac{eR}{|x|}))^{\frac{\gamma-1}{q}}, & \text{when} \quad \frac{1}{\ell} \leq |x| \leq \frac{R}{2}, \\ (\log(\log(2e)))^{\frac{\gamma-1}{q}}\frac{2}{R}(R-|x|), & \text{when} \quad \frac{R}{2} \leq |x| \leq R. \end{cases}$$

It is clear that $f_\ell \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(B(0,R))$. Letting $\widetilde{f}_\ell(r) := f_\ell(x)$ with $r = |x| \geq 0$, we get

$$\frac{d}{dr}\widetilde{f}_\ell(r) = \begin{cases} 0, & \text{when} \quad r < \frac{1}{\ell}, \\ -\frac{\gamma-1}{q}r^{-1}(\log(\log\frac{eR}{r}))^{\frac{\gamma-1}{q}-1}(\log\frac{eR}{r})^{-1}, & \text{when} \quad \frac{1}{\ell} < r < \frac{R}{2}, \\ -\frac{2}{R}(\log(\log(2e)))^{\frac{\gamma-1}{q}}, & \text{when} \quad \frac{R}{2} < r < R. \end{cases}$$

Denoting by $|\wp|$ the $Q-1$-dimensional surface measure of the unit sphere with respect to the quasi-norm $|\cdot|$, by a direct calculation we have

$$\begin{aligned}
&\int_{B(0,R)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx \\
&= |\wp|\int_0^R r^{q-1}\left|\log\frac{eR}{r}\right|^{q-1}\left|\log\left|\log\frac{eR}{r}\right|\right|^{q-\gamma}\left|\frac{d}{dr}\widetilde{f}_\ell(r)\right|^q dr \\
&= |\wp|\left(\frac{\gamma-1}{q}\right)^q\int_{\frac{1}{\ell}}^{\frac{R}{2}} r^{-1}\left(\log\frac{eR}{r}\right)^{-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-1} dr \\
&\quad + (\log(\log(2e)))^{\gamma-1}\left(\frac{2}{R}\right)^q|\wp|\int_{\frac{R}{2}}^R r^{q-1}\left(\log\frac{eR}{r}\right)^{q-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{q-\gamma} dr
\end{aligned}$$

$$= -|\wp| \left(\frac{\gamma-1}{q}\right)^q \int_{\frac{1}{\ell}}^{\frac{R}{2}} \frac{d}{dr}\left(\log\left(\log\left(\log\frac{eR}{r}\right)\right)\right)$$
$$+ (\log(\log(2e)))^{\gamma-1}\left(\frac{2}{R}\right)^q |\wp| \int_{\frac{R}{2}}^{R} r^{q-1}\left(\log\frac{eR}{r}\right)^{q-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{q-\gamma} dr$$
$$=: |\wp| \left(\frac{\gamma-1}{q}\right)^q (\log(\log(\log \ell eR)) - \log(\log(\log 2e))) + |\wp| C_{\gamma,q}, \tag{10.21}$$

where

$$C_{\gamma,q} := (2e)^q (\log(\log(2e)))^{\gamma-1} \int_0^{(\log(\log(2e)))} s^{q-\gamma} e^{q(s-e^s)} ds.$$

The assumption $q-\gamma+1>0$ implies that $C_{\gamma,q} < +\infty$. Moreover, we have

$$\int_{B(0,R)} \frac{|f(x)|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q} = |\wp| \int_0^R \frac{|\widetilde{f}_\ell(r)|^q}{\left|\log\left|\log\frac{eR}{r}\right|\right|^{\gamma}\left|\log\frac{eR}{r}\right|}\frac{dr}{r}$$
$$= |\wp| (\log(\log(\ell eR)))^{\gamma-1} \int_0^{\frac{1}{\ell}} r^{-1}\left(\log\frac{eR}{r}\right)^{-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma} dr$$
$$+ |\wp| \int_{\frac{1}{\ell}}^{\frac{R}{2}} r^{-1}\left(\log\frac{eR}{r}\right)^{-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-1} dr$$
$$+ |\wp| (\log(\log(2e)))^{\gamma-1}\left(\frac{2}{R}\right)^q$$
$$\times \int_{\frac{R}{2}}^{R} r^{-1}(R-r)^q\left(\log\frac{eR}{r}\right)^{-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma} dr$$
$$= \frac{|\wp|}{\gamma-1} + |\wp|(\log(\log(\log(\ell eR)) - \log(\log(\log(2e))) + |\wp| C_{R,\gamma,q}, \tag{10.22}$$

where

$$C_{R,\gamma,q} := (\log(\log(2e)))^{\gamma-1}\left(\frac{2}{R}\right)^q$$
$$\times \int_{\frac{R}{2}}^{R} r^{-1}(R-r)^q\left(\log\frac{eR}{r}\right)^{-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\gamma} dr.$$

The inequality $\log(\log\frac{eR}{r}) \geq \frac{R-r}{R}$ for all $r \leq R$ and the assumption $q-\gamma > -1$ imply $C_{R,\gamma,q} < +\infty$. Then, by (10.21) and (10.22), we arrive at

$$\int_{B(0,R)} |x|^{q-Q}\left|\log\frac{eR}{|x|}\right|^{q-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{q-\gamma}\left|\frac{1}{|x|}\mathbb{E}f\right|^q dx$$
$$\times \left(\int_{B(0,R)} \frac{|f(x)|^q}{\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\gamma}\left|\log\frac{eR}{|x|}\right|}\frac{dx}{|x|^Q}\right)^{-1} \to \left(\frac{\gamma-1}{q}\right)^q$$

as $\ell \to \infty$, which implies that the constant $\frac{q}{\gamma-1}$ in (10.20) is sharp. $\square$

10.3 Generalized Morrey spaces

In the next sections, we develop the theory of Morrey and Campanato spaces on general homogeneous groups. Moreover, we analyse properties of Bessel–Riesz operators, maximal operators, and fractional integral operators on these spaces.

A brief discussion of the general theory of Morrey and Campanato spaces was given in the introduction, and we can refer there also for references.

10.3.1 Bessel–Riesz kernels on homogeneous groups

The classical Bessel–Riesz operators on the Euclidean space $\mathbb{R}^n$ are the operators of the form

$$I_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y)f(y)dy = \int_{\mathbb{R}^n} \frac{|x-y|^{\alpha-n}}{(1+|x-y|)^{\gamma}} f(y)dy, \tag{10.23}$$

where $\gamma \geq 0$ and $0 < \alpha < n$. Here, $I_{\alpha,\gamma}$ and $K_{\alpha,\gamma}$ are called a *Bessel–Riesz operator* and a *Bessel–Riesz kernel*, respectively. The original works on these operators go back to Hardy and Littlewood in [HL27, HL32] and Sobolev in [Sob38]. We refer to Section 5.3 for the appearance of the related operators.

Definition 10.3.1 (Bessel–Riesz kernels)**.** A natural analogue of the operators (10.23) in the setting of homogeneous groups are operators of the form

$$I_{\alpha,\gamma}f(x) := \int_{\mathbb{G}} K_{\alpha,\gamma}(xy^{-1})f(y)dy = \int_{\mathbb{G}} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^{\gamma}} f(y)dy, \tag{10.24}$$

with the Bessel–Riesz kernels defined by

$$K_{\alpha,\gamma}(z) = \frac{|z|^{\alpha-Q}}{(1+|z|)^{\gamma}}, \tag{10.25}$$

where $|\cdot|$ is a homogeneous quasi-norm on a homogeneous group $\mathbb{G}$.

First we calculate the L^p-norms of the Bessel–Riesz kernels.

Theorem 10.3.2 (L^p-norms of Bessel–Riesz kernels)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. If $0 < \alpha < Q$ and $\gamma > 0$ then $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$ and*

$$\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \sim \left(\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1},$$

for $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

We will use the following result in some proofs.

Lemma 10.3.3 ([IGLE15]). *If $b > a > 0$ then for every $u > 1$ and $R > 0$ we have*

$$\sum_{k\in\mathbb{Z}} \frac{(u^k R)^a}{(1+u^k R)^b} < \infty.$$

Proof of Theorem 10.3.2. By using the polar coordinates decomposition from Proposition 1.2.10, for any $R > 0$ we have

$$\begin{aligned}\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx &= \int_{\mathbb{G}} \frac{|x|^{(\alpha-Q)p_1}}{(1+|x|)^{\gamma p_1}} dx \\ &= \int_0^\infty \int_{\wp} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} d\sigma(y) dr \\ &= |\wp| \sum_{k\in\mathbb{Z}} \int_{2^k R \le r < 2^{k+1} R} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} dr,\end{aligned}$$

where $|\wp|$ is the $Q-1$-dimensional surface measure of the unit sphere. This implies

$$\begin{aligned}\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx &\le |\wp| \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma p_1}} \int_{2^k R \le r < 2^{k+1} R} r^{(\alpha-Q)p_1+Q-1} dr \\ &= \frac{|\wp|(2^{(\alpha-Q)p_1+Q}-1)}{(\alpha-Q)p_1+Q} \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx &\ge \frac{|\wp|}{2^{\gamma p_1}} \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma p_1}} \int_{2^k R \le r < 2^{k+1} R} r^{(\alpha-Q)p_1+Q-1} dr \\ &= \frac{|\wp|(2^{(\alpha-Q)p_1+Q}-1)}{2^{\gamma p_1}((\alpha-Q)p_1+Q)} \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}.\end{aligned}$$

Combining these two inequalities, for each $R > 0$ we have

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx \sim \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}.$$

For $p_1 \in \left(\frac{Q}{Q+\gamma-\alpha}, \frac{Q}{Q-\alpha}\right)$ using Lemma 10.3.3 with $u = 2$, $a = (\alpha-Q)p_1 + Q$ and $b = \gamma p_1$, we arrive at

$$\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} < \infty,$$

which also implies that $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$. $\square$

Since

$$I_{\alpha,\gamma} f = K_{\alpha,\gamma} * f,$$

the Young inequality in Proposition 1.2.13 immediately implies

Corollary 10.3.4 (L^p-L^q boundedness of Bessel–Riesz operators)**.** *Let* $0 < \alpha < Q$ *and* $\gamma > 0$. *Assume that* $1 \leq p, q, p_1 \leq \infty$, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{p_1}$ *and* $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$. *Then we have*

$$\|I_{\alpha,\gamma} f\|_{L^q(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{L^p(\mathbb{G})}$$

for all $f \in L^p(\mathbb{G})$.

This means that $I_{\alpha,\gamma}$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ and that its operator norm can be estimated as

$$\|I_{\alpha,\gamma}\|_{L^p(\mathbb{G}) \to L^q(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

10.3.2 Hardy–Littlewood maximal operator in Morrey spaces

We now define Morrey and generalized Morrey spaces on homogeneous groups and show the boundedness of the Hardy–Littlewood maximal operator in these spaces.

As usual throughout this section $\mathbb{G}$ is a homogeneous group of homogeneous dimension Q.

Definition 10.3.5 (Local Morrey spaces)**.** Let us define the *local Morrey spaces* $LM^{p,q}(\mathbb{G})$ by

$$LM^{p,q}(\mathbb{G}) := \{f \in L^p_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{LM^{p,q}(\mathbb{G})} < \infty\}, \quad 1 \leq p \leq q, \tag{10.26}$$

where

$$\|f\|_{LM^{p,q}(\mathbb{G})} := \sup_{r>0} r^{Q(1/q-1/p)} \left(\int_{B(0,r)} |f(x)|^p dx \right)^{1/p}.$$

Sometimes these spaces are called central Morrey spaces in the literature.

Definition 10.3.6 (Generalized local Morrey spaces)**.** Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $1 \leq p < \infty$. We define the *generalized local Morrey space* $LM^{p,\phi}(\mathbb{G})$ by

$$LM^{p,\phi}(\mathbb{G}) := \{f \in L^p_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{LM^{p,\phi}(\mathbb{G})} < \infty\}, \tag{10.27}$$

where

$$\|f\|_{LM^{p,\phi}(\mathbb{G})} := \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |f(x)|^p dx \right)^{1/p}.$$

Let us first formulate the assumptions for the function ϕ in the above definition.

Assumptions on ϕ

From now on we will assume that ϕ is nonincreasing and $t^{Q/p}\phi(t)$ is nondecreasing, so that ϕ satisfies the doubling condition, i.e., there exists a constant $C_1 > 0$ such that we have

$$\frac{1}{2} \leq \frac{r}{s} \leq 2 \Longrightarrow \frac{1}{C_1} \leq \frac{\phi(r)}{\phi(s)} \leq C_1. \tag{10.28}$$

Definition 10.3.7 (Hardy–Littlewood maximal operator)**.** For every $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ we define the Hardy–Littlewood maximal operator $\mathscr{M}$ by

$$\mathscr{M} f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \tag{10.29}$$

where $|B(x,r)|$ denotes the Haar measure of the quasi-ball $B(x,r)$.

Using the definition (10.26) of local Morrey spaces one can readily obtain the following estimate:

Lemma 10.3.8. *For any* $1 \leq p_2 \leq p_1$ *and* $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$ *we have*

$$\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{LM^{p_1,p_1}(\mathbb{G})} = \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}. \tag{10.30}$$

We now prove the boundedness of the Hardy–Littlewood maximal operator on generalized local Morrey spaces.

Theorem 10.3.9 (Hardy–Littlewood maximal operator on generalized local Morrey spaces)**.** *Let* $1 < p < \infty$. *Then there exists some* $C_p > 0$ *such that for all* $f \in LM^{p,\phi}(\mathbb{G})$ *we have*

$$\|\mathscr{M} f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_p \|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{10.31}$$

Remark 10.3.10. In the Euclidean space $\mathbb{R}^n$ this was shown by Nakai in [Nak94]. On stratified groups (or homogeneous Carnot groups) it was shown in [GAM13, Corollary 3.2]. For general homogeneous groups this and other results of this section were shown in [RSY18c].

Proof of Theorem 10.3.9. From the definition of the norm of the generalized local Morrey space (10.27) it follows that

$$\left(\int_{B(0,r)} |f(x)|^p dx \right)^{1/p} \leq \phi(r) r^{\frac{Q}{p}} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{10.32}$$

for all $r > 0$.

On the other hand, using Corollary 2.5 (b) from Folland and Stein [FS82] we have the general property of the maximal function

$$\left(\int_{B(0,r)} |\mathscr{M} f(x)|^p dx \right)^{1/p} \leq C_p \left(\int_{B(0,r)} |f(x)|^p dx \right)^{1/p}. \tag{10.33}$$

Combining (10.32) and (10.33) we arrive at

$$\frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |\mathscr{M} f(x)|^p dx \right)^{1/p} \leq C_p \|f\|_{LM^{p,\phi}(\mathbb{G})},$$

for all $r > 0$. But this is exactly (10.31). $\square$

10.3.3 Bessel–Riesz operators in Morrey spaces

In this section, we show the boundedness of the Bessel–Riesz operators on the generalized local Morrey space from Definition 10.3.6. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$ with the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ the results of this section have been obtained in [IGE16]. The exposition of this section follows the results obtained in [RSY18c].

Theorem 10.3.11 (Boundedness of Bessel–Riesz operators on Morrey spaces, I)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $\gamma > 0$ and $0 < \alpha < Q$. Let $\beta < -\alpha$, $1 < p < \infty$, and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$. Assume that $0 < \phi(r) \le Cr^\beta$ for all $r > 0$. Then for all $f \in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \le C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{10.34}$$

where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 10.3.11. For every $f \in LM^{p,\phi}(\mathbb{G})$, let us write $I_{\alpha,\gamma} f(x)$ in the form

$$I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x),$$

where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^\gamma} dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1+|xy^{-1}|)^\gamma} dy,$$

for some $R > 0$.

By using the dyadic decomposition for I_1 we obtain

$$\begin{aligned} |I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^k R \le |xy^{-1}| < 2^{k+1} R} \frac{|xy^{-1}|^{\alpha-Q} |f(y)|}{(1+|xy^{-1}|)^\gamma} dy \\ &\le \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^\gamma} \int_{2^k R \le |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\ &\le C \mathscr{M} f(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q+Q/p_1} (2^k R)^{Q/p_1'}}{(1+2^k R)^\gamma}. \end{aligned}$$

From this using Hölder's inequality for $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ we get

$$|I_1(x)| \le C \mathscr{M} f(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \left(\sum_{k=-\infty}^{-1} (2^k R)^Q \right)^{1/p_1'}.$$

Since

$$\left(\sum_{k=-\infty}^{-1}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}\le\left(\sum_{k\in\mathbb{Z}}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}\sim\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})},\tag{10.35}$$

we arrive at

$$|I_1(x)|\le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\mathscr{M}f(x)R^{Q/p_1'}.\tag{10.36}$$

For the second term I_2, by using Hölder's inequality for $\frac{1}{p}+\frac{1}{p'}=1$ we obtain that

$$\begin{aligned}|I_2(x)|&\le\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|dy\\&\le\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}dy\right)^{\frac{1}{p'}}\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|^pdy\right)^{\frac{1}{p}}\\&=\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}\left(\int_{2^kR}^{2^{k+1}R}\int_{\wp}r^{Q-1}d\sigma(y)dr\right)^{\frac{1}{p'}}\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|^pdy\right)^{\frac{1}{p}}\\&\le C\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}(2^kR)^{Q/p'}\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|^pdy\right)^{\frac{1}{p}}.\end{aligned}$$

This implies that

$$|I_2(x)|\le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q+Q/p_1}}{(1+2^kR)^{\gamma}}\phi(2^kR)(2^kR)^{Q/p_1'}.$$

Since $\phi(r)\le Cr^{\beta}$ by assumption, we can estimate

$$|I_2(x)|\le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q+Q/p_1}}{(1+2^kR)^{\gamma}}(2^kR)^{\beta+Q/p_1'}.$$

Applying Hölder's inequality again we get

$$|I_2(x)|\le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\left(\sum_{k=0}^{\infty}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}\left(\sum_{k=0}^{\infty}(2^kR)^{\beta p_1'+Q}\right)^{1/p_1'}.$$

From the conditions $p_1<\frac{Q}{Q-\alpha}$ and $\beta<-\alpha$ we have $\beta p_1'+Q<0$. By Theorem 10.3.2, we also have

$$\left(\sum_{k=0}^{\infty}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}\le\left(\sum_{k\in\mathbb{Z}}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}\sim\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

Using these, we arrive at

$$|I_2(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}R^{Q/p_1'+\beta}. \tag{10.37}$$

Combining estimates (10.36) and (10.37) we get

$$|I_{\alpha,\gamma}f(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}(\mathscr{M}f(x)R^{Q/p_1'} + \|f\|_{LM^{p,\phi}(\mathbb{G})}R^{Q/p_1'+\beta}).$$

Assuming that f is not identically 0 and that $\mathscr{M}f$ is finite everywhere, we can choose $R>0$ such that $R^\beta = \frac{\mathscr{M}f(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, that is,

$$|I_{\alpha,\gamma}f(x)| \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p_1'}}(\mathscr{M}f(x))^{1+\frac{Q}{\beta p_1'}},$$

for every $x\in\mathbb{G}$. Setting $q=\frac{\beta p_1' p}{\beta p_1'+Q}$, for any $r>0$ we get

$$\left(\int_{|x|<r}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q} \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\left(\int_{|x|<r}|\mathscr{M}f(x)|^p dx\right)^{1/q}.$$

Then we divide both sides by $(\phi(r))^{p/q}r^{Q/q}$ to get

$$\frac{\left(\int_{|x|<r}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q}}{\psi(r)r^{Q/q}} \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\frac{\left(\int_{|x|<r}|\mathscr{M}f(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q}r^{Q/q}},$$

where $\psi(r)=(\phi(r))^{p/q}$. Now by taking the supremum over $r>0$ we obtain that

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \le C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\|\mathscr{M}f\|_{LM^{p,\phi}(\mathbb{G})}^{p/q},$$

which gives (10.34) after applying estimate (10.31). □

Lemma 10.3.8 states, in particular, that the Bessel–Riesz kernel belongs to local Morrey spaces. This fact will be used in the following statement refining estimate (10.34) in Theorem 10.3.11.

Theorem 10.3.12 (Boundedness of Bessel–Riesz operators on Morrey spaces, II)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $\gamma>0$, $0<\alpha<Q$ and $1<p<\infty$. Let $\beta<-\alpha$, $\frac{Q}{Q+\gamma-\alpha}<p_2\le p_1<\frac{Q}{Q-\alpha}$ and $p_2\ge 1$. Assume that $0<\phi(r)\le Cr^\beta$ for all $r>0$. Then for all $f\in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} \le C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{10.38}$$

where $q=\frac{\beta p_1' p}{\beta p_1'+Q}$ and $\psi(r)=(\phi(r))^{p/q}$.

Indeed, this refines Theorem 10.3.11 since by Lemma 10.3.8 we can estimate

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}\\ &\leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}.\end{aligned}$$

Proof of Theorem 10.3.12. Similarly to the proof of Theorem 10.3.11 we write $I_{\alpha,\gamma}f(x)$ in the form

$$I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x),$$

where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^\gamma}dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^\gamma}dy,$$

for some $R > 0$ to be chosen later.

As before, we estimate the first term I_1 by using the dyadic decomposition:

$$\begin{aligned}|I_1(x)| &\leq \sum_{k=-\infty}^{-1}\int_{2^kR\leq|xy^{-1}|<2^{k+1}R}\frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^\gamma}dy\\ &\leq \sum_{k=-\infty}^{-1}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma}\int_{2^kR\leq|xy^{-1}|<2^{k+1}R}|f(y)|dy\\ &\leq CMf(x)\sum_{k=-\infty}^{-1}\frac{(2^kR)^{\alpha-Q+Q/p_2}(2^kR)^{Q/p_2'}}{(1+2^kR)^\gamma},\end{aligned}$$

where $1 \leq p_2 \leq p_1$. From this using Hölder's inequality for $\frac{1}{p_2}+\frac{1}{p_2'}=1$, we get

$$|I_1(x)| \leq C\mathscr{M}f(x)\left(\sum_{k=-\infty}^{-1}\frac{(2^kR)^{(\alpha-Q)p_2+Q}}{(1+2^kR)^{\gamma p_2}}\right)^{1/p_2}\left(\sum_{k=-\infty}^{-1}(2^kR)^Q\right)^{1/p_2'}.$$

In view of (10.35) we have

$$\begin{aligned}|I_1(x)| &\leq C_2\mathscr{M}f(x)\left(\int_{0<|x|<R}K_{\alpha,\gamma}^{p_2}(x)dx\right)^{1/p_2}R^{Q/p_2'}\\ &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\mathscr{M}f(x)R^{Q/p_1'}.\end{aligned}\tag{10.39}$$

Now for I_2 by using Hölder's inequality for $\frac{1}{p}+\frac{1}{p'}=1$ we have

$$|I_2(x)| \leq \sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^\gamma}(2^kR)^{Q/p'}\left(\int_{2^kR\leq|xy^{-1}|<2^{k+1}R}|f(y)|^pdy\right)^{1/p},$$

that is,

$$|I_2(x)| \leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R)}{(1+2^k R)^\gamma} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} dy\right)^{1/p_2}}{(2^k R)^{Q/p_2}}$$
$$\leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \phi(2^k R)(2^k R)^{Q/p_1'} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{(2^k R)^{Q/p_2 - Q/p_1}},$$

where we have used the inequality

$$\begin{aligned} &\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2} \\ &\sim \frac{(2^k R)^{(\alpha-Q)+Q/p_2}}{(1+2^k R)^\gamma} \geq C\frac{(2^k R)^{(\alpha-Q)}}{(1+2^k R)^\gamma} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} dy\right)^{1/p_2}. \end{aligned} \tag{10.40}$$

Since we have $\phi(r) \leq Cr^\beta$ by assumption and since

$$\frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1}R} K_{\alpha,\gamma}^{p_2}(xy^{-1})dy\right)^{1/p_2}}{(2^k R)^{Q/p_2 - Q/p_1}} \lesssim \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}$$

for every $k = 0, 1, 2, \ldots$, we get

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{\beta + Q/p_1'}.$$

Using that $\beta + Q/p_1' < 0$ we obtain

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} R^{\beta + Q/p_1'}. \tag{10.41}$$

Combining estimates (10.39) and (10.41) we get

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}(\mathscr{M}f(x)R^{Q/p_1'} + \|f\|_{LM^{p,\phi}(\mathbb{G})} R^{\beta + Q/p_1'}).$$

Assuming that f is not identically 0 and that $\mathscr{M}f$ is finite everywhere, we can choose $R > 0$ such that $R^\beta = \frac{\mathscr{M}f(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, which yields

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p_1'}} (\mathscr{M}f(x))^{1+\frac{Q}{\beta p_1'}}.$$

Now by using that $q = \frac{\beta p_1' p}{\beta p_1' + Q}$, for every $r > 0$ we obtain

$$\begin{aligned} &\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q} \\ &\quad \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q} \left(\int_{|x|<r} |\mathscr{M}f(x)|^p dx\right)^{1/q}. \end{aligned}$$

Then we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$\frac{\left(\int_{|x|<r} |I_{\alpha,\gamma} f(x)|^q dx\right)^{1/q}}{\psi(r) r^{Q/q}} \leq C \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \|f\|^{1-p/q}_{LM^{p,\phi}(\mathbb{G})} \frac{\left(\int_{|x|<r} |\mathscr{M} f(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q} r^{Q/q}},$$

where $\psi(r) = (\phi(r))^{p/q}$. Taking the supremum over $r > 0$ and then using (10.31), we obtain the desired result

$$\begin{aligned}\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \|f\|^{1-p/q}_{LM^{p,\phi}(\mathbb{G})} \|\mathscr{M} f\|^{p/q}_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})},\end{aligned}$$

proving (10.38). □

To refine and extend the obtained boundedness statements we first show that the Bessel–Riesz kernel $K_{\alpha,\gamma}$ belongs to the generalized local Morrey space $LM^{p_2,\omega}(\mathbb{G})$ for some $p_2 \geq 1$ and some function ω.

Lemma 10.3.13 (Bessel–Riesz kernel in generalized local Morrey space)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q. Let $\gamma > 0$, $p_2 \geq 1$ and $Q - \frac{Q}{p_2} < \alpha < Q$. If $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\omega(r) \geq C r^{\alpha - Q}$ for all $r > 0$, then $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$.*

Proof of Lemma 10.3.13. It is sufficient to evaluate the following integral around zero, and using polar decomposition from Proposition 1.2.10, we have

$$\begin{aligned}\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x) dx &= \int_{|x|\leq R} \frac{|x|^{(\alpha-Q)p_2}}{(1+|x|)^{\gamma p_2}} dx \\ &\leq |\wp| \int_{0<r\leq R} r^{(\alpha-Q)p_2+Q-1} dr \leq C \omega^{p_2}(R) R^Q.\end{aligned}$$

By dividing both sides of this inequality by $\omega^{p_2}(R) R^Q$ and taking p_2th-root, we obtain

$$\frac{\left(\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x) dx\right)^{1/p_2}}{\omega(R) R^{Q/p_2}} \leq C^{1/p_2}.$$

Then, we take the supremum over $R > 0$ to get

$$\sup_{R>0} \frac{\left(\int_{|x|\leq R} K^{p_2}_{\alpha,\gamma}(x) dx\right)^{1/p_2}}{\omega(R) R^{Q/p_2}} < \infty,$$

which implies that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$. □

Theorem 10.3.14 (Boundedness of Bessel–Riesz operators on Morrey spaces, III)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $0<\alpha<Q$, $\gamma>0$ and $1<p<\infty$. Let $\omega:\mathbb{R}^+\to\mathbb{R}^+$ satisfy the doubling condition (10.28) and assume that $0<\omega(r)\le Cr^{-\alpha}$ for all $r>0$, so that $K_{\alpha,\gamma}\in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha}<p_2<\frac{Q}{Q-\alpha}$ and $p_2\ge 1$. Let $\beta<-\alpha<-Q-\beta$ and assume that $0<\phi(r)\le Cr^{\beta}$ for all $r>0$. Then for all $f\in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})}\le C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{10.42}$$

where $q=\frac{\beta p}{\beta+Q-\alpha}$ and $\psi(r)=(\phi(r))^{p/q}$.

Proof of Theorem 10.3.14. As in the proof of Theorem 10.3.11, we write

$$I_{\alpha,\gamma}f(x):=I_1(x)+I_2(x),$$

where

$$I_1(x):=\int_{B(x,R)}\frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^{\gamma}}dy$$

and

$$I_2(x):=\int_{B^c(x,R)}\frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^{\gamma}}dy,$$

for some $R>0$ to be chosen later.

First, we estimate I_1 by using the dyadic decomposition

$$\begin{aligned}|I_1(x)|&\le\sum_{k=-\infty}^{-1}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}\frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^{\gamma}}dy\\&\le\sum_{k=-\infty}^{-1}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|dy\\&\le C\mathscr{M}f(x)\sum_{k=-\infty}^{-1}\frac{(2^kR)^{\alpha-Q+Q/p_2}(2^kR)^{Q/p_2'}}{(1+2^kR)^{\gamma}}.\end{aligned}$$

From this using Hölder's inequality for $\frac{1}{p_2}+\frac{1}{p_2'}=1$ we get

$$|I_1(x)|\le C\mathscr{M}f(x)\left(\sum_{k=-\infty}^{-1}\frac{(2^kR)^{(\alpha-Q)p_2+Q}}{(1+2^kR)^{\gamma p_2}}\right)^{1/p_2}\left(\sum_{k=-\infty}^{-1}(2^kR)^{Q}\right)^{1/p_2'}.$$

By (10.35) we have

$$\begin{aligned}|I_1(x)|&\le C\mathscr{M}f(x)\left(\int_{0<|x|<R}K_{\alpha,\gamma}^{p_2}(x)dx\right)^{1/p_2}R^{Q/p_2'}\\&\le C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\mathscr{M}f(x)\omega(R)R^{Q},\end{aligned}$$

and using that $\omega(r) \le Cr^{-\alpha}$ by assumption, we arrive at

$$|I_1(x)| \le C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\mathscr{M}f(x)R^{Q-\alpha}. \tag{10.43}$$

Now let us estimate the second term I_2 as follows

$$\begin{aligned}|I_2(x)| &\le \sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|dy\\ &\le C\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha-Q}}{(1+2^kR)^{\gamma}}(2^kR)^{Q/p'}\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}|f(y)|^p dy\right)^{1/p}\\ &\le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha}\phi(2^kR)}{(1+2^kR)^{\gamma}}\frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}dy\right)^{1/p_2}}{(2^kR)^{Q/p_2}},\end{aligned}$$

where we have used that $\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}dy\right)^{1/p_2}\sim(2^kR)^{Q/p_2}$. Using (10.40) we obtain

$$|I_2(x)| \le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}\frac{(2^kR)^{\alpha}\phi(2^kR)}{(2^kR)^{\alpha-Q}}\frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}K^{p_2}_{\alpha,\gamma}(xy^{-1})dy\right)^{\frac{1}{p_2}}}{(2^kR)^{Q/p_2}}.$$

Taking into account that $\phi(r)\le Cr^{\beta}$ and $\omega(r)\le Cr^{-\alpha}$ for all $r>0$ we have

$$|I_2(x)| \le C\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}(2^kR)^{Q-\alpha+\beta}\frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}K^{p_2}_{\alpha,\gamma}(xy^{-1})dy\right)^{\frac{1}{p_2}}}{\omega(2^kR)(2^kR)^{Q/p_2}}.$$

Since we have

$$\frac{\left(\int_{2^kR\le|xy^{-1}|<2^{k+1}R}K^{p_2}_{\alpha,\gamma}(xy^{-1})dy\right)^{1/p_2}}{\omega(2^kR)(2^kR)^{Q/p_2}}\lesssim\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}$$

for every $k=0,1,2,\dots$, it follows that

$$|I_2(x)| \le C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}\sum_{k=0}^{\infty}(2^kR)^{Q-\alpha+\beta}.$$

Since $Q-\alpha+\beta<0$, it implies that

$$|I_2(x)| \le C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}R^{Q-\alpha+\beta}. \tag{10.44}$$

Combining estimates (10.43) and (10.44) we get

$$|I_{\alpha,\gamma}f(x)| \le C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}(\mathscr{M}f(x)R^{Q-\alpha}+\|f\|_{LM^{p,\phi}(\mathbb{G})}R^{Q-\alpha+\beta}).$$

Assuming that f is not identically 0 and that $\mathscr{M}f$ is finite everywhere, we can choose $R > 0$ such that $R^\beta = \frac{\mathscr{M}f(x)}{\|f\|_{LM^{p,\phi}(\mathbb{G})}}$, that is

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{(\alpha-Q)/\beta}(\mathscr{M}f(x))^{1+(Q-\alpha)/\beta}.$$

Since $q = \frac{\beta p}{\beta+Q-\alpha}$, for all $r > 0$ we get

$$\begin{aligned}\left(\int_{|x|<r}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q} \\ \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\left(\int_{|x|<r}|\mathscr{M}f(x)|^p dx\right)^{1/q}.\end{aligned}$$

Then we divide both sides by $(\phi(r))^{p/q}r^{Q/q}$ to get

$$\begin{aligned}\frac{\left(\int_{|x|<r}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q}}{\psi(r)r^{Q/q}} \\ \leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\frac{\left(\int_{|x|<r}|\mathscr{M}f(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q}r^{Q/q}},\end{aligned}$$

where $\psi(r) = (\phi(r))^{p/q}$. Finally, taking the supremum over $r > 0$ and using (10.31), we obtain the desired result

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{1-p/q}\|Mf\|_{LM^{p,\phi}(\mathbb{G})}^{p/q} \\ &\leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},\end{aligned}$$

which gives (10.42). □

Remark 10.3.15. We can make the following comparison between the obtained estimates, similarly to the Euclidean case [IGE16, Section 3], namely, that also in the case of general homogeneous groups, Theorem 10.3.14 gives the best estimate among the three. Indeed, if we take

$$\omega(R) := (1 + R^{Q/q_1})R^{-Q/p_1}$$

for some $q_1 > p_1$, then

$$\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}.$$

By Theorem 10.3.14 and Lemma 10.3.8 we then obtain

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}.\end{aligned}$$

10.3.4 Generalized Bessel–Riesz operators

In this section we investigate properties of some generalizations of Bessel–Riesz operators.

Definition 10.3.16 (Generalized Bessel–Riesz operators)**.** We define the *generalized Bessel–Riesz operator* $I_{\widetilde{\rho},\gamma}$ by

$$I_{\widetilde{\rho},\gamma} f(x) := \int_{\mathbb{G}} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}} f(y)dy, \tag{10.45}$$

where $\gamma \geq 0$, $\widetilde{\rho} : \mathbb{R}^+ \to \mathbb{R}^+$, $\widetilde{\rho}$ satisfies the doubling condition (10.28) and the condition

$$\int_0^1 \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}} dt < \infty. \tag{10.46}$$

We denote its kernel by

$$K_{\widetilde{\rho},\gamma}(z) = \frac{\widetilde{\rho}(|z|)}{(1+|z|)^{\gamma}}.$$

We recover the usual Bessel–Riesz kernels from Definition 10.3.1 by taking $\widetilde{\rho}(t) = t^{\alpha-Q}$ for $0 < \alpha < Q$, in which case we have

$$K_{\widetilde{\rho},\gamma}(z) = K_{\alpha,\gamma}(z) = \frac{|z|^{\alpha-Q}}{(1+|z|)^{\gamma}}.$$

Theorem 10.3.17 (Boundedness of generalized Bessel–Riesz operators on Morrey spaces)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $\gamma > 0$. Let $\widetilde{\rho}$ and ϕ satisfy the doubling condition* (10.28)*. Assume that ϕ is surjective and for some $1 < p < q < \infty$ satisfies*

$$\int_r^{\infty} \frac{(\phi(t))^p}{t} dt \leq C_1(\phi(r))^p, \tag{10.47}$$

and

$$\phi(r)\int_0^r \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}} dt + \int_r^{\infty} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \leq C_2(\phi(r))^{p/q}, \tag{10.48}$$

for all $r > 0$. Then we have

$$\|I_{\widetilde{\rho},\gamma} f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{10.49}$$

Proof of Theorem 10.3.17*.* For every $R > 0$, let us write $I_{\widetilde{\rho},\gamma} f(x)$ in the form

$$I_{\widetilde{\rho},\gamma} f(x) = I_{1,\widetilde{\rho}}(x) + I_{2,\widetilde{\rho}}(x),$$

where

$$I_{1,\widetilde{\rho}}(x) := \int_{B(x,R)} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}} f(y)dy$$

and

$$I_{2,\widetilde{\rho}}(x) := \int_{B^c(x,R)} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}} f(y)dy.$$

For $I_{1,\widetilde{\rho}}(x)$, we have

$$\begin{aligned}
|I_{1,\widetilde{\rho}}(x)| &\le \int_{|xy^{-1}|<R} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}} |f(y)|dy \\
&\le \int_{|xy^{-1}|<R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}} |f(y)|dy \\
&= \sum_{k=-\infty}^{-1} \int_{2^k R \le |xy^{-1}| < 2^{k+1}R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}} |f(y)|dy.
\end{aligned}$$

In view of (10.28) we can estimate

$$\begin{aligned}
|I_{1,\widetilde{\rho}}(x)| &\le C \sum_{k=-\infty}^{-1} \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma}} \int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy \\
&\le CMf(x) \sum_{k=-\infty}^{-1} \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q}} \\
&\le CMf(x) \sum_{k=-\infty}^{-1} \int_{2^k R}^{2^{k+1}R} \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}} dt \\
&= CMf(x) \int_0^R \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}} dt,
\end{aligned}$$

where we have used the fact that

$$\int_{2^k R}^{2^{k+1}R} \frac{\widetilde{\rho}(t)}{t^{\gamma-Q+1}} dt \ge C \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q+1}} 2^k R \ge C \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q}}. \tag{10.50}$$

Now, using (10.48) we obtain

$$|I_{1,\widetilde{\rho}}(x)| \le CMf(x)(\phi(R))^{(p-q)/q}. \tag{10.51}$$

For $I_{2,\widetilde{\rho}}(x)$ can estimate

$$\begin{aligned}
|I_{2,\widetilde{\rho}}(x)| &\le \int_{|xy^{-1}|\ge R} \frac{\widetilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^{\gamma}} |f(y)|dy \\
&\le \int_{|xy^{-1}|\ge R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}} |f(y)|dy \\
&= \sum_{k=0}^{\infty} \int_{2^k R \le |xy^{-1}| < 2^{k+1}R} \frac{\widetilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^{\gamma}} |f(y)|dy.
\end{aligned}$$

Applying (10.28), we get

$$|I_{2,\widetilde{\rho}}(x)| \le C \sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma}} \int_{|xy^{-1}|<2^{k+1}R} |f(y)| dy.$$

From this using the Hölder inequality, we obtain

$$\begin{aligned}
|I_{2,\widetilde{\rho}}(x)| &\le C \sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma}} \left(\int_{|xy^{-1}|<2^{k+1}R} dy \right)^{1-1/p} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)| dy \right)^{1/p} \\
&\le C \sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^k R)}{(2^k R)^{\gamma - Q + \frac{Q}{p}}} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)| dy \right)^{1/p} \\
&\le C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^k R)^{\gamma - Q}} \\
&\le C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \\
&= C \|f\|_{LM^{p,\phi}(\mathbb{G})} \int_{R}^{\infty} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt,
\end{aligned}$$

where we have used the fact that

$$\int_{2^k R}^{2^{k+1}R} \frac{\widetilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \ge C \frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^{k+1}R)^{\gamma-Q+1}} 2^k R \ge C \frac{\widetilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^k R)^{\gamma-Q}}.$$

Now, using (10.48) we obtain

$$|I_{2,\widetilde{\rho}}(x)| \le C \|f\|_{LM^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}. \tag{10.52}$$

Combining estimates (10.51) and (10.52) we get

$$|I_{\widetilde{\rho},\gamma} f(x)| \le C (Mf(x)(\phi(R))^{(p-q)/q} + \|f\|_{LM^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}).$$

Assuming that f is not identically 0 and that $\mathscr{M} f$ is finite everywhere and then using the fact that ϕ is surjective, we can choose $R > 0$ such that

$$\phi(R) = \mathscr{M} f(x) \cdot \|f\|^{-1}_{LM^{p,\phi}(\mathbb{G})}.$$

Thus, for every $x \in \mathbb{G}$, we have

$$|I_{\widetilde{\rho},\gamma} f(x)| \le C (\mathscr{M} f(x))^{\frac{p}{q}} \|f\|^{\frac{q-p}{q}}_{LM^{p,\phi}(\mathbb{G})}.$$

It follows that

$$\left(\int_{B(0,r)} |I_{\widetilde{\rho},\gamma} f(x)|^q \right)^{1/q} \le C \left(\int_{B(0,r)} |\mathscr{M} f(x)|^p \right)^{1/q} \|f\|^{\frac{q-p}{q}}_{LM^{p,\phi}(\mathbb{G})}.$$

Now dividing both sides by $(\phi(r))^{p/q} r^{Q/q}$ we get

$$\begin{aligned}&\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^q\right)^{1/q}\\&\leq C\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|\mathscr{M}f(x)|^p\right)^{1/q}\|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.\end{aligned}$$

Taking the supremum over $r>0$ and using the boundedness of the maximal operator $\mathscr{M}$ on $LM^{p,\phi}(\mathbb{G})$ from (10.31), we obtain

$$\|I_{\widetilde{\rho},\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})}\leq C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

This gives (10.49). □

10.3.5 Olsen type inequalities for Bessel–Riesz operator

In this section, we show the Olsen type inequalities for the generalized Bessel–Riesz operator $I_{\rho,\gamma}$. Another type of this inequality will be shown in Theorem 10.3.22.

Theorem 10.3.18 (Olsen type inequalities for generalized Bessel–Riesz operators)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $\gamma>0$. Let $\widetilde{\rho}$ and ϕ satisfy the doubling condition* (10.28)*. Assume that ϕ is surjective and satisfies* (10.47) *and* (10.48)*. Then we have*

$$\|WI_{\widetilde{\rho},\gamma}f\|_{LM^{p,\phi}(\mathbb{G})}\leq C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},\tag{10.53}$$

provided that $1<p<p_2<\infty$ and $W\in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 10.3.18. By using Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\widetilde{\rho},\gamma}f(x)|^p dx\\&\leq\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{p/p_2}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{(p_2-p)/p_2}.\end{aligned}$$

Now let us take the pth roots and then divide both sides by $\phi(r)$, so that

$$\begin{aligned}&\frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\widetilde{\rho},\gamma}f(x)|^p dx\right)^{1/p}\\&\leq\frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2}\\&\quad\times\frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\widetilde{\rho},\gamma}f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{(p_2-p)/pp_2}.\end{aligned}$$

By taking the supremum over $r > 0$ and using the inequality (10.49), we get

$$\|WI_{\widetilde{\rho},\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|I_{\widetilde{\rho},\gamma}f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Taking into account that $1 < p < \frac{pp_2}{p_2-p} < \infty$ and putting $q = \frac{pp_2}{p_2-p}$ in (10.49), we obtain (10.53). $\square$

10.3.6 Fractional integral operators in Morrey spaces

In this section we extend the previous results to more general fractional integral operators, in particular establishing their boundedness and the Olsen type inequality on generalized Morrey spaces on homogeneous groups.

Definition 10.3.19 (Generalized fractional integral operator)**.** We define the *generalized fractional integral operator* T_ρ by

$$T_\rho f(x) := \int_{\mathbb{G}} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} f(y)dy, \tag{10.54}$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the doubling condition (10.28) and the condition

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty. \tag{10.55}$$

As in the standard Euclidean case, taking $\rho(t) = t^\alpha$ with $0 < \alpha < Q$, we have the classical Riesz transform

$$T_\rho f(x) = I_\alpha f(x) = \int_{\mathbb{G}} \frac{1}{|xy^{-1}|^{Q-\alpha}} f(y)dy.$$

Theorem 10.3.20 (Fractional integral operators on Morrey spaces)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q *with a homogeneous quasi-norm* $|\cdot|$*. Let* ρ *and* ϕ *satisfy the doubling condition* (10.28)*. Assume that* ϕ *is surjective and for some* $1 < p < q < \infty$ *satisfies the inequalities*

$$\int_r^\infty \frac{\phi(t)^p}{t} dt \le C_1(\phi(r))^p, \tag{10.56}$$

and

$$\phi(r)\int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le C_2(\phi(r))^{p/q}, \tag{10.57}$$

for all $r > 0$*. Then we have*

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \le C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}. \tag{10.58}$$

Proof of Theorem 10.3.20. For every $R > 0$, let us write $T_\rho f(x)$ in the form

$$T_\rho f(x) = T_1(x) + T_2(x),$$

where

$$T_1(x) := \int_{B(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y)dy$$

and

$$T_2(x) := \int_{B^c(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y)dy.$$

For $T_1(x)$, we have

$$\begin{aligned} |T_1(x)| &\leq \int_{|xy^{-1}|<R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)|dy \\ &= \sum_{k=-\infty}^{-1} \int_{2^kR \leq |xy^{-1}| < 2^{k+1}R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)|dy. \end{aligned}$$

By view of (10.28) we can estimate

$$\begin{aligned} |T_1(x)| &\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^kR)}{(2^kR)^Q} \int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy \\ &\leq C\mathscr{M}f(x) \sum_{k=-\infty}^{-1} \rho(2^kR) \\ &\leq C\mathscr{M}f(x) \sum_{k=-\infty}^{-1} \int_{2^kR}^{2^{k+1}R} \frac{\rho(t)}{t} dt \\ &= C\mathscr{M}f(x) \int_0^R \frac{\rho(t)}{t} dt. \end{aligned}$$

Here we have used the fact that

$$\int_{2^kR}^{2^{k+1}R} \frac{\rho(t)}{t} dt \geq C\rho(2^kR) \int_{2^kR}^{2^{k+1}R} \frac{1}{t} dt = C\rho(2^kR) \ln 2. \tag{10.59}$$

Now, using (10.57), we obtain

$$|T_1(x)| \leq C\mathscr{M}f(x)(\phi(R))^{(p-q)/q}. \tag{10.60}$$

For $T_2(x)$ we estimate it as

$$\begin{aligned} |T_2(x)| &\leq \int_{|xy^{-1}|\geq R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)|dy \\ &= \sum_{k=0}^{\infty} \int_{2^kR \leq |xy^{-1}| < 2^{k+1}R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)|dy. \end{aligned}$$

Applying (10.28) we get

$$|T_2(x)| \leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^Q} \int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy.$$

From this using Hölder's inequality we obtain

$$\begin{aligned}
|T_2(x)| &\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^Q} \left(\int_{|xy^{-1}|<2^{k+1}R} dy \right)^{1-1/p} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy \right)^{1/p} \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^{Q/p}} \left(\int_{|xy^{-1}|<2^{k+1}R} |f(y)|dy \right)^{1/p} \\
&\leq C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \rho(2^{k+1}R)\phi(2^{k+1}R) \\
&\leq C \|f\|_{LM^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} \frac{\rho(t)\phi(t)}{t} \\
&= C \|f\|_{LM^{p,\phi}(\mathbb{G})} \int_R^{\infty} \frac{\rho(t)\phi(t)}{t},
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\int_{2^k R}^{2^{k+1}R} \frac{\rho(t)\phi(t)}{t} dt &\geq C\rho(2^{k+1}R)\phi(2^{k+1}R) \int_{2^k R}^{2^{k+1}R} \frac{1}{t} dt \\
&= C\rho(2^{k+1}R)\phi(2^{k+1}R) \ln 2.
\end{aligned}$$

Now, in view of (10.57) we obtain

$$|T_2(x)| \leq C\|f\|_{LM^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}. \tag{10.61}$$

Combining estimates (10.60) and (10.61) we get

$$|T_\rho f(x)| \leq C(\mathscr{M} f(x)(\phi(R))^{(p-q)/q} + \|f\|_{LM^{p,\phi}(\mathbb{G})}(\phi(R))^{p/q}).$$

Assuming that f is not identically 0 and that $\mathscr{M} f$ is finite everywhere and then using the fact that ϕ is surjective, we can choose $R > 0$ such that

$$\phi(R) = \mathscr{M} f(x) \cdot \|f\|_{LM^{p,\phi}(\mathbb{G})}^{-1}.$$

Thus, for every $x \in \mathbb{G}$ we have

$$|T_\rho f(x)| \leq C(\mathscr{M} f(x))^{\frac{p}{q}} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

It follows that

$$\left(\int_{B(0,r)} |T_\rho f(x)|^q\right)^{1/q} \le C\left(\int_{B(0,r)} |\mathscr{M} f(x)|^p\right)^{1/q} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

Dividing both sides by $(\phi(r))^{p/q} r^{Q/q}$ we obtain

$$\begin{aligned}&\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)} |T_\rho f(x)|^q\right)^{1/q}\\ &\quad\le C\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)} |\mathscr{M} f(x)|^p\right)^{1/q} \|f\|_{LM^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.\end{aligned}$$

Taking the supremum over $r > 0$ and using the boundedness of the maximal operator $\mathscr{M}$ on $LM^{p,\phi}(\mathbb{G})$ shown in (10.31), we obtain

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \le C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

This gives (10.58). □

10.3.7 Olsen type inequalities for fractional integral operators

We now show Olsen type inequalities for the generalized fractional integral operator T_ρ and the Bessel–Riesz operator $I_{\alpha,\gamma}$. This continues the analysis from Section 10.3.5.

Theorem 10.3.21 (Olsen type inequalities for fractional integral operators)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let ρ and ϕ satisfy the doubling condition* (10.28)*. Assume that ϕ is surjective and satisfies* (10.56) *and* (10.57)*. Then we have*

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \tag{10.62}$$

provided that $1 < p < p_2 < \infty$ and $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 10.3.21*.* By using Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{r^Q}\int_{B(0,r)} |W(x)T_\rho f(x)|^p dx\\ &\quad\le \left(\frac{1}{r^Q}\int_{B(0,r)} |W(x)|^{p_2} dx\right)^{p/p_2}\left(\frac{1}{r^Q}\int_{B(0,r)} |T_\rho f(x)|^{\frac{pp_2}{p_2-p}} dx\right)^{(p_2-p)/p_2}.\end{aligned}$$

Now let us take the pth roots and then divide both sides by $\phi(r)$ to obtain

$$\begin{aligned}&\frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)T_\rho f(x)|^p dx\right)^{1/p}\\&\quad\le\frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2}\\&\qquad\times\frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}}\left(\frac{1}{r^Q}\int_{B(0,r)}|T_\rho f(x)|^{\frac{pp_2}{p_2-p}}dx\right)^{(p_2-p)/pp_2}.\end{aligned}$$

By taking the supremum over $r>0$ and using the inequality (10.58) we get

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})}\le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|T_\rho f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Taking into account that $1<p<\frac{pp_2}{p_2-p}<\infty$ and putting $q=\frac{pp_2}{p_2-p}$ in (10.58) we obtain (10.62). ☐

Theorem 10.3.22 (Olsen type inequalities for Bessel–Riesz operators)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let $1<p<\infty$, $0<\alpha<Q$ and $\gamma>0$. Let $\omega:\mathbb{R}^+\to\mathbb{R}^+$ satisfy the doubling condition and assume that $0<\omega(r)\le Cr^{-\alpha}$ for every $r>0$, so that $K_{\alpha,\gamma}\in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha}<p_2<\frac{Q}{Q-\alpha}$ and $p_2\ge 1$, where $q=\frac{\beta p}{\beta+Q-\alpha}$. Assume that $0<\phi(r)\le Cr^\beta$ for all $r>0$, where $\beta<-\alpha<-Q-\beta$. Then we have*

$$\|WI_{\alpha,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})}\le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},\tag{10.63}$$

provided that $W\in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$, where $\frac{1}{p_2}=\frac{1}{p}-\frac{1}{q}$.

Proof of Theorem 10.3.22*.* As in Theorem 10.3.21, by using Hölder's inequality for $\frac{1}{p_2}+\frac{1}{q}=\frac{1}{p}$, we have

$$\begin{aligned}&\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\alpha,\gamma}f(x)|^p dx\\&\quad\le\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{p/p_2}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\alpha,\gamma}f(x)|^q dx\right)^{p/q}.\end{aligned}$$

Now we take the pth roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned}&\frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)I_{\alpha,\gamma}f(x)|^p dx\right)^{1/p}\\&\leq\frac{1}{(\phi(r))^{p/p_2}}\left(\frac{1}{r^Q}\int_{B(0,r)}|W(x)|^{p_2}dx\right)^{1/p_2}\\&\times\frac{1}{(\phi(r))^{p/q}}\left(\frac{1}{r^Q}\int_{B(0,r)}|I_{\alpha,\gamma}f(x)|^q dx\right)^{1/q}.\end{aligned}$$

By taking the supremum over $r>0$ we have

$$\|WI_{\alpha,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})}\leq C\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|I_{\alpha,\gamma}f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})}.$$

Putting $\psi(r)=(\phi(r))^{p/q}$ and using Theorem 10.3.14 we obtain (10.63). ☐

10.3.8 Summary of results

Let us now briefly summarize and collect the main results shown in this section, for a clearer overview of the statements.

Corollary 10.3.23. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Then we have the following properties:*

1. *Let $K_{\alpha,\gamma}(x):=\frac{|x|^{\alpha-n}}{(1+|x|)^\gamma}$. If $0<\alpha<Q$ and $\gamma>0$, then $K_{\alpha,\gamma}\in L^{p_1}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha}<p_1<\frac{Q}{Q-\alpha}$, and*

$$\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\sim\left(\sum_{k\in\mathbb{Z}}\frac{(2^kR)^{(\alpha-Q)p_1+Q}}{(1+2^kR)^{\gamma p_1}}\right)^{1/p_1}$$

   for any $R>0$.
2. *For any $f\in LM^{p,\phi}(\mathbb{G})$ and $1<p<\infty$, we have*

$$\|\mathscr{M}f\|_{LM^{p,\phi}(\mathbb{G})}\leq C_p\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

   where generalized local Morrey space $LM^{p,\phi}(\mathbb{G})$ and the Hardy–Littlewood maximal operator $\mathscr{M}$ are defined in (10.27) and (10.29), respectively.
3. *Let $I_{\alpha,\gamma}$ be a Bessel–Riesz operator on a homogeneous group defined in (10.24). Let $\gamma>0$ and $0<\alpha<Q$. If $0<\phi(r)\leq Cr^\beta$ for every $r>0$, $\beta<-\alpha$, $1<p<\infty$, and $\frac{Q}{Q+\gamma-\alpha}<p_1<\frac{Q}{Q-\alpha}$, then for all $f\in LM^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{LM^{q,\psi}(\mathbb{G})}\leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

   where $q=\frac{\beta p_1'p}{\beta p_1'+Q}$ and $\psi(r)=(\phi(r))^{p/q}$.

4. *Let* $\gamma > 0$, $1 < p < \infty$ *and* $0 < \alpha < Q$. *If* $0 < \phi(r) \leq Cr^{\beta}$ *for all* $r > 0$, $\beta < -\alpha$, $\frac{Q}{Q+\gamma-\alpha} < p_2 \leq p_1 < \frac{Q}{Q-\alpha}$ *and* $p_2 \geq 1$, *then for all* $f \in LM^{p,\phi}(\mathbb{G})$ *we have*

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{LM^{p_2,p_1}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$ *and* $\psi(r) = (\phi(r))^{p/q}$.

5. *Let* $1 < p < \infty$. *Let* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy the doubling condition and assume that* $0 < \omega(r) \leq Cr^{-\alpha}$ *for all* $r > 0$, *so that* $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ *for* $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ *and* $p_2 \geq 1$, *where* $0 < \alpha < Q$ *and* $\gamma > 0$. *If* $0 < \phi(r) \leq Cr^{\beta}$ *for all* $r > 0$, *where* $\beta < -\alpha < -Q - \beta$, *then for all* $f \in LM^{p,\phi}(\mathbb{G})$ *we have*

$$\|I_{\alpha,\gamma} f\|_{LM^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{LM^{p_2,\omega}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})},$$

where $q = \frac{\beta p}{\beta + Q - \alpha}$ *and* $\psi(r) = (\phi(r))^{p/q}$.

6. *Let* $I_{\rho,\gamma}$ *be the generalized Bessel–Riesz operator defined in* (10.45). *Let* $\gamma > 0$ *and let* ρ *and* ϕ *satisfy the doubling condition* (10.28). *Let* $1 < p < q < \infty$. *Assume that* ϕ *is surjective and satisfies*

$$\int_r^{\infty} \frac{\phi(t)^p}{t} dt \leq C_1 (\phi(r))^p,$$

and

$$\phi(r) \int_0^r \frac{\rho(t)}{t^{\gamma-Q+1}} dt + \int_r^{\infty} \frac{\rho(t)\phi(t)}{t^{\gamma-Q+1}} dt \leq C_2 (\phi(r))^{p/q},$$

for all $r > 0$. *Then we have*

$$\|I_{\rho,\gamma} f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q} \|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

7. *Let* ρ *and* ϕ *satisfy the doubling condition* (10.28). *Let* $\gamma > 0$ *and assume that* ϕ *is surjective and satisfies* (10.47) *and* (10.48). *Then we have*

$$\|W I_{\rho,\gamma} f\|_{LM^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|f\|_{LM^{p,\phi}(\mathbb{G})}, \quad 1 < p < p_2 < \infty,$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

8. *Let* T_{ρ} *be the generalized fractional integral operator defined in* (10.54). *Let* ρ *and* ϕ *satisfy the doubling condition* (10.28). *Let* $1 < p < q < \infty$. *Assume that* ϕ *is surjective and satisfies*

$$\int_r^{\infty} \frac{\phi(t)^p}{t} dt \leq C_1 (\phi(r))^p,$$

and

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^{\infty} \frac{\rho(t)\phi(t)}{t} dt \leq C_2 (\phi(r))^{p/q},$$

for all $r > 0$. Then we have

$$\|T_\rho f\|_{LM^{q,\phi^{p/q}}(\mathbb{G})} \le C_{p,q,\phi,Q}\|f\|_{LM^{p,\phi}(\mathbb{G})}.$$

9. *Let ρ and ϕ satisfy the doubling condition (10.28). Assume that ϕ is surjective and satisfies (10.56) and (10.57). Then we have*

$$\|WT_\rho f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})}, \quad 1 < p < p_2 < \infty,$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

10. *Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the doubling condition and assume that $0 < \omega(r) \le Cr^{-\alpha}$ for all $r > 0$, so that $K_{\alpha,\gamma} \in LM^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ and $p_2 \ge 1$, where $0 < \alpha < Q$, $1 < p < \infty$, $q = \frac{\beta p}{\beta+Q-\alpha}$ and $\gamma > 0$. If $0 < \phi(r) \le Cr^\beta$ for all $r > 0$, where $\beta < -\alpha < -Q - \beta$, then we have*

$$\|WI_{\alpha,\gamma}f\|_{LM^{p,\phi}(\mathbb{G})} \le C_{p,\phi,Q}\|W\|_{LM^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|f\|_{LM^{p,\phi}(\mathbb{G})},$$

provided that $W \in LM^{p_2,\phi^{p/p_2}}(\mathbb{G})$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{q}$.

11. *Let the generalized local Campanato space $\mathcal{LM}^{p,\psi}(\mathbb{G})$ and operator $\widetilde{T}_\rho$ be defined in (10.67) and (10.70), respectively. Let ρ satisfy (10.55), (10.28), (10.68), (10.69), and let ϕ satisfy the doubling condition (10.28) and $\int_1^\infty \frac{\phi(t)}{t}dt < \infty$. If*

$$\int_r^\infty \frac{\phi(t)}{t}dt \int_0^r \frac{\rho(t)}{t}dt + r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}dt \le C_3\psi(r) \quad \textit{for all} \quad r > 0,$$

then we have

$$\|\widetilde{T}_\rho f\|_{\mathcal{LM}^{p,\psi}(\mathbb{G})} \le C_{p,\phi,Q}\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}, \quad 1 < p < \infty.$$

10.4 Besov type space: Gagliardo–Nirenberg inequalities

Now we discuss a family of Gagliardo–Nirenberg type inequalities on homogeneous groups. The formulation is based on the Besov type space $B^\alpha(\mathbb{R} \times \wp)$, which we define as the space of all tempered distributions f on $\mathbb{R} \times \wp$ with the norm

$$\|g\|_{B^\alpha(\mathbb{R}\times\wp)} := \sup_{t>0}\{t^{-\alpha/2}\|Fe^{-tA^2}F^{-1}f\|_{L^\infty(\mathbb{R}\times\wp)}\} < \infty, \tag{10.64}$$

where the operators F and A are defined in Section 1.3.4, and $\wp$ is the quasi-sphere from the polar decomposition in Proposition 1.2.10.

Now we state the Gagliardo–Nirenberg type inequalities:

Theorem 10.4.1 (Gagliardo–Nirenberg type inequalities with Besov norms)**.** *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* Q*. Let* $1 \le p < q < \infty$*. Then there exists a positive constant* $C = C(p,q) > 0$ *such that we have*

$$\|f\|_{L^q(\mathbb{R}\times\wp)} \le C\|\mathcal{R}f\|^{p/q}_{L^p(\mathbb{R}\times\wp)}\|f\|^{1-p/q}_{B^{p/(p-q)}(\mathbb{R}\times\wp)} \tag{10.65}$$

for all $f \in B^{p/(p-q)}(\mathbb{R}\times\wp)$ *such that* $\mathcal{R}f \in L^p(\mathbb{R}\times\wp)$*.*

The following one-dimensional result will be useful for the proof:

Theorem 10.4.2 ([Led03, Theorem 1])**.** *Let* $1 \le p < q < \infty$*. Then for every function* $f \in L^p_1(\mathbb{R}^n)$ *there exists a positive constant* $C = C(p,q,n) > 0$ *such that*

$$\|f\|_{L^q(\mathbb{R}^n)} \le C\|\nabla f\|^{p/q}_{L^p(\mathbb{R}^n)}\|f\|^{1-p/q}_{B^{p/(p-q)}_{\infty,\infty}(\mathbb{R}^n)}, \tag{10.66}$$

where

$$\|f\|_{B^\alpha_{\infty,\infty}(\mathbb{R}^n)} := \sup_{t>0,\ x\in\mathbb{R}^n}\left\{t^{-\alpha/2}\left|\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} f(y)e^{-|x-y|^2/4t}dy\right|\right\}.$$

Proof of Theorem 10.4.1*.* Using Theorem 10.4.2 with $n = 1$ and Theorem 1.3.5, we obtain

$$\begin{aligned}\int_{\mathbb{R}}|f(r,y)|^q dr &\le C^q\int_{\mathbb{R}}\left|\frac{\partial f(r,y)}{\partial r}\right|^p dr\\ &\quad\times\left(\sup_{t>0,r\in\mathbb{R}} t^{p/(2(q-p))}\left|\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(r-s)^2/(4t)}f(s,y)ds\right|\right)^{q-p}\\ &= C^q\int_{\mathbb{R}}|\mathcal{R}f(r,y)|^p dr\left(\sup_{t>0,r\in\mathbb{R}} t^{p/(2(q-p))}\left|Fe^{-tA^2}F^{-1}f(r,y)\right|\right)^{q-p}\\ &\le C^q\int_{\mathbb{R}}|\mathcal{R}f(r,y)|^p dr\left(\sup_{t>0} t^{p/(2(q-p))}\left\|Fe^{-tA^2}F^{-1}f\right\|_{L^\infty(\mathbb{R}\times\wp)}\right)^{q-p}\\ &= C^q\int_{\mathbb{R}}|\mathcal{R}f(r,y)|^p dr\|f\|^{q-p}_{B^{p/(p-q)}(\mathbb{R}\times\wp)},\end{aligned}$$

for any $y \in \wp$, in view of (10.64). One obtains (10.65) after integrating the above inequality with respect to y over $\wp$. $\square$

10.5 Generalized Campanato spaces

In this section we show the boundedness of the extended version of fractional integral operators in Campanato spaces on homogeneous groups.

As before, throughout this section $\mathbb{G}$ is a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$.

Definition 10.5.1 (Generalized local Campanato spaces)**.** Assume that the function $\frac{\phi(r)}{r}$ is nonincreasing. Denote

$$f_B = f_{B(0,r)} := \frac{1}{r^Q}\int_{B(0,r)} f(y)dy,$$

where $B(0,r) := \{x \in \mathbb{G} : |x| < r\}$. We define the *generalized local Campanato space* by

$$\mathcal{LM}^{p,\phi}(\mathbb{G}) := \{f \in L^p_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} < \infty\}, \tag{10.67}$$

where

$$\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} := \sup_{r>0} \frac{1}{\phi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)} |f(x) - f_B|^p dx\right)^{1/p}.$$

Sometimes these spaces are called central Camponato spaces in the literature.

Definition 10.5.2 (Modified generalized fractional integral operators)**.** Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy (10.55), (10.28) and the following conditions:

$$\int_r^\infty \frac{\rho(t)}{t^2}dt \le C_1\frac{\rho(r)}{r} \quad \text{for all } r > 0; \tag{10.68}$$

$$\frac{1}{2} \le \frac{r}{s} \le 2 \Longrightarrow \left|\frac{\rho(r)}{r^Q} - \frac{\rho(s)}{s^Q}\right| \le C_2|r-s|\frac{\rho(s)}{s^{Q+1}}. \tag{10.69}$$

We define the *modified version of the generalized fractional integral operator* T_ρ by

$$\widetilde{T_\rho} f(x) := \int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q}\right) f(y)dy, \tag{10.70}$$

where $\chi_{B(0,1)}$ is the characteristic function of $B(0,1)$.

For instance, the function $\rho(r) = r^\alpha$ satisfies (10.55), (10.28) and (10.69) for $0 < \alpha < Q$, and also satisfies (10.68) for $0 < \alpha < 1$.

Theorem 10.5.3 (Fractional integral operators on Camponato spaces)**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q with a homogeneous quasi-norm $|\cdot|$. Let ρ satisfy* (10.55), (10.28), (10.68), (10.69), *and let ϕ satisfy the doubling condition* (10.28) *and $\int_1^\infty \frac{\phi(t)}{t}dt < \infty$. If*

$$\int_r^\infty \frac{\phi(t)}{t}dt \int_0^r \frac{\rho(t)}{t}dt + r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}dt \le C_3\psi(r) \quad \textit{for all } r > 0, \tag{10.71}$$

then we have

$$\|\widetilde{T}_\rho f\|_{\mathcal{LM}^{p,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}, \quad 1<p<\infty. \tag{10.72}$$

Proof of Theorem 10.5.3. For every $x \in B(0,r)$ and $f \in \mathcal{LM}^{p,\phi}(\mathbb{G})$, let us write $\widetilde{T}_\rho f$ in the form

$$\begin{aligned}\widetilde{T}_\rho f(x) &= \widetilde{T}_{B(0,r)}(x) + C^1_{B(0,r)} + C^2_{B(0,r)}\\ &= \widetilde{T}^1_{B(0,r)}(x) + \widetilde{T}^2_{B(0,r)}(x) + C^1_{B(0,r)} + C^2_{B(0,r)},\end{aligned}$$

where

$$\widetilde{T}_{B(0,r)}(x) := \int_{\mathbb{G}} (f(y)-f_{B(0,2r)})\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,2r)}(y))}{|y|^Q}\right)dy,$$

$$C^1_{B(0,r)} := \int_{\mathbb{G}} (f(y)-f_{B(0,2r)})\left(\frac{\rho(|y|)(1-\chi_{B(0,2r)}(y))}{|y|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q}\right)dy,$$

$$C^2_{B(0,r)} := \int_{\mathbb{G}} f_{B(0,2r)}\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q}\right)dy,$$

$$\widetilde{T}^1_{B(0,r)}(x) := \int_{B(0,2r)} (f(y)-f_{B(0,2r)})\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}dy,$$

$$\widetilde{T}^2_{B(0,r)}(x) := \int_{B^c(0,2r)} (f(y)-f_{B(0,2r)})\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q}\right)dy.$$

Since

$$\begin{aligned}&\left|\frac{\rho(|y|)(1-\chi_{B(0,2r)}(y))}{|y|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q}\right|\\ &\qquad \leq \begin{cases} 0, \ |y|<\min(1,2r) \text{ or } |y|\geq \max(1,2r);\\ \frac{\rho(|y|)}{|y|^Q} = \text{const}, \ \text{otherwise},\end{cases}\end{aligned}$$

we have that $C^1_{B(0,r)}$ is finite.

Now let us show that $C^2_{B(0,r)}$ is finite. For this it is enough to prove that the following integral is finite:

$$\begin{aligned}&\int_{\mathbb{G}}\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1-\chi_{B(0,1)}(y))}{|y|^Q}\right)dy\\ &\quad = \int_{\mathbb{G}}\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q}\right)dy + \int_{B(0,1)}\frac{\rho(|y|)}{|y|^Q}dy.\end{aligned}$$

Let us denote

$$A := \int_{\mathbb{G}}\left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q}\right)dy.$$

For large $R>0$, we write A in the form

$$A = A_1 + A_2 + A_3,$$

where

$$A_1 = \int_{B(x,R)} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy - \int_{B(0,R)} \frac{\rho(|y|)}{|y|^Q} dy,$$

$$A_2 = \int_{B(x,R+r)\setminus B(x,R)} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} dy - \int_{B(x,R+r)\setminus B(0,R)} \frac{\rho(|y|)}{|y|^Q} dy,$$

$$A_3 = \int_{B^c(x,R+r)} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right) dy.$$

Since we have $\int_0^1 \frac{\rho(t)}{t} dt < +\infty$, it follows that

$$\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}, \frac{\rho(|y|)}{|y|^Q} \in L^1_{\mathrm{loc}}(\mathbb{G}),$$

and hence $A_1 = 0$.

For A_2, we have

$$|A_2| \leq \int_{B(x,R+r)\setminus B(x,R-r)} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} + \frac{\rho(|y|)}{|y|^Q} \right) dy$$
$$\sim ((R+r)^Q - (R-r)^Q) \frac{\rho(R)}{R^Q} \leq Cr \frac{\rho(R)}{R}.$$

In view of conditions (10.28) and (10.68) we have that

$$|A_2| \leq Cr \frac{\rho(R)}{R} \to 0 \text{ as } R \to +\infty.$$

By (10.69) we have

$$|A_3| \leq \int_{B^c(x,R+r)} \left| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)}{|y|^Q} \right| dy$$
$$\leq C \int_{B^c(x,R+r)} ||xy^{-1}| - |y|| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^{Q+1}} dy.$$

By using the triangle inequality from Proposition 1.2.4 and the symmetric property of homogeneous quasi-norms, we get

$$|A_3| \leq C||x| + |y^{-1}| - |y|| \int_{R+r}^{+\infty} \int_{\wp} \frac{\rho(t)}{t^{Q+1}} t^{Q-1} d\sigma(y) dt \leq C|\wp| r \int_{R+r}^{+\infty} \frac{\rho(t)}{t^2} dt.$$

The inequality (10.68) implies that the last integral is integrable and $|A_3| \to 0$ as $R \to +\infty$.

Summarizing these properties of A_1, A_2, A_3 we have that $A \to 0$ as $R \to +\infty$, so that $A = 0$ and hence

$$\int_{\mathbb{G}} \left(\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} - \frac{\rho(|y|)(1 - \chi_{B(0,1)}(y))}{|y|^Q} \right) dy = \int_{B(0,1)} \frac{\rho(|y|)}{|y|^Q} dy < \infty,$$

which implies that $C^2_{B(0,r)}$ is finite.

Now before estimating $\widetilde{T}^1_{B(0,r)}$, let us denote

$$\widetilde{f} := (f - f_{B(0,2r)})\chi_{B(0,2r)} \qquad \text{and} \qquad \widetilde{\phi}(r) := \int_r^\infty \frac{\phi(t)}{t}dt.$$

Then we have

$$\begin{aligned}|\widetilde{T}^1_{B(0,r)}(x)| &\leq \int_{B(0,2r)} |\widetilde{f}(y)| \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}dy \\ &= \sum_{k=-\infty}^{0} \int_{2^k r \leq |xy^{-1}| < 2^{k+1}r} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}|\widetilde{f}(y)|dy.\end{aligned}$$

By using (10.28) and (10.59) we get

$$\begin{aligned}|\widetilde{T}^1_{B(0,r)}(x)| &\leq C \sum_{k=-\infty}^{0} \frac{\rho(2^k r)}{(2^k r)^Q} \int_{|xy^{-1}|<2^{k+1}r} |\widetilde{f}(y)|dy \\ &\leq C\mathscr{M}\widetilde{f}(x) \sum_{k=-\infty}^{0} \rho(2^k r) \leq C\mathscr{M}\widetilde{f}(x) \sum_{k=-\infty}^{0} \rho(2^{k-1}r) \\ &\leq C\mathscr{M}\widetilde{f}(x) \sum_{k=-\infty}^{0} \int_{2^{k-1}r}^{2^k r} \frac{\rho(t)}{t}dt = C\mathscr{M}\widetilde{f}(x) \int_0^r \frac{\rho(t)}{t}dt.\end{aligned}$$

Now using (10.71) we have

$$|\widetilde{T}^1_{B(0,r)}(x)| \leq C\frac{\psi(r)}{\widetilde{\phi}(r)}\mathscr{M}\widetilde{f}(x).$$

Using (10.33) we have

$$\begin{aligned}&\frac{1}{\psi(r)}\left(\frac{1}{r^Q}\int_{B(0,r)} |\widetilde{T}^1_{B(0,r)}(x)|^p dx\right)^{1/p} \\ &\quad \leq C\frac{1}{\widetilde{\phi}(r)r^{Q/p}}\left(\int_{B(0,r)} |\mathscr{M}\widetilde{f}(x)|^p dx\right)^{1/p} \leq C\frac{1}{\widetilde{\phi}(r)r^{Q/p}}\|\widetilde{f}\|_{L^p(\mathbb{G})}.\end{aligned}$$

By the Minkowski inequality, we have

$$\begin{aligned}&\frac{1}{\widetilde{\phi}(r)r^{Q/p}}\|\widetilde{f}\|_{L^p(\mathbb{G})} = \frac{1}{\widetilde{\phi}(r)r^{Q/p}}\|(f - f_{B(0,2r)})\chi_{B(0,2r)}\|_{L^p(\mathbb{G})} \\ &\quad \leq C\frac{1}{\widetilde{\phi}(r)r^{Q/p}}(\|(f-\sigma(f))_{\chi_{B(0,2r)}}\|_{L^p(\mathbb{G})} + (2r)^{Q/p}|f_{B(0,2r)} - \sigma(f)|),\end{aligned}$$

where $\sigma(f) = \lim\limits_{r\to\infty} f_{B(0,r)}$.

Moreover, we obtain the following inequalities exactly in the same way as in the standard Euclidean case (see [EN04, Section 6]):

$$\|f-\sigma(f)\|_{LM^{p,\widetilde{\phi}}(\mathbb{G})} \leq C_1\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})},$$

and

$$|f_{B(0,r)}-\sigma(f)| \leq C_2\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\widetilde{\phi}(r).$$

Finally, using these inequalities we get the estimate for $\widetilde{T}^1_{B(0,r)}$ as

$$|\widetilde{T}^1_{B(0,r)}(x)| \leq C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}. \tag{10.73}$$

Now let us estimate $\widetilde{T}^2_{B(0,r)}$. By (10.28) and (10.69) we have

$$\begin{aligned}|\widetilde{T}^2_{B(0,r)}(x)| &\leq \int_{B^c(0,2r)}|f(y)-f_{B(0,2r)}|\left|\frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q}-\frac{\rho(|y|)}{|y|^Q}\right|dy \\ &\leq C||xy^{-1}|-|y||\int_{|y|\geq 2r}|f(y)-f_{B(0,2r)}|\frac{\rho(|y|)}{|y|^{Q+1}}dy.\end{aligned}$$

By using the triangle inequality from Proposition 1.2.4 and symmetric property of homogeneous quasi-norms we get

$$\begin{aligned}|\widetilde{T}^2_{B(0,r)}(x)| &\leq C||x|+|y^{-1}|-|y||\int_{|y|\geq 2r}|f(y)-f_{B(0,2r)}|\frac{\rho(|y|)}{|y|^{Q+1}}dy \\ &\leq C|x|\int_{|y|\geq 2r}|f(y)-f_{B(0,2r)}|\frac{\rho(|y|)}{|y|^{Q+1}}dy \\ &= C|x|\sum_{k=2}^{\infty}\int_{2^{k-1}r\leq|y|<2^k r}\frac{\rho(|y|)|f(y)-f_{B(0,2r)}|}{|y|^{Q+1}}dy.\end{aligned}$$

By using (10.28) and Hölder's inequality we obtain

$$\begin{aligned}|\widetilde{T}^2_{B(0,r)}(x)| &\leq C|x|\sum_{k=2}^{\infty}\frac{\rho(2^k r)}{(2^k r)^{Q+1}}\int_{|y|<2^k r}|f(y)-f_{B(0,2r)}|dy \\ &\leq C|x|\sum_{k=2}^{\infty}\frac{\rho(2^k r)}{2^k r}\left(\frac{1}{(2^k r)^Q}\int_{|y|<2^k r}|f(y)-f_{B(0,2r)}|^p dy\right)^{1/p}.\end{aligned}$$

As in the Abelian case ([EN04]) we have

$$\left(\frac{1}{(2^k r)^Q}\int_{B(0,2^k r)}|f(y)-f_{B(0,2r)}|^p dy\right)^{1/p} \leq C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}\int_{2r}^{2^{k+1}r}\frac{\phi(s)}{s}ds,$$

for every $k\geq 2$. The inequality (10.59) implies that

$$\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t^2}dt \geq \frac{1}{2^{k+1}r}\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t}dt \geq C\frac{\rho(2^k r)}{2^k r}.$$

By using the last two inequalities we get

$$
\begin{aligned}
|\widetilde{T}^2_{B(0,r)}(x)| &\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \sum_{k=2}^{\infty} \frac{\rho(2^k r)}{2^k r} \int_{2r}^{2^{k+1}r} \frac{\phi(s)}{s} ds \\
&\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \sum_{k=2}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\rho(t)}{t^2} \left(\int_{2r}^{t} \frac{\phi(s)}{s} ds \right) dt \\
&\le C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \int_{2r}^{\infty} \frac{\rho(t)}{t^2} \left(\int_{2r}^{t} \frac{\phi(s)}{s} ds \right) dt \\
&= C|x|\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \int_{2r}^{\infty} \left(\int_{s}^{\infty} \frac{\rho(t)}{t^2} dt \right) \frac{\phi(s)}{s} ds.
\end{aligned}
$$

Using (10.68) and then (10.71), it implies that

$$
|\widetilde{T}^2_{B(0,r)}(x)| \le Cr\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})} \int_{2r}^{\infty} \frac{\rho(s)\phi(s)}{s^2} ds \le C\psi(r)\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}.
$$

Finally, it follows that

$$
\frac{1}{\psi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |\widetilde{T}^2_{B(0,r)}(x)|^p dx \right)^{1/p} \le C\|f\|_{\mathcal{LM}^{p,\phi}(\mathbb{G})}. \tag{10.74}
$$

Combining estimates (10.73) and (10.74) we obtain (10.72). $\square$