Introduction

The present book is devoted to the exposition of the research developments at the intersection of two active fields of mathematics: Hardy inequalities and related analysis, and the noncommutative analysis in the setting of nilpotent Lie groups of different types. Both subjects are very broad and deserve separate monograph presentations on their own, and many good books are already available.

However, the recent active research in the area does allow one to make a consistent treatment of 'anisotropic' Hardy inequalities, their numerous features, and a number of related topics. This brings many new insights to the subject, also allowing to underline the interesting character of its subelliptic features. The progress in this field is facilitated by the rapid developments in both areas of Hardy inequalities and related topics, and in the noncommutative analysis on Folland and Stein's homogeneous groups.

We will now give some short insights into both fields and into the scope of this book. Here we only give a general overview, with more detailed references and explanations of different features presented throughout the monograph.

Hardy inequalities and related topics

The classical L^2-Hardy inequality in the modern literature in the Euclidean space $\mathbb{R}^n$ with $n \geq 3$ can be written in the form

$$\left\| \frac{f}{|x|_E} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} \left\| \nabla f \right\|_{L^2(\mathbb{R}^n)}, \tag{1}$$

where ∇ is the standard gradient in $\mathbb{R}^n$,

$$|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$$

is the Euclidean norm, $f \in C_0^\infty(\mathbb{R}^n)$, and where the constant $\frac{2}{n-2}$ is known to be sharp. In addition to references in the preface, the multidimensional version of the Hardy inequality was proved by J. Leray [Ler33].

It has numerous applications in different fields, for example in the spectral theory, leading to the lower bounds for the quadratic form associated to the Laplacian operator. It is also related to many other areas and fields, notably to

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the uncertainty principles. The uncertainty principle in physics is a fundamental concept going back to Heisenberg's work on quantum mechanics [Hei27, Hei85], as well as to its mathematical justification by Hermann Weyl [Wey50]. In the simplest Euclidean setting it can be stated as the inequality

$$\left(\int_{\mathbb{R}^n} |\nabla\phi|^2 dx\right)\left(\int_{\mathbb{R}^n} |x|_E^2\phi^2 dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} \phi^2 dx\right)^2, \tag{2}$$

for all real-valued functions $\phi \in C_0^\infty(\mathbb{R}^n)$, where the constant $\frac{n^2}{4}$ is sharp. It can be shown to be a consequence of (1). There are good surveys on the mathematical aspects of uncertainty principles by Fefferman [Fef83] and by Folland and Sitaram [FS97]. We also note that the uncertainty principle can be also obtained without Hardy inequalities, see, e.g., Ciatti, Ricci and Sundari [CRS07].

The inequality (1) can be extended to L^p-spaces, taking the form

$$\left\|\frac{f}{|x|_E}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n, \tag{3}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, and where the constant $\frac{p}{n-p}$ is known to be sharp.

As mentioned in the preface, such inequalities go back to Hardy [Har19], and have been evolving and growing over the years. In fact, the subject is so deep and broad at the same time that it would be impossible to give justice to all the authors who have made their contributions. To this end, we can refer to several extensive presentations of the subject in the books and surveys and the references therein: Opic and Kufner [OK90] in 1990, Davies [Dav99] in 1999, Edmunds and Evans [EE04] in 2004, part of Mazya's books [Maz85, Maz11], Ghoussoub and Moradifam [GM13] in 2013, and the recent book by Balinsky, Evans and Lewis [BEL15]. Hardy type inequalities have been very intensively studied, see, e.g., also Davies and Hinz [DH98], Davies [Dav99] as well as Ghoussoub and Moradifam [GM11] for reviews and applications.

One further extension of the Hardy inequality is the now classical result by Rellich appearing at the 1954 ICM in Amsterdam [Rel56] with the inequality

$$\left\|\frac{f}{|x|_E^2}\right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)}\|\Delta f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 5, \tag{4}$$

with the sharp constant. We can refer, for example, to Davies and Hinz [DH98] (see also Brézis and Vázquez [BV97]) for further history and later extensions, including the derivation of sharp constants.

Higher-order Hardy inequalities have been also intensively investigated. Some of such results go back to 1961 to Birman [Bir61, p. 48] who has shown, for functions $f \in C_0^k(0,\infty)$, the family of inequalities

$$\left\|\frac{f}{x^k}\right\|_{L^2(0,\infty)} \leq \frac{2^k}{(2k-1)!!}\left\|f^{(k)}\right\|_{L^2(0,\infty)}, \quad k \in \mathbb{N}, \tag{5}$$

where $(2k-1)!! = (2k-1)\cdot(2k-3)\cdots 3\cdot 1$. For $k=1$ and $k=2$ this reduces to one-dimensional Hardy and Rellich inequalities, respectively. Such one-dimensional inequalities have recently found new life and one can find their historical discussion by Gesztesy, Littlejohn, Michael and Wellman in [GLMW17].

There is now a whole scope of related inequalities playing fundamental roles in different branches of mathematics, in particular, in the theory of linear and nonlinear partial differential equations. For example, the analysis of more general weighted Hardy–Sobolev type inequalities has also a long history, initiated by Caffarelli, Kohn and Nirenberg [CKN84] as well as by Brézis and Nirenberg in [BN83], and then Brézis and Lieb [BL85] with a mixture with Sobolev inequalities, Brézis and Vázquez in [BV97, Section 4], also [BM97], with many subsequent works in this direction. We also refer to more recent paper of Hoffmann-Ostenhof and Laptev [HOL15] on this subject and to further references therein. Many of these inequalities will be also appearing in the present book.

Of course, there are many more aspects to Hardy inequalities. In particular, working in domains, one can establish inequalities under certain boundary conditions. For example, for Hardy inequalities for Robin Laplacians and p-Laplacians see [KL12] and [EKL15], respectively, or [LW99, BLS04] for magnetic versions, or [BM97, HOHOL02] for versions involving the distance to the boundary. For Hardy inequalities for discrete Laplacians see, e.g., [KL16], or [HOHOLT08] for many-particle versions.

Homogeneous groups of different types

The harmonic analysis on homogeneous groups goes back to 1982 to Folland and Stein who in their book [FS82] laid down the foundations of 'anisotropic' harmonic analysis, that is, the harmonic analysis that depends only on the group and dilation structures of the group.

Such homogeneous groups are necessarily nilpotent, and provide a unified framework including many well-known classes of (nilpotent) Lie groups: *the Euclidean space, the Heisenberg group, H-type groups, polarizable Carnot groups, stratified groups* (*homogeneous Carnot groups*), *graded groups*. All of these groups are homogeneous and have the rational weights for their dilations.

The class of homogeneous groups is closer to the classical analysis than one might first think: in fact, any homogeneous group can be identified with some space $\mathbb{R}^n$ with a polynomial group law. The simplest examples are $\mathbb{R}^n$ itself, where the group law is linear, or the Heisenberg group, where the group law is quadratic in the last variable.

An important feature of homogeneous groups is that they do not have to allow for homogeneous hypoelliptic left invariant partial differential operators. In fact, if such an operator exists, the group has to be *graded* and its weights of dilations are rational. The class of stratified groups is a particularly important class of graded groups allowing for a homogeneous second-order sub-Laplacian. In general, nilpotent Lie groups provide local models for many questions in subelliptic analysis and sub-Riemannian geometry, their importance widely recognized since the essential role they played in deriving sharp subelliptic estimates for differential operators on manifolds, starting from the seminal paper by Rothschild and Stein [RS76] (see also [Fol77, Rot83]).

In order to facilitate the exposition in the sequel, in Chapter 1 we will recall all the necessary facts needed for the analysis in this book.

We note, however, that the general scope of techniques available on such groups is much more extensive than presented in Chapter 1. The fundamental paper by Folland [Fol75] developed the rich functional analysis on stratified groups. Further functional spaces (e.g., of Besov type) on stratified groups have been analysed by Saka [Sak79]. There are many sources with rather comprehensive and deep treatments of general nilpotent Lie groups, for example, the books by Goodman [Goo76] or Corwin and Greenleaf [CG90]. Good sources of information are the notes by Fulvio Ricci [Ric] and Folland's books [Fol89, Fol95, Fol16].

As a side remark we can note that there is also a number of recent works developing function spaces on graded groups extending Folland and Saka's constructions in the stratified case, see [FR17] and [FR16] for Sobolev, and [CR16] and [CR17] for Besov spaces, respectively.

In our presentation and approach to the basic analysis on homogeneous groups of different types we mostly rely on the recent *open access* book [FR16]. Moreover, the exposition of the topics in this book is done more in the spirit of the classical potential theory, without much reference to the Fourier analysis. However, here we should mention that the noncommutative Fourier analysis on nilpotent Lie groups is extremely rich, with many powerful approaches available, such as Kirillov's orbit method [Kir04], Mackey general description of the unitary dual, or the von Neumann algebra approaches of Dixmier [Dix77, Dix81]. We can refer to [FR16, Appendix B] for a workable summary of these methods.

The recently developed noncommutative quantization theories on nilpotent Lie groups, in particular, the global theory of pseudo-differential operators on graded groups, indeed heavily rely on such Fourier analysis. We refer the interested reader to [FR16] for the thorough exposition and application of such methods. A good exposition of the analysis of questions not requiring the Fourier analysis, in the setting of stratified groups, can be found in the book [BLU07] by Bonfiglioli, Lanconelli and Uguzzoni.

Hardy inequalities and potential theory on stratified groups

The study of the subelliptic Hardy inequalities has also begun more than 40 years ago due to their importance for many questions involving subelliptic partial differential equations, unique continuation, sub-Riemannian geometry, subelliptic spectral theory, etc. Not surprisingly, here the work started with the most important example of the Heisenberg group, where we can mention a fundamental contribution by Garofalo and Lanconelli [GL90].

There is a deep link with the properties of the fundamental solutions for the sub-Laplacian on stratified groups. In general, the understanding of the fundamental solutions of differential operators is one of the keys for solving boundary value problems for differential equations in a domain, and this idea has a long history dating back to the works of mathematicians such as Gauss [Gau77, Gau29] and Green [Gre28].

In general, the sub-Laplacians on stratified (and on more general graded) groups play important roles not only in theoretical settings (see, e.g., Gromov [Gro96] or Danielli, Garofalo and Nhieu [DGN07] for general expositions from different points of view), but also in applications of mathematics, for example in mathematical models of crystal material and human vision (see, for example, [Chr98] and [CMS04]).

The fundamental solution for the sub-Laplacian on stratified groups behaves well and was already understood by Folland [Fol75]. In particular, one always has its existence and uniqueness, an advantageous feature when compared to higher-order operators on stratified groups, or more general hypoelliptic operators on graded groups, see Geller [Gel83], and an exposition in [FR16, Section 3.2.7].

Roughly speaking, there are three versions of Hardy type inequalities on stratified groups available in the literature:

(A) Using the homogeneous quasi-norm, sometimes called the $\mathcal{L}$-gauge, given by the appropriate power of the fundamental solution of the sub-Laplacian $\mathcal{L}$. Thus, if $d(x)$ is the $\mathcal{L}$-gauge, then $d(x)^{2-Q}$ is a constant multiple of Folland's [Fol75] fundamental solution of the sub-Laplacian $\mathcal{L}$, with Q being the homogeneous dimension of the stratified group $\mathbb{G}$; these will be discussed in Chapter 7.

(B) Using the Carnot–Carathéodory distance, i.e., the control distance associated to the sub-Laplacian.

(C) Using the Euclidean distance on the first stratum of the group.

One can note that if one is not interested in best constants in such inequalities one can work with any of these equivalent quasi-norms. In fact, in such a case one can also work with fractional-order derivatives expressed as arbitrary powers of the sub-Laplacian, see, e.g., Ciatti, Cowling and Ricci [CCR15] for such an analysis on stratified groups, as well as Yafaev [Yaf99] for some Euclidean considerations also with best constants, or Hoffmann-Ostenhof and Laptev [HOL15] and references therein.

However, the best constants in the corresponding inequalities in cases (A)–(C) above may depend on the quasi-norm that one is using.

Thus, in the case (A) there is an extensive literature on Hardy inequalities and related topics on stratified groups relating them to the fundamental solution to the sub-Laplacian. Here we can briefly mention some papers [GL90, GK08, Gri03, NZW01, HN03, D'A04b, WN08, DGP11, Kom10, JS11, Lia13, Yan13, CCR15, Yen16, GKY17], with more details and acknowledgements given throughout the book. Here, the Hardy inequality typically takes the form

$$\left\|\frac{f}{d(x)}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p}\left\|\nabla_H f\right\|_{L^p(\mathbb{G})}, \quad Q \geq 3,\ 1<p<Q, \tag{6}$$

where Q is the homogeneous dimension of the stratified group $\mathbb{G}$, ∇_H is the horizontal gradient, and $d(x)$ is the so-called $\mathcal{L}$-gauge related to the fundamental solution of the sub-Laplacian, and the constant is sharp. The analysis in the case (A) in terms of the fundamental solution of the sub-Laplacian will be the subject of Chapter 11 of this book.

The results on Hardy and other inequalities for the case (B) are less extensive, mostly devoted to the case of the Heisenberg group. However, the case (C) has recently attracted a lot of attention due to its geometrically clear nature and importance for questions in partial differential equations, see, e.g., [BT02a] and [D'A04b]. A typical horizontal Hardy inequality would take the form

$$\left\|\frac{f}{|x'|}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{N-p}\left\|\nabla_H f\right\|_{L^p(\mathbb{G})}, \quad 1<p<N, \tag{7}$$

where N is the dimension of the first stratum, x' denotes the variables in the first stratum of $\mathbb{G}$, and

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

is the Euclidean norm on the first stratum of $\mathbb{G}$, which can be identified with $\mathbb{R}^N$. The constant $\frac{p}{N-p}$ in (7) is also sharp.

In Chapter 6 we aim at giving a comprehensive treatment of such horizontal estimates based on the divergence relations and on the potential theory on the stratified groups.

Another ingredient that we find to be missing in the literature in the setting of stratified groups is the *classical style potential theory* working with layer potential operators. Indeed, nowadays the appearing boundary layer operators and elements of the potential theory serve as the main apparatus for the analysis and construction of solutions to boundary value problems. That have led to a vast literature concerning modern theory of boundary layer operators and potential theory in $\mathbb{R}^n$ as well as their important applications. In the subelliptic setting we can mention the works of Jerison [Jer81] and Romero [Rom91] on the Heisenberg group. We refer to [GL03], [GN88], [GW92], [LU97], [RS17d] and [WN16] as well as to references therein for more general Green function analysis of second order subelliptic (and weighted degenerate) operators. In this book, we follow a more geometric approach of our recent paper [RS17c] to present such a subject in the setting of general stratified groups, and to give its applications to several questions, such as boundary value problems for the sub-Laplacian, traces of Newton potential operators, and Hardy inequalities with boundary terms. All these topics will be the subject of discussions in Chapter 11.

The boundary value problems in the subelliptic settings are substantially more complicated than in the elliptic case due to the appearance of so-called *characteristic points* at the boundary – some problems of such a type are well explained, e.g., in [DGN06]. However, there is still a particular type of boundary conditions (Kac' boundary value problem) which can be viewed as a subelliptic version of M. Kac's question: *is there any boundary value problem for the Laplacian which is explicitly solvable in the classical sense for any smooth domain?*

An answer to M. Kac's question was given in [RS16c] for the Heisenberg group and in [RS17c] for general stratified groups. The appearing boundary conditions are, however, nonlocal and the corresponding boundary value problem can be called Kac's boundary value problem. One interesting fact is that the explicit solutions that one constructs for Kac's boundary value problem for the sub-Laplacian work also well in the presence of characteristic points on the boundary.

In Section 11.5 we also discuss another version of such a question: *is there a class of domains in which the Dirichlet boundary value problem for the sub-Laplacian is explicitly solvable in the classical sense?* This is discussed in the setting of H-type groups following our recent paper [GRS17] with Nicola Garofalo.

Furthermore, in Chapter 12 we will give an exposition of the potential theory and related Hardy–Rellich inequalities for more general Hörmander's sums of squares, based on the properties of the fundamental solutions rather than those of the $\mathcal{L}$-gauge. This has a definitive advantage of eliminating the need to use Folland's formula relating the $\mathcal{L}$-gauge with the fundamental solution. As a result, we can extend the analysis to more general settings, also those without any group structure, dealing with Hörmander's sums of squares beyond the setting of the stratified groups.

In general, there are several ways to obtain improvements of Hardy inequalities by including boundary terms. For the Laplacians such problems have been considered in [ACR02] by using variational method and in [WZ03] by using conformal transformation method. The methods described in this book are based on the potential theory and, compared to other approaches, do not rely so much on the particular structure of the Euclidean space. Certain Hardy and Rellich inequalities for sums of squares have been considered by Grillo [Gri03], compared to which our approach provides refinements from several points of view, based on the generalized representation formulae (Green's formulae) of non-subharmonic functions for improved Hardy and Rellich type inequalities with boundary terms.

Hardy inequalities and related topics on homogeneous groups

The lack of homogeneous hypoelliptic left invariant differential operators on general homogeneous groups is compensated by several other advantageous properties, such as a good polar decomposition which, combined with dilations, still allows one to explore the radial structure of the group. For this purpose, we extensively work with the radial derivative operator $\mathcal{R}$ and the Euler operator $\mathbb{E}$ which we describe in Section 1.3.

This fits well with the structure of Hardy and other inequalities as their maximisers are often achieved on radial functions. An advantage of such an approach is that one can also work with arbitrary quasi-norms and anisotropic structures still yielding similar properties and best constants in the inequalities. In addition, in Section 1.3.3 we demonstrate how the Hardy type inequalities for radial functions often imply similar inequalities for functions of general (non-radial) type.

Thus, Chapter 2 is devoted to Hardy inequalities on homogeneous groups, their weighted and critical versions, stability and remainder estimates. Furthermore, Chapter 3 is devoted to Rellich, Caffarelli–Kohn–Nirenberg and Sobolev type inequalities on homogeneous groups. In Chapter 9 we present different versions of uncertainty principles on homogeneous groups. We follow an abstract approach by defining abstract position and momentum operators satisfying minimal structural properties, already allowing one to establish a number of uncertainty type relations. Consequently, different choices of such abstract position and momentum operators are possible based on the additional structural properties available on the group.

In Chapter 10 we discuss different function spaces on homogeneous groups, with or without differentiability properties. The spaces involving radial derivatives are the Euler–Hilbert–Sobolev and Sobolev–Lorentz–Zygmund spaces. We investigate their basic properties and embeddings. The spaces not involving the differentiable structure are the Morrey and Campanato spaces. There, we discuss the boundedness of integral operators, namely, the Hardy–Littlewood maximal operator, Bessel–Riesz operators, generalized Bessel–Riesz operators, generalized fractional integral operators and Olsen type inequalities in generalized Morrey spaces on homogeneous groups.

Incidentally, all these and other results on homogeneous groups in this book give new statements already in the Euclidean setting of $\mathbb{R}^n$ when we are working with anisotropic differential structure in view of the arbitrariness of the choice of any homogeneous quasi-norm.

Let us consider, for instance, the following Bessel–Riesz operators

$$I_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n} \frac{|x-y|_E^{\alpha-n}}{(1+|x-y|_E)^{\gamma}} f(y)dy, \tag{8}$$

where $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p \geq 1$, $\gamma \geq 0$ and $0 < \alpha < n$. Classical results on the Bessel–Riesz operators are due to Hardy, Littlewood and Sobolev, precisely, the boundedness of the Bessel–Riesz operators on Lebesgue spaces was shown by Hardy and Littlewood in [HL27], [HL32] and by Sobolev in [Sob38]. In the case of $\mathbb{R}^n$, the Hardy–Littlewood maximal operator, the Riesz potential $I_{\alpha,0} = I_\alpha$, the generalized fractional integral operators, which are a generalized form of the Riesz potential $I_{\alpha,0} = I_\alpha$, Bessel–Riesz operators and Olsen type inequalities are widely analysed on Lebesgue spaces, Morrey spaces and generalized Morrey spaces. For further discussions in this direction we refer to [Ada75, CF87, BNC14, Nak94, EN04, Eri02, KNS99, Nak01, Nak02, GE09, SST12, IGLE15, IGE16], as well as to

[Bur13] for a recent survey. Morrey spaces for non-Euclidean distances find their applications in many problems, see, e.g., [GS15a, GS15b] and [GS16]. A natural analogue of the Bessel–Riesz operator (8) on homogeneous groups is the operator

$$I_{\alpha,\gamma}f(x) := \int_{\mathbb{G}} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^{\gamma}} f(y)dy.$$

In the setting of graded Lie groups the connections between these operators and the Sobolev spaces have been investigated in [FR16, Chapter 4] using the heat kernel methods. Here, in Chapter 10 we concentrate on the harmonic analysis aspects in the framework of Morrey and Campanato spaces on homogeneous groups.

In addition, we can mention an overview of constructions for Morrey–Campanato spaces in [RSS13] by Rafeiro, N. Samko and S. Samko, or in [RT15] by Rosenthal and Triebel. It is worth noting that Morrey–Campanato spaces can be interpolated [VS14]. One also considered variable exponent versions of Morrey spaces and maximal and singular operators there, see [GS13, GS16] and references therein.

In Chapter 4 we look at the Hardy inequalities from the point of view of operators of fractional orders. Certainly, fractional powers of Laplacians and sub-Laplacians can be defined in different ways, e.g., using the Fourier or spectral analysis. However, here, we first adopt the integral representation that turns out to make perfect sense on general homogeneous groups. More specifically, for $p > 1$ and $s \in (0, 1)$, we consider the *fractional p-sub-Laplacian* $(-\Delta_p)^s$ on a general homogeneous group $\mathbb{G}$ defined by the formula

$$(-\Delta_p)^s u(x) := 2 \lim_{\delta \searrow 0} \int_{\mathbb{G}\setminus B(x,\delta)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|y^{-1}x|^{Q+sp}} dy, \quad x \in \mathbb{G},$$

where $B(x,\delta) = B_{|\cdot|}(x,\delta)$ is a quasi-ball with respect to the quasi-norm $|\cdot|$, with radius δ centred at $x \in \mathbb{G}$. It turns out that this operator has many advantageous properties similar to those exhibited by the usual p-Laplacians on the Euclidean spaces, and in Chapter 4 we present their analysis and some applications to ‘partial differential’ functional equations and related spectral questions. Consequently, we look at operators of fractional orders from a different point of view, and in Section 4.7 we discuss the boundedness of the operator

$$T_\alpha f(x) := |x|^{-\alpha} \mathcal{L}^{-\alpha/2} f(x),$$

on L^p-spaces on stratified Lie groups, where $\mathcal{L}$ is a sub-Laplacian. The analysis is based on the Riesz kernel representation of such operators, and we also supplement it with several versions of the Landau–Kolmogorov inequalities.

In Chapter 5 we discuss integral versions of Hardy inequalities. In fact, such a point of view goes back to one of the original versions of such inequalities by Hardy [Har20], where he has shown the inequality

$$\int_b^\infty \left(\frac{\int_b^x f(t)dt}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_b^\infty f(x)^p dx,$$

where $p > 1$, $b > 0$, and $f \geq 0$ is a non-negative function. It turns out that such an inequality can be also put in the framework of general homogeneous groups, especially since it does not involve derivatives, so that one does not need to specify one's analysis to a particular choice of the gradient. Thus, in Chapter 5 we present inequalities of such a type in weighted and unweighted settings, actually providing characterizations of weights for which integral Hardy inequalities hold true. We also present inequalities in the convolution form which, in turn, can be used for the derivation of Hardy–Littlewood–Sobolev and Stein–Weiss inequalities. The latter can be then established both on general homogeneous groups as well as on stratified/graded groups using Riesz kernels of hypoelliptic differential operators.

In Chapter 8 we discuss the so-called geometric versions of Hardy inequalities. By this one usually means Hardy inequalities on domains when the distance to the boundary enters the inequality as a weight. For example, if Ω is a convex open set of the Euclidean space $\mathbb{R}^n$, then a geometric version of the Hardy inequality can take a form

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{\operatorname{dist}(x, \partial\Omega)^2} dx,$$

for all $u \in C_0^\infty(\Omega)$, with the sharp constant $1/4$. In the case of half-spaces and, more generally, convex domains, in Chapter 8 we present such inequalities in the setting of stratified groups.