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SCALE RENORMALIZATION AND RANDOM SOLUTIONS OF THE BURGERS EQUATION

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Abstract

Solutions of the Burgers equation with a stationary (spatially) stochastic initial condition are considered. A class of limit laws for the solution which correspond to a scale renormalization is considered.

STATIONARY INITIAL CONDITION; SHORT-RANGE DEPENDENCE; LONG-RANGE DEPENDENCE

Introduction

The Burgers equation

$$u_t + uu_x = \mu u_{xx}, \qquad \mu > 0, \tag{1}$$

$u = u(x, t)$ is a simple example of a non-linear partial differential equation that has been suggested as a simple model for turbulence (see Burgers (1948)) as well as for shock waves when $\mu \downarrow 0$ (see Whitham (1974)). Spatially stochastic solutions of the Navier–Stokes equation have been used in discussions of turbulence (see Batchelor (1953)) and Vishik et al., (1979)). We shall consider solutions of the Burgers equation with spatially stationary stochastic initial conditions.

Both Hopf (1950) and Cole (1951) have shown how the discussion of the Burgers equation can be reduced to that of the diffusion equation by a simple non-linear transformation. We shall give Hopf's formulation in the following theorem.

Theorem 0 (Hopf). *Let* $u(x, t)$ *be a solution of Equation* (1) *with continuous*

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initial condition $u_0(x) = u(x, 0)$. *If*

(2) $$\int_0^x u_0(\xi)d\xi = o(x^2)$$

as $|x| \to \infty$, *then*

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left[-\frac{1}{2\mu}F(x,y,t)\right]dy}{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\mu}F(x,y,t)\right]dy}$$

with

$$F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(\xi)d\xi.$$

Notice that if we set

$$\varphi_0(y) = \exp\left[-\frac{1}{2\mu}\int_0^y u_0(y)dy\right]$$

and

$$\varphi(x,t) = (4\pi\mu t)^{-1/2}\int_{-\infty}^{\infty} \varphi_0(y)\exp\left[-\frac{(x-y)^2}{4\mu t}\right]dy$$

then

(3) $$u(x,t) = -2\mu\frac{\partial}{\partial x}\log\varphi(x,t).$$

Random solutions. We shall be interested in the case in which $u_0(x)$ is a continuous and stationary process in x. To use Hopf's representation of the solution and its consequences, we shall require that the stationary process $u_0(\cdot)$ satisfy (2). For the moment assume that this is the case. Simple conditions on the process $u_0(\cdot)$ that imply (2) will be given a bit later. Our interest is in the asymptotics of

(4) $$\int_0^R u(x,t)dx = -2\mu\log\frac{\varphi(R,t)}{\varphi(0,t)}$$

as $R \to \infty$ with $t > 0$ fixed. Notice that

$$\varphi(R,t) = (4\pi\mu t)^{-1/2}\int_{-\infty}^{\infty} \varphi_0(R+v)\exp\left[-\frac{v^2}{4\mu t}\right]dv$$

with

$$\varphi_0(R+v) = \exp\left[-\frac{1}{2\mu}\int_0^R u_0(y)dy\right]\exp\left[-\frac{1}{2\mu}\int_R^{R+v} u_0(y)dy\right]$$

so that

$$\log\varphi(R,t) = -\frac{1}{2\mu}\int_0^R u_0(y)dy$$

(5)

$$+\log\left\{(4\pi\mu t)^{-1/2}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\mu}\int_R^{R+v} u_0(y)dy - \frac{v^2}{4\mu t}\right]dv\right\}.$$

The following simple remark can be made.

Proposition 1. *If* $u_0(x)$ *is a continuous stationary process in* x *that satisfies* (2), *then the process* $u(x,t)$ *is stationary in* x *for each fixed* $t>0$.

The proposition can be seen to be valid in the following way. If (2) is satisfied by $u_0(\cdot)$, the expression

$$(6) \qquad (4\pi\mu t)^{-1/2}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\mu}\int_x^{x+v} u_0(y)dy - \frac{v^2}{4\mu t}\right]dv$$

is well defined and determines a process that is jointly stationary in x with $u_0(\cdot)$. On differentiating $\log\varphi(x,t)$ with respect to x and making use of (4) and (5), one sees that $u(x,t)$ is stationary in x.

As an immediate consequence of the ergodic theorem we have the following remark.

Proposition 2. *If* u_0 *is a stationary process with*

$$E|u_0(x)| < \infty$$

then

$$(7) \qquad \int_0^x u_0(y)dy = O(x)$$

as $|x|\to\infty$ *and consequently* (2) *is satisfied.*

Under the assumptions of this proposition

$$\frac{1}{x}\int_0^x u_0(y)dy \to E(u_0(0)|\mathscr{I})$$

with probability 1 as $|x|\to\infty$ where $\mathscr{I}$ is the σ-field of invariant sets for the process $u_0(\cdot)$. However, this immediately yields (7).

Let us now consider stationary processes $u_0(\cdot)$ with finite second moment

$Eu_0(x)^2$. For most such processes the variance

$$\sigma^2(R) = \text{var}\left(\int_0^R u_0(y)dy\right)$$

will diverge as $R \to \infty$ though it is of course easy to construct processes for which this variance remains bounded as $R \to \infty$. Nonetheless, the most interesting processes are those for which the variance diverges and we shall assume that this is the case. Consider the scale renormalization

$$\sigma(R)^{-1} \int_0^{Ry} u(\eta, t)d\eta$$

as $R \to \infty$. Given a process $z(x)$ consider

$$h(R)^{-1} \int_0^{Ry} \{z(x) - c\}dx = T_R(y)$$

where $h(R) \to \infty$ as $R \to \infty$. We shall say that $T_R(y)$ converges weakly (to some process $W(y)$) if for any given continuous functional ψ on $C[0, 1]$ the distribution of

$$\psi(T_R(y), 0 \leqq y \leqq 1)$$

tends to the distribution of $\psi(W(y), 0 \leqq y \leqq 1)$. We then have the following theorem.

Theorem 1. *Let $u_0(\cdot)$ be a continuous stationary process with finite second moment such that $\sigma(R)$ diverges as $R \to \infty$. Assume that $u_0(\cdot)$ is the initial condition for the Burgers equation and $u(x, t)$ the solution of the Burgers equation. If the processes*

$$\sigma(R)^{-1} \int_0^{Ry} \{u_0(x) - Eu_0(x)\}dx, \tag{8}$$

$$\sigma(R)^{-1} \int_0^{Ry} \{u(x, t) - Eu_0(x)\}dx \tag{9}$$

both converge weakly as $R \to \infty$, they must converge weakly to the same random process.

The theorem follows from the stationarity of (6) in x for each $t > 0$, together with relations (4) and (5).

Theorem 1 is of interest because it implies that the low frequency behavior of the spatial spectrum of $u(x, t)$ in the immediate neighborhood of 0 is the same independently of $t > 0$ and is the same as that of $u_0(x)$. Remarks suggesting something like this have been made in the case of homogeneous turbulence (see Batchelor (1953), Section 3.3).

We shall mention a number of conditions on the process $u_0(\cdot)$ (the initial condition) that allow one to determine the asymptotic behavior of (8) and (9) as $R \to \infty$ (but with $\mu > 0$ and t fixed). Let

$$\mathscr{B}_x = \mathscr{B}\{u_0(w), w \leqq x\},$$
$$\mathscr{F}_x = \mathscr{B}\{u_0(w), w \geqq x\},$$

be the σ-fields of events generated by $u_0(w)$, $w \leqq x$, and $u_0(w)$, $w \geqq x$, respectively. Thus, $\mathscr{B}_x$ and $\mathscr{F}_x$ correspond to the spatial information to the left of x, and the spatial information to the right of x for the process $u_0(\cdot)$. Let $\alpha(d)$, $d > 0$ be the coefficient

$$\alpha(d) = \sup_{\substack{A \in \mathscr{B}_x \\ B \in \mathscr{F}_{x+d}}} |P(A \cap B) - P(A)P(B)|.$$

Notice that $\alpha(d)$ is independent of x because of the assumption that $u_0(x)$ is stationary in x. The process $u_0(\cdot)$ is said to be strongly mixing if

$$\alpha(d) \to 0$$

as $d \to \infty$. The strong mixing property is one of the many non-equivalent ways of formulating a concept of short-range dependence. Let $u_0(\cdot)$ be stationary with

$$Eu_0(x) \equiv 0, \qquad E|u_0(x)|^{2+\delta} < \infty \tag{10}$$

with $0 < \delta \leqq 1$, and strongly mixing with

$$\alpha(d) = O(d^{-(1+\varepsilon)(1+2/\delta)})$$

for some $0 < \varepsilon \leqq 1$. If $\sigma(R) \to \infty$ as $R \to \infty$, these conditions are enough to guarantee that (8) and (9) converge weakly to a Brownian motion process $B(y)$, $0 \leqq y$. This and even stronger results are directly implied by Theorem 1 and remarks in Dehling and Philipp (1982).

A more restricted but less complicated result can be formulated in terms of the concept of an asymptotically uncorrelated process. If $\mathscr{A}$ is a σ-field, let $L^2(\mathscr{A})$ be the collection of random variables measurable with respect to $\mathscr{A}$ that have finite second moment. Let

$$\beta(d) = \sup_{\substack{f \in L^2(\mathscr{B}_x) \\ g \in L^2(\mathscr{F}_{x+d})}} |\operatorname{corr}(f, g)|$$

with corr(f, g) the correlation of f and g. The process $u_0(\cdot)$ is said to be asymptotically uncorrelated if $\beta(d) \to 0$ as $d \to \infty$. The asymptotically uncorrelated condition is an even stronger formulation of a concept of short-range dependence than is that of strong mixing.

Theorem 2. *Let* $u_0(\cdot)$ *be a stationary process that is asymptotically uncorrelated, satisfies* (10), *and is such that* $\sigma(R) \to \infty$ *as* $R \to \infty$. *It then follows that* (8) *converges weakly to a Brownian motion process* $B(y)$ *as* $R \to \infty$.

Theorem 2 follows directly from our Theorem 1 and a result of Ibragimov (1975) concerning asymptotic normality for asymptotically uncorrelated processes. It is easy to see how to modify these results if the mean of $u_0(\cdot)$ is non-zero.

Let us now consider a special class of processes $u_0(\cdot)$ as initial condition which imply a weak limiting process different from the Brownian motion. These processes all exhibit a long-range dependence as contrasted with those processes dealt with earlier. This long-range dependence shows itself in terms of a singular mass accumulation in the neighborhood of 0 for the spectrum of the process $u_0(\cdot)$. All the processes $u_0(\cdot)$ we exhibit are obtained by an instantaneous function of a Gaussian process. Let $G(x)$ be a stationary Gaussian process with mean zero and covariance function

$$r(x) = EG(0)G(x) \cong x^{-D}L(x) \tag{11}$$

as $x \to \infty$, $L(\cdot)$ slowly varying and $0 < D < 1$ For convenience, assume that $r(0) = 1$. Let $h(\cdot)$ be a function with

$$\begin{aligned} &Eh(G(x)) \equiv 0, \\ &Eh(G(x))^2 < \infty. \end{aligned} \tag{12}$$

Our object is to take the initial condition for the Burgers equation $u(x, 0) = u_0(x)$ a process of the form

$$u_0(x) = h(G(x)). \tag{13}$$

Consider the Fourier–Hermite expansion of $h(\cdot)$

$$h(u) = \sum_{q=1}^{\infty} \frac{J(q)}{q!} h_j(u)$$

and let m be the first index with non-zero coefficient $J(m)$. The Hermite polynomials $h_j(u)$ have leading coefficient 1. We shall be interested in the range

$$0 < D < \frac{1}{m}.$$

The process $u_0(\cdot)$ given by (13) satisfied the conditions of Theorem 1 and so the asymptotic behavior of (8) will be the same as that of (9). Now

$$\sigma^2(R) = E\left(\int_0^R h(G(\dot{x}))dx\right)^2$$

$$\cong \left(\frac{J(m)}{m!}\right)^2 \frac{2}{(1-mD)(2-mD)} R^{2-mD} L^m(R)$$

as $R \to \infty$. Let $W(\cdot)$ be the complex Gaussian process with

$$EW(\Delta) \equiv 0$$

$$W(\Delta) = \overline{W(-\Delta)}$$

for Δ an interval,

$$W\left(\bigcup_{i=1}^n \Delta_i\right) = \sum_{i=1}^n W(\Delta_i)$$

for disjoint intervals Δ_i, and covariance

$$E(W(\Delta_1)\overline{W(\Delta_2)}) = |\Delta_1 \cap \Delta_2|$$

($|\Delta|$is the length of Δ). Let the symbol$\int_m'$ denote integration over all of m-dimensional Euclidean space except for $\lambda_i = \pm\lambda_j$, $i \neq j$, $i, j = 1, \cdots, m$. We introduce the process

$$Z_m(x) = C(m, D) \int_m' \frac{\exp(i(\lambda_1 + \cdots + \lambda_m)x) - 1}{i(\lambda_1 + \cdots + \lambda_m)}$$
$$\times |\lambda_1|^{(D-1)/2} \cdots |\lambda_m|^{(D-1)/2} W(d\lambda_1) \cdots W(d\lambda_m)$$

with

$$C(m, D) = \left\{\frac{(1-mD)(2-mD)}{2(m!)(2\Gamma(D)\cos D\pi/2)^m}\right\}^{1/2}.$$

Notice that the process is Gaussian only in the case $m = 1$.

Theorem 3. *Let* $u_0(\cdot)$ *be the process* (13) *satisfying* (11) *and* (12). *Then*

$$\sigma(R)^{-1} \int_0^{Ry} u(x, t)dx$$

will converge weakly to the process

$$Z_m(y)$$

as $R \to \infty$.

Theorem 3 follows directly from our Theorem 1 and results of Taqqu (1981) (his Theorem 3.2) and Dobrushin and Major (1979) (their Theorem 1). The limiting process $Z_m(y)$ is Gaussian when $m = 1$ and is then a fractional Brownian motion.

A counterexample for a growth conditon. The growth condition (2) has played an important part in our discussion. It is of some interest to give a simple example of a stationary process $u_0(\cdot)$ that does not satisfy the condition. The process will understandably not have moments in view of Proposition 2. It will be defined in terms of a stable symmetric process $\xi(\cdot)$ with coefficient α and with stationary independent increments. Let $F(y)$ be the distribution function of $\xi(1)$. It is well known that

$$1 - F(y) \cong \frac{C}{y^{\alpha}}$$

as $y \to \infty$ *with some constant* $c > 0$ and $0 < \alpha < 2$. The characteristic function of $\xi(1)$ is $\exp[-|u|^{\alpha}]$. Since the characteristic function of $\xi(\tau)$, $\tau > 0$, is $\exp[-\tau|u|^{\alpha}]$ it follows that the distribution function $\xi(\tau)$ is $F(\tau^{-1/2}y)$ with

$$1 - F(\tau^{-1/2}y) \cong \frac{c\tau}{y^{\alpha}} \tag{14}$$

as $y \to \infty$. Let $g(t)$ be an increasing continuous non-negative function defined for $t > 0$. Now $g(t)$ is called an upper function for $\xi(\cdot)$ if

$$P\left\{\limsup_{t\to\infty} \xi(t)/g(t) < 1\right\} = 1$$

and a lower function for $\xi(\cdot)$ if

$$\mathrm{P}\left\{\limsup_{t\to\infty} \xi(t)/g(t) > 1\right\} = 1.$$

There are sufficient conditions for $g(t)$ to be an upper or lower function for $\xi(\cdot)$ (see Gikhman and Skorokhod (1969)). A sufficient condition for $g(\cdot)$ to be an upper function for $\xi(\cdot)$ is

$$\int_1^{\infty} \frac{1}{t} P\{\xi(t) > g(t)\} dt < \infty. \tag{15}$$

A sufficient condition for $g(\cdot)$ to be a lower function for $\xi(\cdot)$ is that

$$\sum_{k=1}^{\infty} P\{\xi(a^k) > g(a^k)\} \tag{16}$$

diverge for all $a > 1$. By using (14) and (15) it is clear that

$$g(t) = t^{1/\alpha}(\log t)^{1/\alpha+\varepsilon}$$

or

$$g(t) = t^{1/\alpha}(\log t)^{1/\alpha}(\log\ \log t)^{1/\alpha+\varepsilon}$$

are upper functions for $\xi(\cdot)$ for any $\varepsilon > 0$.

Similarly (14) and (16) imply that

$$g(t) = t^{1/\alpha}(\log \log t)^{1/\alpha}$$

or

$$g(t) = t^{1/\alpha}(\log t)^{1/\alpha}(\log\ \log t)^{1/\alpha}$$

are lower functions for $\xi(\cdot)$. We shall now generate a stationary process $u_0(\cdot)$ with continuous sample functions from the stable process $\xi(\cdot)$ with independent stationary increments as follows. Let

$$u_0(x) = \int_x^{x+1} [\xi(w+1) - \xi(w)]dw. \tag{17}$$

Now

$$\int_0^x u_0(y)dy = \int_0^x \int_y^{y+1} [\xi(w+1) - \xi(w)]dw\, dy$$

$$= \int_0^1 y[\xi(y+1) - \xi(y)]dy + \int_{x-1}^x (x-y)[\xi(y+1) - \xi(y)]dy \tag{18}$$

$$+ \int_1^{x-1} [\xi(y+1) - \xi(y)]dy$$

while

$$\int_1^{x-1} [\xi(y+1) - \xi(y)]dy = \int_{x-1}^x \xi(y)dy - \int_1^2 \xi(y)dy. \tag{19}$$

From (18) and (19) it follows that the functions $g(x)$ mentioned above as upper and lower functions for $\xi(x)$ as $x \to \infty$ are also upper and lower functions for

$$\int_0^x u_0(y)dy$$

with $u_0(\cdot)$ given by (17). Notice that this implies that (17) satisfies the growth condition (2) if $\alpha > \frac{1}{2}$ but does not if $\alpha \leqq \frac{1}{2}$.

Varying μ. In the discussion of asymptotics in the earlier part of the paper one considered letting $R \to \infty$ in (4) but with μ and t fixed. It's clearly also of interest to look at the case in which $R \to \infty$ and $\mu \downarrow 0$ but with t fixed or at least bounded. A simple estimate will show us that the asymptotics obtained earlier are still valid in such a circumstance if one can assume that for the initial condition one has the finite bound

$$|u_0(x)| \leqq c$$

for all x with probability 1. For then

$$\int_x^{x+v} u_0(y)dy + \frac{v^2}{2t} \geqq \begin{cases} -cv + \dfrac{v^2}{2t} & \text{if } v \geqq 0 \\ cv + \dfrac{v^2}{2t} & \text{if } v \leqq 0. \end{cases} \tag{20}$$

This implies that an upper bound for (6) is given by

$$\frac{1}{\sqrt{4\pi\mu t}} \exp\left(\frac{c^2 t}{4\mu}\right)\left[\int_0^\infty \exp\left(-\frac{(v-ct)^2}{4\mu t}\right)dv + \int_{-\infty}^0 \exp\left(-\frac{(v+ct)^2}{4\mu t}\right)dv\right]$$

$$\leqq 2\exp\left(\frac{c^2 t}{4\mu}\right).$$

(21)

A lower bound for (6) can be obtained by noting that

$$\int_x^{x+v} u_0(y)dy + \frac{v^2}{2t} \leqq \begin{cases} cv + \dfrac{v^2}{2t} & \text{if } v \geqq 0 \\ -cv + \dfrac{v^2}{2t} & \text{if } v \leqq 0. \end{cases} \tag{20'}$$

The lower bound for (6) is then

$$\frac{1}{\sqrt{4\pi\mu t}} \exp\left(\frac{c^2 t}{4\mu}\right)\left[\int_0^\infty \exp\left(-\frac{(v+ct)^2}{4\mu t}dv\right) + \int_{-\infty}^0 \exp\left(-\frac{(v-ct)^2}{4\mu t}\right)dv\right]$$

$$= \exp\left(\frac{c^2 t}{4\mu}\right)2\left[1-\Phi\left(\frac{ct}{\sqrt{2\mu t}}\right)\right] \tag{22}$$

$$\leqq \begin{cases} \dfrac{1}{c}\left(\dfrac{\mu}{\pi t}\right)^{1/2} & \text{if } c\left(\dfrac{t}{2\mu}\right)^{1/2} > 1 \\ \exp\left(\dfrac{c^2 t}{4\mu}\right)[1-\Phi(1)] & \text{if } c\left(\dfrac{t}{2\mu}\right)^{1/2} \leqq 1. \end{cases}$$

The upper and lower bounds (21) and (22) for (6) together with the relations (4) and (5) imply that for a bounded initial condition $u_0(\cdot)$ the results already obtained for $R \to \infty$ but with μ and t fixed are still valid when $R \to \infty$, $\mu \downarrow 0$ and t is bounded.

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