*Asymptotic Behavior of Eigenvalues of Toeplitz Forms**

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1. Introduction. Let $F(\lambda) = \{F_{jk}(\lambda); j, k = 1, \cdots, s\}$ be an $s \times s$ matrix-valued function of bounded variation on $[-\pi, \pi]$. By this we mean that every complex-valued element $F_{jk}(\lambda)$ is of bounded variation. Further, let every difference $F(\lambda_1) - F(\lambda_2)$ be Hermitian. It will be convenient and in no way less general to take $F(-\pi) = 0$, the null matrix. Let

$$r_k(dF) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda}\, dF(\lambda), \qquad k = 0, \pm 1, \cdots, \tag{1}$$

be the sequence of Fourier-Stieltjes coefficients of F. The sequence $r_k(dF)$ is a sequence of $s \times s$ matrices. Let the Hermitian matrix

$$R_n(dF) = \{r_{j-k}(dF); j, k = 1, \cdots, n\} \tag{2}$$

be the $ns \times ns$ block Toeplitz matrix generated by dF. Since F is of bounded variation, its derivative

$$f(\lambda) = \frac{dF(\lambda)}{d\lambda} \tag{3}$$

exists almost everywhere and is Hermitian. Let $\mu_1(\lambda) \leqq \mu_2(\lambda) \leqq \cdots \leqq \mu_s(\lambda)$ be the eigenvalues of $f(\lambda)$ in order of magnitude. The functions $\mu_j(\lambda)$, $j = 1, \cdots, s$, are measurable (see GYIRES [3]). Let $\mu_{j,n}$, $j = 1, \cdots, ns$, be the eigenvalues of $R_n(dF)$ and $N_n(x)$ the number of eigenvalues of $R_n(dF)$ less than or equal to x. We show that

$$\lim_{n\to\infty} \frac{N_n(x)}{ns} = \frac{1}{2\pi s} \sum_{j=1}^{s} m\{\lambda \mid \mu_j(\lambda) \leqq x\} \tag{4}$$

at every point of continuity of the right hand side, where m is Lebesgue measure on $[-\pi, \pi]$. The case of Toeplitz matrices arises when $s = 1$ and was treated by SZEGÖ and GRENANDER & SZEGÖ (see [2]). The result obtained in this paper is a generalization of that given by GYIRES [3] and ROSENBLATT [6] for block Toeplitz matrices.

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A crude upper bound is obtained for the maximal eigenvalue of Toeplitz matrices generated by an absolutely continuous F. This upper bound is used to obtain a result on the asymptotic distribution of covariance estimates of normal stationary processes.

2. Nondecreasing $F(\lambda)$. We first prove the result on the asymptotic distribution of eigenvalues of block Toeplitz forms in the case of nondecreasing $F(\lambda)$. The function $F(\lambda)$ is said to be nondecreasing if $F(\lambda_1) - F(\lambda_2)$ is a positive definite matrix whenever $\lambda_1 \geqq \lambda_2$.

Lemma 1. *Let $F(\lambda)$ be an $s \times s$ nondecreasing matrix-valued function of bounded variation on $[-\pi, \pi]$. The limiting distribution of the eigenvalues of the block Toeplitz matrix $R_n(dF)$ as $n \to \infty$ is given by*

$$\frac{1}{2\pi s} \sum_{j=1}^{s} m\{\lambda \mid \mu_j(\lambda) \leqq x\}. \tag{5}$$

Let $G_n(x)$ be the distribution function of the eigenvalues of $R_n(dF)$, that is,

$$G_n(x) = \frac{N_n(x)}{sn}. \tag{6}$$

We need only consider the distribution functions $G_n(x)$ on $[0, \infty)$. Now

$$\int_0^\infty x \, dG_n(x) = \frac{1}{ns} \sum_{j=1}^{ns} \mu_{j,n} = \frac{1}{ns} \operatorname{sp}(R_n(dF)) = \frac{1}{s} \operatorname{sp}(r_0(dF)) \tag{7}$$

where sp (M) denotes the trace of the matrix M. Notice that

$$y[1 - G_n(y)] \leqq \int_y^\infty x \, dG_n(x) \leqq \frac{1}{s} \operatorname{sp}(r_0) = c, \tag{8}$$

so that

$$1 - G_n(y) \leqq \frac{c}{y}. \tag{9}$$

Consider any weak limit $G(y)$ of a subsequence of the sequence $G_n(y)$. Clearly any such limit must also satisfy

$$1 - G(y) \leqq \frac{c}{y} \tag{10}$$

and hence is a distribution function. Let z be a nonnegative number. The basic result on the determinant of the prediction error covariance matrix in the multidimensional prediction problem tells us that

$$\begin{aligned} \frac{1}{ns} \sum_{j=1}^{ns} \log(1 + z\mu_{j,n}) &= \int_0^\infty \log(1 + zx) \, dG_n(x) \\ &\to \frac{1}{2\pi s} \int_{-\pi}^{\pi} \log \det(I + zf(\lambda)) \, d\lambda = \frac{1}{2\pi s} \sum_{j=1}^{s} \int_{-\pi}^{\pi} \log(1 + z\mu_j(\lambda)) \, d\lambda \end{aligned} \tag{11}$$

as $n \to \infty$ (see [5] and [6]). On integrating by parts we see that

$$\int_0^\infty \log(1 + zx)\, dG_n(x) = \log(1 + zx)(G_n(x) - 1)\,\Big|_0^\infty + \int_0^\infty \frac{z}{1 + zx}(1 - G_n(x))\, dx = \int_0^\infty \frac{z}{1 + zx}(1 - G_n(x))\, dx. \tag{12}$$

Relations (11) and (12) imply that for any weak limit $G(y)$ of a subsequence of $G_n(y)$

$$\int_0^\infty \frac{z}{1 + zx}(1 - G(x))\, dx = \frac{1}{2\pi s} \sum_{j=1}^{s} \int_{-\pi}^{\pi} \log(1 + z\mu_j(\lambda))\, d\lambda. \tag{13}$$

The distribution function (5) obviously satisfies (13). To complete the proof for nondecreasing F we only have to show that G is uniquely determined by the transform (13). Consider the uniqueness problem for the transform

$$\int_0^\infty \frac{z}{1 + zx} H(x)\, dx, \qquad z \geqq 0, \tag{14}$$

where H is of bounded variation and satisfies $|H(x)| \leqq k/(1 + x)$ for some constant k. This is equivalent to the uniqueness problem for the Stieltjes transform

$$\int_0^\infty \frac{1}{z + x} H(x)\, dx, \qquad z > 0. \tag{15}$$

A proof of uniqueness is easily given. Suppose

$$\int_0^\infty \frac{1}{z + x} H(x)\, dx \equiv 0 \tag{16}$$

for all $z > 0$. Equation (16) can be rewritten as

$$\int_0^\infty e^{-zu} \int_0^\infty e^{-xu} H(x)\, dx\, du \equiv 0. \tag{17}$$

But this implies that

$$\int_0^\infty e^{-xu} H(x)\, dx \equiv 0. \tag{18}$$

It follows that $H(x) \equiv 0$ almost everywhere. The proof of Lemma 1 is complete.

It is readily seen that the conclusion of Lemma 1 holds for every F of bounded variation such that $F(\lambda) + a(\lambda + \pi)I$ is nondecreasing for some real value a. It similarly holds for every F of bounded variation such that $F(\lambda) + a(\lambda + \pi)I$ is nonincreasing for some real value a.

3. General F. In this section it is shown that the conclusion of Lemma 1 holds for general Hermitian F of bounded variation. The extension will depend on the following remarks. Given two $n \times n$ Hermitian matrices A, B with

$A - B \geqq 0$ (difference positive definite), the eigenvalues λ_j and μ_j of A and B respectively (assume the eigenvalues are indexed in order of magnitude) satisfy the corresponding inequalities, that is,

$$\lambda_j \geqq \mu_j\,, \qquad j = 1, \cdots, n. \tag{19}$$

This follows readily from the minimax property of eigenvalues (see HILBERT & COURANT [1]).

Theorem 1. *Let $F(\lambda)$ $(F(-\pi) = 0)$ be an $s \times s$ Hermitian matrix-valued function of bounded variation on $[-\pi, \pi]$. The limiting distribution of the eigenvalues of the block Toeplitz matrix $R_n(dF)$ as $n \to \infty$ is given by*

$$\frac{1}{2\pi s} \sum_{j=1}^{s} m\{\lambda \mid \mu_j(\lambda) \leqq x\}.$$

We first prove the desired result for such a function F whose singular part is nondecreasing. Let $f(\lambda) = dF(\lambda)/d\lambda$. The singular part of F is given by $F(\lambda) - \int_{-\pi}^{\lambda} f(\mu)\,d\mu$. Let

$$S_a = \{\lambda \mid f(\lambda) + aI \geqq 0\}, \tag{20}$$

and $\bar{S}_a$ be the complement of S_a. The sets S_a are nondecreasing as $a \to \infty$ and such that the Lebesgue measure $m(S_a) \to 2\pi$ as $a \to \infty$. This follows from the fact that $f(\lambda)$ is integrable, that is, every element of f is integrable. Given any value λ, let

$$f^{+}(\lambda) = \sum_{u,v=1}^{s} s\,|f_{u,v}(\lambda)| \cdot I. \tag{21}$$

Note that

$$f(\lambda) \leqq f^{+}(\lambda) \tag{22}$$

($A \geqq B$ means that $A - B \geqq 0$). Inequality (22) is to be understood as holding almost everywhere where the elements of the matrices are finite and hence well defined. Let

$$F^{a}(\lambda) = F(\lambda) - \int_{-\pi}^{\lambda} f(\mu)\,d\mu + \int_{-\pi}^{\lambda} f^{a}(\mu)\,d\mu \tag{23}$$

where

$$f^{a}(\lambda) = \begin{cases} f(\lambda) & \text{on } S_a \\ f^{+}(\lambda) & \text{on } \bar{S}_a. \end{cases} \tag{24}$$

Because of (22) it follows that

$$R_n(dF) \leqq R_n(dF^{a}) \tag{25}$$

for all a. Further, $R_n(dF^{a})$ is nonincreasing as $a \to \infty$. Let $G^{(n)}(x)$, $G_a^{(n)}(x)$ be the distribution functions of the eigenvalues of $R_n(dF)$ and $R_n(dF^{a})$ respectively.

It then follows that

$$G_a^{(n)}(x) \leqq G^{(n)}(x) \tag{26}$$

for all n, a, and x. The integrability of $f(\lambda)$, $f^+(\lambda)$, inequality (25), and the fact that $m(\bar{S}_a) \to 0$ as $a \to \infty$ imply that for every $\epsilon > 0$ there is an $a(\epsilon) > 0$ such that for $a > a(\epsilon)$

$$\begin{aligned} 0 &\leqq \frac{1}{sn} \operatorname{sp} (R_n(dF^a) - R_n(dF)) \\ &= \int_{-\infty}^{\infty} x \, d[G_a^{(n)}(x) - G^{(n)}(x)] = \int_{-\infty}^{\infty} [G^{(n)}(x) - G_a^{(n)}(x)] \, dx < \epsilon \end{aligned} \tag{27}$$

independently of n. By the remarks at the end of section 2, it follows that $G_a^{(n)}(x)$ has the limiting distribution

$$G_a(x) = \frac{1}{2\pi s} \sum_{j=1}^{s} m\{\lambda \mid \mu_j^a(\lambda) \leqq x\} \tag{28}$$

as $n \to \infty$ where the $\mu_j^a(\lambda)$ are the eigenvalues of $f^a(\lambda)$. Let $H(x)$ be the limit of any subsequence of the functions $G^{(n)}(x)$. It follows that

$$G_a(x) \leqq H(x) \tag{29}$$

for all x and a. Inequality (29) implies that

$$0 \leqq \int_{-\infty}^{\infty} [H(x) - G_a(x)] \, dx \leqq \epsilon \tag{30}$$

for all $a > a(\epsilon)$. The family of functions $G_a(x)$ is nondecreasing at each x as $a \to \infty$ and has as limit as $a \to \infty$

$$G(x) = \frac{1}{2\pi s} \sum_{j=1}^{s} m\{\lambda \mid \mu_j(\lambda) \leqq x\}. \tag{31}$$

Thus

$$H(x) \geqq G(x) \tag{32}$$

and

$$0 \leqq \int_{-\infty}^{\infty} [H(x) - G(x)] \, dx \leqq \epsilon \tag{33}$$

for every ϵ. But this implies that $H(x) = G(x)$. Since this is true for the limit $H(x)$ of any subsequence of $G^{(n)}(x)$, there is a limiting distribution which is given by $G(x)$. The proof is complete for F with a nondecreasing singular part.

We now complete the proof of Theorem 1 by showing that (5) is the asymptotic distribution of eigenvalues of $R_n(dF)$ for a general Hermitian F of bounded variation. The singular part of F is

$$H(\lambda) = F(\lambda) - \int_{-\pi}^{\lambda} f(\mu) \, d\mu \tag{34}$$

where $f(\lambda) = dF(\lambda)/d\lambda$. Let

$$H^+(\lambda) = \sum_{u,v=1}^{s} s \int_{-\pi}^{\lambda} |dH_{u,v}(\mu)| \cdot I. \tag{35}$$

Set

$$F_1(\lambda) = \int_{-\pi}^{\lambda} f(\mu)\, d\mu + H^+(\lambda), \qquad F_2(\lambda) = \int_{-\pi}^{\lambda} f(\mu)\, d\mu - H^+(\lambda). \tag{36}$$

F_1 has a nondecreasing singular part H^+ and F_2 a nonincreasing singular part $-H^+$. Further,

$$R_n(dF_1) \geqq R_n(dF) \geqq R_n(dF_2) \tag{37}$$

for all n. Since $R_n(dF_1)$, $R_n(dF_2)$ have the same asymptotic eigenvalue distribution (5) by the result on the asymptotic distribution of eigenvalues for a function with a nondecreasing singular part given in the last paragraph, it follows from (37) that $R_n(dF)$ has the asymptotic eigenvalue distribution (5).

4. Asymptotic distribution of covariance estimates. It will be useful to have an upper bound on the maximal eigenvalue of the ordinary Toeplitz matrix $R_n(f)$ generated by a nonnegative function $f \in L^2$ in our derivation of the asymptotic distribution of the covariance estimates of a normal stationary process. There are refined estimates of $\lambda_{n,n}$ when f is a bounded function satisfying certain regularity conditions (see [2]). However, we wish to allow f to be unbounded. Let

$$G(x) = \frac{1}{2\pi} m\{\lambda \mid f(\lambda) \leqq x\}.$$

Lemma 2. *The maximal eigenvalue $\lambda_{n,n}$ of the Toeplitz matrix $R_n(f)$ generated by $f \in L$ is bounded above by*

$$n \int_{b(n)}^{\infty} x\, dG(x) \tag{38}$$

where $b(n)$ is the supremum of the values α such that $1 - G(\alpha) > 1/n$. If $f \in L^2$, this upper bound is of smaller order than $n^{\frac{1}{2}}$ as $n \to \infty$.

The eigenvalue $\lambda_{n,n}$ is the maximal value attained by the quadratic form cR_nc' with c an n-vector subject to the restraint $cc' = 1$. But

$$cR_nc' = \frac{1}{2\pi} \int_{-\pi}^{\pi} |c_n(e^{i\lambda})|^2 f(\lambda)\, d\lambda, \qquad c_n(e^{i\lambda}) = \sum_{j=1}^{n} c_j e^{ij\lambda} \tag{39}$$

and the absolute maximum of $|c_n(e^{i\lambda})|^2$ subject to the restraint $cc' = 1$ is n. Let

$$S_x = \{\lambda \mid f(\lambda) \geqq x\}. \tag{40}$$

We certainly overestimate $\lambda_{n,n}$ by replacing $|c_n(e^{i\lambda})|^2$ by a function equal to n on the set $S_{b(n)}$. Now

$$\frac{1}{2\pi} n \int_{S_{b(n)}} f(\lambda)\, d\lambda = n \int_{b(n)}^{\infty} x\, dG(x).$$

If $f \varepsilon L^2$ then $\int x^2\, dG(x) < \infty$ and by the Schwarz inequality

$$n \int_{b(n)}^{\infty} x\, dG(x) \leqq n\left[\int_{b(n)}^{\infty} dG(x) \int_{b(n)}^{\infty} x^2\, dG(x)\right]^{1/2} = o(n^{1/2}).$$

An upper bound for the maximal eigenvalue of a block Toeplitz matrix can be obtained by similar techniques.

We now consider the asymptotic distribution of the estimates

$$r_k^* = \frac{1}{N} \sum_{j=1}^{N-|k|} X_j X_{j+k}, \qquad k = 0, 1, \cdots, s, \tag{41}$$

of the covariances r_k, $k = 0, 1, \cdots, s$, of a normal stationary process $\{X_k\}$.

Theorem 2. *Let $\{X_k ; k = 0, \pm 1, \cdots\}$ be a normal stationary process with mean* $\mathrm{E}\, X_k \equiv 0$ *and covariances*

$$r_k = \operatorname{cov}(X_j, X_{j+k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda \qquad k = 0, \pm 1, \cdots, \tag{42}$$

with $f \varepsilon L^2$. Then

$$\sqrt{N}\,(r_k^* - \mathrm{E}\, r_k^*), \qquad k = 0, 1, \cdots, s, \tag{43}$$

are asymptotically jointly normally distributed with mean zero and covariances

$$c_{j,k} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos j\lambda \cos k\lambda\, f^2(\lambda)\, d\lambda, \qquad j, k = 0, 1, \cdots, s. \tag{44}$$

The joint characteristic function of the random variables (43) is

$$\begin{aligned} \mathrm{E}\left(\exp\left\{\sum_{k=0}^{s} it_k (r_k^* - \mathrm{E}\, r_k^*) \sqrt{N}\right\}\right) &= \exp\left\{-\sum_{k=0}^{s} it_k \sqrt{N}\, \mathrm{E}\, r_k^*\right\} (2\pi)^{-1/2N} |R_N|^{-1/2} \\ &\quad \cdot \int_{-\infty}^{\infty} \cdots \int \exp\{ixT_N x'/\sqrt{N} - \tfrac{1}{2} x R_N^{-1} x'\}\, dx \\ &= |I - 2iR_N T_N/\sqrt{N}|^{-1/2} \exp\left\{-\sum_{k=0}^{s} it_k \sqrt{N}\, \mathrm{E}\, r_k^*\right\} \end{aligned} \tag{45}$$

where x is a row N-vector, $R_N = R_N(f)$, and $T_N = \{t_{j,k} ; j, k = 1, \cdots, N\}$ is the Toeplitz matrix with $t_{i,i} \equiv t_0$, $t_{i,i\pm k} = \frac{1}{2} t_{|k|}$ if $0 < k \leqq s$, and all other elements zero. Now expression (45) can be written as

$$\exp\left\{-\tfrac{1}{2} \sum_{j=1}^{N} [\log(1 - 2i\mu_{j,N}/\sqrt{N}) + 2\, i\mu_{j,N}/\sqrt{N}]\right\}$$

where the $\mu_{j,N}$ are the eigenvalues of $R_N^{1/2} T_N R_N^{1/2}$. Here $R_N^{\frac{1}{2}}$ is the positive definite square root of R_N. However,

$$\mu_{j,N} = O(\max_j \alpha_{j,N} \sum_j |t_j|),$$

where the $\alpha_{j,N}$ are the eigenvalues of R_N. By Lemma 2 the eigenvalues of R_N are $o(N^{\frac{1}{2}})$ uniformly in j so that $\mu_{j,N}/\sqrt{N} = o(1)$ uniformly in j. Therefore (45) can be given as

$$\exp\left\{- \text{sp}\,(R_N T_N)^2/N + O\left(\sum_{j=1}^{N} |\mu_{j,N}|^3/N^{3/2}\right)\right\}. \tag{46}$$

But

$$\sum |\mu_{j,N}|^3 \leqq (\sum \mu_{j,N}^2 \sum \mu_{j,N}^4)^{1/2} = [\text{sp}\,(R_N T_N)^2\, \text{sp}\,(R_N T_N)^4]^{1/2}. \tag{47}$$

Furthermore

$$0 \leqq \text{sp}\,(R_N T_N)^{2k} \leqq \text{sp}\,(M_N)^{2k}, \qquad k = 1, 2, \cdots, \tag{48}$$

where M_N is the $N \times N$ Toeplitz matrix with elements

$$m_{i-j} = \sum_{|u| \leqq s} |r_{i-j-u}| \cdot |t_u|.$$

Notice that M_N is generated by a real-valued function $g \in L^2$ since $m_k = m_{-k}$ and $\sum |r_k|^2 < \infty$. The fact that $\sum m_k^2 < \infty$ implies that sp $(M_N^2) = O(N)$. Since $g \in L^2$, Lemma 2 indicates that the eigenvalues $\eta_{j,N}$ of M_N are all uniformly $o(N^{\frac{1}{2}})$. But then

$$\text{sp}\,(M_N)^4 = \sum \eta_{j,N}^4 = o(N)\, \text{sp}\,(M_N)^2 = o(N^2). \tag{49}$$

This implies that

$$\sum_{j=1}^{N} |\mu_{j,N}|^3/N^{3/2} = o(1) \tag{50}$$

as $N \to \infty$. A simple computation indicates that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\, \text{sp}\,(R_N T_N)^2 = \sum_{j,k=-s}^{s} \tfrac{1}{4}(1 + \delta_{j,0})(1 + \delta_{k,0}) t_{|j|} t_{|k|} \frac{1}{2\pi} \\ \cdot \int_{-\pi}^{\pi} e^{i(j-k)\lambda} f^2(\lambda)\, d\lambda = \sum_{j,k=0}^{s} \tfrac{1}{2} t_j t_k c_{j,k}\end{aligned} \tag{51}$$

where $c_{j,k}$ is given by (44). The proof of asymptotic normality is complete by the correspondence between characteristic functions and distribution functions, for the limit of (45) as $N \to \infty$

$$\exp\left\{-\tfrac{1}{2} \sum_{j,k=0}^{s} c_{j,k} t_j t_k\right\}$$

is the characteristic function of $s + 1$ jointly normal random variables with mean zero and covariances $c_{j,k}$.

Some results related to Theorem 2 are discussed in [4]. Relatively simple examples can be constructed to show that one need no longer have asymptotic normality if $f \notin L^2$ (see [7]). In fact, given any p with $1 \leqq p < 2$ one can construct an $f \in L^p$ but not in L^2 such that one does not have asymptotic normality of the covariance estimates.

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