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A NOTE ON SUFFICIENCY IN COHERENT MODELS

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ABSTRACT. Partly of an expository nature, this article brings together a number of notions related to sufficiency in an abstract measure theoretic setting. The notion of a coherent statistical model, as introduced by Hasegawa and Perlman [6], is studied in some details. A few results are generalized and their earlier proofs simplified. Among other things, it is shown that a coherent model can be connected in the sense of Basu [2] if and only if no splitting set (Koehn and Thomas, [7]) exists.

KEY WORDS AND PHRASES. Sufficiency, coherent model, splitting set.

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1. INTRODUCTION.

This article is partly of an expository nature and is written mainly for its pedagogical interest. Given a mathematical model (X, A, P) for a statistical experiment, it is of some theoretical interest to inquire whether the family of sufficient sub-σ-fields (subfields) C has a minimum element in some sense. It is now known that if the model is coherent in the sense of Hasegawa and Perlman [6] then such a minimum sufficient subfield exists. The case of a coherent model is

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studied in some details in this article. Among other things it is demonstrated that for a coherent model the notion of connectedness (Basu, [2]) and that of the nonexistence of a splitting set (Koehn and Thomas, [7]) coincide.

2. NOTATION AND DEFINITIONS.

The basic statistical model is denoted by (X, A, P), where X is the sample space, A a σ-field of subsets of X, and $P = \{P_\theta: \theta \varepsilon \Theta\}$ a family of probability measures on A. By an A-measurable function we mean a measurable map of (X, A) into (R_1, B_1). Any sub-σ-field C of A will be referred to as a subfield. The function f is C-measurable if $f^{-1}B_1$ is contained in C. Given a family $\{A_t: t \varepsilon T\}$ of measurable sets, we write $\sigma\{A_t: t \varepsilon T\}$ for the subfield generated by the family of sets. Likewise, $\sigma\{f_t: t \varepsilon T\}$ will stand for the smallest subfield C such that each f_t is C-measurable. By $C \vee D$ we denote $\sigma(C \cup D)$, that is, the smallest subfield containing both C and D.

A set N in A is P-null if $P_\theta(N) = 0$ for all $\theta \varepsilon \Theta$. Let N denote the class of all P-null sets. For A, B ε A, the statement "A = B [P]" means that the symmetric difference A Δ B is P-null. Similarly, for any two A-measurable functions f and g, we write f = g [P] to indicate that $\{x: f(x) \neq g(x)\} \varepsilon N$.

The completion $\overline{C}$ of a subfield C is defined as $\overline{C} = C \vee \sigma(N)$. Accordingly, a subfield C is called complete if $N \subset C$. Let C and D be two subfields. We write $C \subset D[P]$ if $C \subset \overline{D}$, that is, corresponding to each set C in C, there exists a set D ε D such that C = D[P]. If $C \subset D[P]$ and $D \subset C[P]$, then $C = D[P]$. By a P-essentially $\overline{C}$-measurable function we mean a function f such that $f^{-1}B_1 \subset C\ [P]$, i.e., f is $\overline{C}$-measurable. This is also equivalent to the statement that there exists a C-measurable function g such that f = g [P]. An A-measurable function is P-integrable if $\int_X |f| d P_\theta < \infty$ for all $\theta \varepsilon \Theta$.

DEFINITION 1. (Halmos and Savage, [5]). The statistical model (X, A, P) is called <u>dominated</u> if every probability measure in P is absolutely continuous with respect to a fixed σ-finite measure λ on A.

In this case, we say that the family is dominated by λ and write $P << \lambda$.

DEFINITION 2. (Basu and Ghosh, [3]). The statistical model $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is called discrete if

(i) each P_θ is a discrete probability measure,

(ii) $\mathcal{A}$ is the class of all subsets of $\mathcal{X}$, and

(iii) for each $x \in \mathcal{X}$, there exists a $\theta \in \Theta$ such that $P_\theta(\{x\}) > 0$.

Condition (iii) implies that the empty set is the only $\mathcal{P}$-null set. A discrete model with a countable sample space $\mathcal{X}$ is clearly dominated. If $\mathcal{P}$ is countable, then the model is dominated. We assume that both $\mathcal{X}$ and $\mathcal{P}$ are uncountable. In this case, the model will be undominated. As we shall see in the next section, dominated and discrete models are particular cases of what Hasegawa and Perlman [6] called a coherent model.

Finally, let us state the notion of sufficiency as follows. A subfield $\mathcal{C}$ is sufficient with respect to the model $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ if, corresponding to each A in $\mathcal{A}$, there exists a $\mathcal{C}$-measurable function $I_A^\dagger$ such that $I_A^\dagger = E_\theta(I_A|\mathcal{C})$ $[P_\theta]$ for all $\theta \in \Theta$. A subfield $\mathcal{C}$ is pairwise sufficient with respect to $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ if, for each A in $\mathcal{A}$ and each pair θ_1, $\theta_2 \in \Theta$, there exists a $\mathcal{C}$-measurable function I_A^* such that $I_A^* = E_{\theta_i}(I_A^*|\mathcal{C})$ $[P_{\theta_i}]$ for $i = 1, 2$. (The function I_A^* may depend on θ_1 and θ_2.)

3. COHERENT STATISTICAL MODEL.

Let $\mathcal{F}$ denote the class of all measurable functions $f: \mathcal{X} \to [0, 1]$ and let

$$s = \{f_\theta : \; f_\theta \; \in \mathcal{F}, \; \theta \in \Theta\}$$

be a collection of members of $\mathcal{F}$ that is indexed by θ. Let $\mathcal{S} = \{s\}$ be the family of all such collections s.

DEFINITION 3. A member $\{f_\theta\}$ of $\mathcal{S}$ is said to be pairwise coherent if, for every pair θ_1, θ_2 in Θ, there exists a function f_{12} in $\mathcal{F}$ such that $f_{\theta_i} = f_{12}$ $[P_{\theta_i}]$ for $i = 1, 2$.

DEFINITION 4. A member $\{f_\theta\}$ of $\mathcal{S}$ is said to be countably coherent if, for every countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \ldots\}$ of Θ, there exists a function f_0 in $\mathcal{F}$ such that $f_{\theta_i} = f_0[P_{\theta_i}]$ for all $i = 1, 2, \ldots$.

DEFINITION 5. A member $\{f_\theta\}$ of $\mathcal{S}$ is said to be coherent if there exists a function f in $\mathcal{F}$ such that $f_\theta = f[P_\theta]$ for all $\theta \in \Theta$.

DEFINITION 6. (Hasegawa and Perlman, [6]). The statistical model $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is said to be coherent if every countably coherent member of $\mathcal{S}$ is coherent.

In the following lemma, we show that the notions of pairwise coherence and countable coherence do coincide.

LEMMA 1. If $\{f_\theta\}$ is a pairwise coherent member of $\mathcal{S}$, then it is countably coherent.

PROFF. Choose and fix a countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \ldots\}$ of Θ. For each pair θ_i, θ_j in Θ_0, there exists a function f_{ij} in $\mathcal{F}$ such that

$$f_{ij} = f_{\theta_i} [P_{\theta_i}] \text{ and } f_{ij} = f_{\theta_j} [P_{\theta_j}].$$

Let

$$g_i = \sup_j f_{ij},\ h_j = \inf_i f_{ij}, \text{ and } f = \inf_i \sup_j f_{ij}.$$

For each fixed i, the functions $f_{i1}, f_{i2}, \ldots$ are P_{θ_i}-equivalent to f_{θ_i}. The supremum of a countable number of P_{θ_i}-equivalent functions is also P_{θ_i}-equivalent to those functions. Thus, $g_i = f_{\theta_i} [P_{\theta_i}]$. Likewise, for each fixed j, we have $h_j = f_{\theta_j} [P_{\theta_j}]$. Therefore, it follows that $g_n = h_n = f_{\theta_n} [P_{\theta_n}]$ for $n = 1, 2, \ldots$. Observe that $h_n \leq f \leq g_n$ for all n. It then follows that

$$f = g_n = h_n = f_{\theta_n} [P_{\theta_n}] \text{ for } n = 1, 2, \ldots .$$

Hence, $\{f_\theta\}$ is countably coherent.

In general, the notions of pairwise coherence and coherence do not coincide as the following example shows.

EXAMPLE 1. (Pitcher, [10]). Let $\mathcal{X}$ be the unit interval [0, 1], $\mathcal{A}$ the σ-field of Borel subsets of $\mathcal{X}$, and $\mathcal{P}$ the family of all probability measures on $\mathcal{A}$ which are either degenerate at a single point of $\mathcal{X}$ or else are absolutely continuous with respect to the Lebesgue measure. For each P_θ in $\mathcal{P}$ and each $x \in \mathcal{X}$, define $f_\theta(x) = P_\theta(\{x\})$. Then $\{f_\theta\}$ is pairwise coherent, but not coherent.

In this model, no proper subfield of $\mathcal{A}$ can be sufficient. To see this, let $\mathcal{C}$ be an arbitrary sufficient subfield and let P_x be the probability measure degenerate at x. Then it follows from the sufficiency of $\mathcal{C}$ that, for all A in

A, there exists a C-measurable function $I_A^\dagger$ such that

$$I_A^\dagger(x) = \int_X I_A^\dagger \, dP_x = \int_X I_A dP_x = I_A(x).$$

Let $C = \{x: I_A^\dagger(x) = 1\}$. Then $A = C \in \mathcal{C}$. Hence, $\mathcal{C} = \mathcal{A}$.

We now show that the model $(X, \mathcal{A}, \mathcal{P})$ is coherent if it is either dominated or discrete.

LEMMA 2. (Hasegawa and Perlman, [6]). If $(X, \mathcal{A}, \mathcal{P})$ is dominated, then it is coherent.

PROOF. Since $\mathcal{P}$ is dominated, it follows that there is a countable subfamily $\Theta_0 = \{\theta_1, \theta_2, \ldots\}$ of Θ such that $\mathcal{P}_0 = \{P_\theta: \theta \in \Theta_0\}$ is equivalent to $\mathcal{P}$. Suppose that $\{f_\theta\}$ is countably coherent. Then there exists a function $f_0 \in \mathcal{F}$ such that $f_0 = f_{\theta_n}[P_{\theta_n}]$ for $n = 1, 2, \ldots$. We now show that $f_0 = f_\theta[P_\theta]$ for all $\theta \in \Theta$, so $\{f_\theta\}$ is coherent. For each $\theta \in \Theta$, consider $\Theta_1 = \Theta_0 \cup \{\theta\}$. Then there exists a function $f_1 \in \mathcal{F}$ such that $f_1 = f_{\theta_n}[P_{\theta_n}]$ for $n = 1, 2, \ldots$. Thus, $f_0 = f_1[\mathcal{P}_0]$ and so $f_0 = f_1[\mathcal{P}]$. Hence, $f_0 = f_1 = f_\theta[P_\theta]$ as required.

LEMMA 3. If $(X, \mathcal{A}, \mathcal{P})$ is discrete, then it is coherent.

PROOF. Let $\{f_\theta\}$ be pairwise coherent. We must show that it is also coherent. For each $\theta \in \Theta$, let $S_\theta = \{x \in X: P_\theta(\{x\}) > 0\}$ denote the countable support of P_θ. For each pair θ_1, θ_2 in Θ, there exists a function f_{12} in $\mathcal{F}$ such that $f_{12} = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Thus, we have

$$f_{\theta_i}(x) = f_{12}(x) \text{ for all } x \in S_{\theta_i},\ i = 1, 2. \tag{1}$$

Hence, $f_{\theta_1}(x) = f_{\theta_2}(x)$ for all $x \in S_{\theta_1} \cap S_{\theta_2}$. Now, choose and fix $x \in X$. Let $\Theta_x = \{\theta \in \Theta: x \in S_\theta\}$. Let θ_0 be a member of Θ_x. Note that $x \in S_{\theta_0} \cap S_\theta$ for all $\theta \in \Theta_x$. In view of (1), we have $f_{\theta_0}(x) = f_\theta(x)$ for all $\theta \in \Theta_x$, that is, $f_\theta(x)$ is constant in $\theta \in \Theta_x$ (for this prefixed x). Let c_x be the common value of f_θ for $\theta \in \Theta_x$, evaluated at x, that is, $f_\theta(x) = c_x$ for all $x \in X$. Since x is arbitrary, define a function f by $f(x) = c_x$ for all $x \in X$. Clearly, $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$ as required.

That the coherent case is not exhausted by the dominated and discrete cases is shown in the following example.

EXAMPLE 2. Let (X_1, A_1, P_1) be a non-discrete dominated model, let (X_2, A_2, P_2) be an undominated discrete model, where X_1 and X_2 are disjoint, $P_1 = \{P_\theta: \theta \in \Theta_1\}$, and $P_2 = \{P_\theta: \theta \in \Theta_2\}$. Let $X = X_1 \cup X_2$, $A = \{A \subset X: AX_i \in A_i$ for $i = 1, 2\}$, and extend P_1 and P_2 to A by defining

$$P_\theta(A) = P_\theta(AX_i) \text{ for all } \theta \in \Theta_i,\ i = 1, 2.$$

It follows from Lemmas 2 and 3 that both (X_1, A_1, P_1) and (X_2, A_2, P_2) are coherent. Now we claim that (X, A, P) is also coherent, where $P = P_1 \cup P_2 = \{P_\theta: \theta \in \Theta_1 \cup \Theta_2 = \Theta\}$. Suppose that $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, A, P). Each f_θ, being an A-measurable function, can be written as

$$f_\theta = \begin{cases} f_\theta^1 & \text{on } X_1 \\ \\ f_\theta^2 & \text{on } X_2, \end{cases}$$

where f_θ^i is A_i-measurable for $i = 1, 2$. Since $\{f_\theta^i: \theta \in \Theta_i\}$, being pairwise coherent with respect to (X_i, A_i, P_i), is coherent, it follows that there exists an A_i-measurable function $0 \leq f_i \leq 1$ such that

$$f_i = f_\theta^i\ [P_\theta] \text{ for all } \theta \in \Theta_i.$$

Define

$$f = \begin{cases} f_1 & \text{on } X_1 \\ \\ f_2 & \text{on } X_2 \end{cases}$$

Clearly, f is A-measurable. Observe that $f_\theta = f_\theta^i[P_\theta]$ for all $\theta \in \Theta_i$, $i = 1, 2$. Therefore, $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. However, it should be noted that (X, A, P) is neither dominated nor discrete.

4. SUFFICIENCY IN THE COHERENT CASE.

For each $\theta \in \Theta$, let N_θ denote the class of all P_θ-null sets, that is, $N_\theta = \{N \in A: P_\theta(N) = 0\}$. For each subfield C of A, define

$$\breve{C} = \bigcap_{\theta \in \Theta} [C \vee \sigma(N_\theta)].$$

Then $\breve{C}$ is also a subfield of A. It is easy to see that a function f is $\breve{C}$-measurable if and only if, for each $\theta \varepsilon \Theta$, there exists a $\mathcal{C}$-measurable function f_θ such that $f = f_\theta[P_\theta]$. Clearly, $\mathcal{C} \subset \overline{\mathcal{C}} \subset \tilde{\mathcal{C}}$.

Let (X, A, P) be a coherent model. Then, so is (X, A, P_0) for any subfamily P_0 of P. However, it is not true that $(X, \mathcal{C}, P)$ is coherent for every subfield $\mathcal{C}$ of A. Such an example is explicitly included in Pitcher's [9] example.

EXAMPLE 3. Let X be the real line and let $\mathcal{B}$ be the σ-field of all Borel subsets of X. Choose and fix a non-empty, non-Borel set E that excludes the origin but is symmetric about the origin, i.e., $E = -E = \{x: -x \varepsilon E\}$. Let Θ be also the real line, and define a family $P = \{P_\theta: \theta \varepsilon \Theta\}$ of probability measures as follows: If $\theta \varepsilon E$, then P_θ is the discrete measure allotting probabilities 1/2 and 1/2 to the two points $-\theta$ and θ. If $\theta \notin E$, then P_θ is degenerate at θ. We claim that $(X, \mathcal{B}, P)$ cannot be coherent. To see this, for each $\theta \varepsilon \Theta$ and each $x \varepsilon X$, define

$$f_\theta(x) = P_\theta(\{x\}).$$

Then $\{f_\theta: \theta \varepsilon \Theta\}$ is pairwise coherent with respect to $(X, \mathcal{B}, P)$. Suppose, on the contrary, that $\{f_\theta: \theta \varepsilon \Theta\}$ is coherent. Then there exists a $\mathcal{B}$-measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \varepsilon \Theta$. This implies that

$$f(x) = \begin{cases} 1/2 & \text{if } x \varepsilon E \\ 1 & \text{if } x \notin E, \end{cases}$$

which obviously contradicts the initial supposition that $E \notin \mathcal{B}$. However, if we consider the class A of all subsets of X, then (X, A, P) is a discrete model and hence is coherent.

In the following lemma, however, we show that if (X, A, P) is a coherent model, then so is $(X, \tilde{\mathcal{C}}, P)$ for any subfield $\mathcal{C}$ of A.

LEMMA 4. If (X, A, P) is coherent and $\mathcal{C}$ is a subfield of A, then $(X, \check{\mathcal{C}}, P)$ is coherent.

PROOF. Let $\{f_\theta: \theta \varepsilon \Theta\}$ be pairwise coherent with respect to $(X, \check{\mathcal{C}}, P)$. Then it is also pairwise coherent with respect to (X, A, P). Since (X, A, P) is coherent, there exists an A-measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \varepsilon \Theta$. Since each f_θ is $\tilde{\mathcal{C}}$-measurable, it is seen that f must be measurable with respect to $\tilde{\tilde{C}} = \tilde{C}$. This proves that $(X, \tilde{C}, P)$ is coherent.

Let $\tilde{P}$ denote the convex hull of P, that is, $\tilde{P} = \{Q: Q = \Sigma a_i P_{\theta_i}, a_i \geq 0, \Sigma a_i = 1, \theta_i \in \Theta\}$.

LEMMA 5. (X, A, P) is coherent if and only if $(X, A, \tilde{P})$ is coherent.

PROOF. The "only if" part is what needs to be proved. Let $\{f_Q: Q \in \tilde{P}\}$ be pairwise coherent with repsect to $(X, A, \tilde{P})$. Then the subset $\{f_\theta: \theta \in \Theta\}$ of $\{f_Q: Q \in \tilde{P}\}$ is pairwise coherent with respect to (X, A, P). Since (X, A, P) is coherent, there exists an A-measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. We now claim that $f = f_Q[Q]$ for all $Q \in \tilde{P}$. Choose and fix $Q \in \tilde{P}$. Then $Q = \Sigma a_i P_{\theta_i}$, where $\theta_i \in \Theta$, $a_i \geq 0$, and $\Sigma a_i = 1$. For each pair Q, P_{θ_i}, there exists an A-measurable function $0 \leq f_i \leq 1$ such that

$$f_i = f_Q[Q]$$

and

$$f_i = f_{\theta_i} = f[P_{\theta_i}].$$

Since $\{P_{\theta_i}: i = 1, 2, \ldots\} \equiv Q$, it follows that

$$f = f_i = f_Q[P_{\theta_i}] \text{ for } i = 1, 2, \ldots .$$

Hence, $f = f_Q[Q]$ as required.

LEMMA 6. Let C be a subfield of A such that (X, C, P) is coherent. If P is closed for countable convex combinations (i.e., $P = \tilde{P}$), then $\overline{C} = \tilde{C}$.

PROOF. Since $\overline{C} \subset \tilde{C}$, it suffices to show that $\tilde{C} \subset \overline{C}$. Let f be a $\tilde{C}$-measurable function such that $0 \leq f \leq 1$. Then, for each $\theta \in \Theta$, there exists a C-measurable function $0 \leq f_\theta \leq 1$ such that $f = f_\theta[P_\theta]$. For each pair $\theta_1, \theta_2 \in \Theta$, let $Q = (P_{\theta_1} + P_{\theta_2})/2$. Then $Q \in \tilde{P} = P$ and so $f = f_Q[Q]$. Since $\{P_{\theta_1}, P_{\theta_2}\} \equiv Q$, we have $f_Q = f = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Thus, we have shown that $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P). Therefore, there exists a C-measurable function $0 \leq f_0 \leq 1$ such that $f_0 = f_\theta[P_\theta]$ for all $\theta \in \Theta$, and hence $f = f_0[P]$. This shows that $\tilde{C} \subset \overline{C}$.

LEMMA 7. (X, C, P) is coherent if and only if $(X, \overline{C}, P)$ is coherent.

PROOF. Let us first prove the "only if" part. Let $\{f_\theta: \theta \in \Theta\}$ be pairwise coherent with respect to $(X, \overline{C}, P)$. For each $\theta \in \Theta$, there exists a C-measurable function $0 \leq g_\theta \leq 1$ such that $f_\theta = g_\theta[P_\theta]$. For each pair $\theta_1, \theta_2 \in \Theta$, there exists a $\overline{C}$-measurable function $0 \leq f_{12} \leq 1$ such that $f_{12} = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Since there is a C-measurable function $0 \leq g_{12} \leq 1$ such that $f_{12} = g_{12}[P]$, it follows that $g_{12} = g_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Hence, $\{g_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P). Since (X, C, P) is coherent, there exists a C-measurable function $0 \leq f \leq 1$ such that $f = g_\theta = f_\theta[P_\theta]$ for all $\theta \in \Theta$. Therefore, $(X, \overline{C}, P)$ is coherent.

To prove the "if" part, let $\{f_\theta: \theta \in \Theta\}$ be pairwise coherent with respect to (X, C, P). Then, it is also pairwise coherent with respect to $(X, \overline{C}, P)$. Thus, there exists a $\overline{C}$-measurable function $0 \leq \overline{f} \leq 1$ such that $\overline{f} = f_\theta[P_\theta]$ for all $\theta \in \Theta$. Since there is a C-measurable function $0 \leq f \leq 1$ such that $f = \overline{f}[P]$, we have $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$ as required.

Under the assumption of coherence, we now prove a number of results on sufficiency.

PROPOSITION 1. Suppose that (X, C, P) is coherent. If C is pairwise sufficient, then C is sufficient.

PROOF. Chose and fix $A \in A$. For each $\theta \in \Theta$, let $0 \leq f_\theta \leq 1$ be a version of $E_\theta(I_A|C)$. Since C is pairwise sufficient, for each pair $\theta_1, \theta_2 \in \Theta$, there exists a C-measurable function $0 \leq f_{12} \leq 1$ such that $f_{12} = f_{\theta_i}[P_{\theta_i}]$ for $i = 1, 2$. Thus, $\{f_\theta: \theta \in \Theta\}$ is pairwise coherent with respect to (X, C, P). Since (X, C, P) is coherent, there exists a C-measurable function $0 \leq f \leq 1$ such that $f = f_\theta[P_\theta]$ for all $\theta \in \Theta$. That is, C is sufficient.

COROLLARY 1. Suppose that (X, A, P) is coherent and $\overline{C} = \hat{C}$. If C is pairwise sufficient, then C is sufficient.

PROOF. Since (X, A, P) is coherent, it follows from Lemma 4 that $(X, \hat{C}, P)$ is coherent. Since $\overline{C} = \hat{C}$, $(X, \overline{C}, P)$ is coherent. In view of Lemma 7, (X, C, P) is coherent. Consequently, the result follows from Proposition 1.

CORALLARY 2. Let $\mathcal{C} \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all pairs $\theta_1, \theta_2 \in \Theta$. If $\mathcal{C}$ is pairwise sufficient and $(\mathcal{X}, \mathcal{D}, \mathcal{P})$ is coherent, then $\mathcal{D}$ is sufficient.

PROOF. Since $\mathcal{C}$ is pairwise sufficient and $\mathcal{C} \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all $\theta_1, \theta_2 \in \Theta$, it is easy to verify that $\mathcal{D}$ is also pairwise sufficient. Since $(\mathcal{X}, \mathcal{D}, \mathcal{P})$ is coherent, it follows from Proposition 1 that $\mathcal{D}$ is sufficient.

COROLLARY 3. Suppose that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is coherent and $\mathcal{C}$ is sufficient. If $\mathcal{C} \subset \mathcal{D}[P_{\theta_1}, P_{\theta_2}]$ for all $\theta_1, \theta_2 \in \Theta$ and $\overline{\mathcal{D}} = \tilde{\mathcal{D}}$, then $\mathcal{D}$ is sufficient.

PROOF. Since $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is coherent and $\overline{\mathcal{D}} = \tilde{\mathcal{D}}$, it follows from Lemmas 4 and 7 that $(\mathcal{X}, \mathcal{D}, \mathcal{P})$ is coherent. In view of Corollary 2, we therefore conclude that $\mathcal{D}$ is sufficient.

We now show that if $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is coherent, then so is $(\mathcal{X}, \mathcal{C}, \mathcal{P})$ for any sufficient subfield $\mathcal{C}$. To this end, we shall need the following lemma.

LEMMA 8. (Pitcher, 1965). If $\mathcal{C}$ is sufficient, then $\overline{\mathcal{C}} = \tilde{\mathcal{C}}$.

PROOF. Let $A \in \tilde{\mathcal{C}}$. Since $\mathcal{C}$ is sufficient, there exists a $\mathcal{C}$-measurable function $I_A^\dagger$ such that $I_A^\dagger = E_\theta(I_A|\mathcal{C})\ [P_\theta]$ for all $\theta \in \Theta$. Note that $I_A - I_A^\dagger$ is $\tilde{\mathcal{C}}$-measurable. Thus, for each $\theta \in \Theta$, there exists a $\mathcal{C}$-measurable function f_θ such that $I_A - I_A^\dagger = f_\theta[P_\theta]$ and so

$$\begin{aligned} \int_{\mathcal{X}}(I_A - I_A^\dagger)^2 dP_\theta &= \int_{\mathcal{X}}(I_A - I_A^\dagger) f_\theta dP_\theta \\ &= \int_{\mathcal{X}} E_\theta[(I_A - I_A^\dagger) f_\theta|\mathcal{C}] dP_\theta \\ &= \int_{\mathcal{X}} f_\theta E_\theta(I_A - I_A^\dagger|\mathcal{C}) dP \\ &= 0. \end{aligned}$$

Hence, $I_A = I_A^\dagger[P_\theta]$ for all $\theta \in \Theta$. Since $I_A^\dagger$ is $\mathcal{C}$-measurable, it follows that $A \in \overline{\mathcal{C}}$. Therefore, $\overline{\mathcal{C}} = \tilde{\mathcal{C}}$.

PROPOSITION 2. If $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is coherent and $\mathcal{C}$ is sufficient, then $(\mathcal{X}, \mathcal{C}, \mathcal{P})$ is coherent.

PROOF. Since $\mathcal{C}$ is sufficient, $\overline{\mathcal{C}} = \tilde{\mathcal{C}}$ in view of Lemma 8. Since $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is coherent, it follows from Lemma 4 and 7 that $(\mathcal{X}, \mathcal{C}, \mathcal{P})$ is coherent.

5. BASU'S THEOREM.

As before, $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is our basic model, where $\mathcal{P} = \{P_\theta: \theta \in \Theta\}$. Two probability measures P_{θ_1} and P_{θ_2} in $\mathcal{P}$ are said to be overlapping if, for any set A in $\mathcal{A}$, $P_{\theta_1}(A) = 1$ implies that $P_{\theta_2}(A) > 0$. We write $\theta_1 <=> \theta_2$ if P_{θ_1} and P_{θ_2} overlap. If there exists a finite number of parameter points $\theta_1, \theta_2, \ldots, \theta_k$ such that

$$\theta_1 <=> \theta_2 <=> \ldots <=> \theta_k <=> \theta',$$

then we say P_θ and $P_{\theta'}$ are connected. The family $\mathcal{P}$ is called connected if every pair of probability measures in the family are connected.

THEOREM 1. (Basu, [2]). If T is a sufficient statistic, $\mathcal{P}$ is connected, and V is a statistic which is independent of T for all $\theta \in \Theta$, then the distribution of V does not depend on θ.

A set A in $\mathcal{A}$ is called splitting set if there is a partition Θ_0, Θ_1, of Θ such that

$$P_\theta(A) = \begin{cases} 0 & \text{if } \theta \in \Theta_0 \\ 1 & \text{if } \theta \in \Theta_1 \end{cases}$$

The above theorem has been generalized by Koehn and Thomas [7] as follows:

THEOREM 2. Let T be a sufficient statistic. There exists a statistic V, independent of T for all $\theta \in \Theta$, whose distribtuion depends on θ if and only if there exists a splitting set.

We now demonstrate that the two notions of connectedness of $\mathcal{P}$ and the nonexistence of a splitting set are equivalent in the coherent case.

LEMMA 9. Let $(X, \mathcal{A}, \mathcal{P})$ be coherent. Then $\mathcal{P}$ is connected if and only if there exists no splitting set.

PROOF. Clearly, the connectedness of $\mathcal{P}$ implies the nonexistence of a splitting set. For each $\theta \in \Theta$, let A_θ be a set such that $P_\theta(A_\theta) = 1$ and $P_\theta(B) > 0$ if $\phi \neq B \subset A_\theta$. Suppose that $\mathcal{P}$ is not connected. Choose and fix $\theta_0 \in \Theta$. Let $\Theta_0 = \{\theta \in \Theta\colon P_\theta \text{ and } P_{\theta_0} \text{ are connected}\}$. Since $\mathcal{P}$ is not connected, $\Theta_1 = \Theta \setminus \Theta_0$ is not empty. For each $\theta \in \Theta$, let $f_\theta = I_{A_\theta}$. It is easy to show that $\{f_\theta\colon \theta \in \Theta\}$ is pairwise coherent. Since $(X, \mathcal{A}, \mathcal{P})$ is coherent, there exists an $\mathcal{A}$-measurable function $0 \leq f \leq 1$ such that $I_{A_\theta} = f[P_\theta]$ for all $\theta \in \Theta$. Let $A = \bigcup_{\theta\in\Theta_0} A_\theta$. Since A_θ is the support of P_θ, it is easily seen that

$$P_\theta(A) = P_\theta(A_\theta) = 1 \text{ for all } \theta \in \Theta_0.$$

Note that $A \bigcap A_\theta = \phi$ if $\theta \in \Theta_1$. Thus,

$$P_\theta(A) = 0 \text{ for all } \theta \in \Theta_1.$$

That nonexistence of a splitting set is weaker than the connectedness property is seen from the following example.

EXAMPLE 4. Let $\mathcal{P}$ be a family consisting of all two-point distributions and the standard normal distribution. Then $\mathcal{P}$ is not connected and this does not possess a splitting set.

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