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ON DEFINING LONG-RANGE DEPENDENCE

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Abstract

Long-range dependence has usually been defined in terms of covariance properties relevant only to second-order stationary processes. Here we provide new definitions, almost equivalent to the original ones in that domain of applicability, which are useful for processes which may not be second-order stationary, or indeed have infinite variances. The ready applicability of this formulation for categorizing the behaviour for various infinite variance models is shown.

LONG-RANGE DEPENDENCE; PERSISTENCE; FRACTAL MODELS; ALLEN VARIANCE; INFINITE VARIANCE MODELS; FRACTIONAL STABLE NOISE

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1. Introduction

The usual definition of long-range dependence is in terms of a second-order stationary process. Let $\{X_t, t=1, 2,\cdots\}$ be a zero mean second-order stationary process with covariances

$$\gamma_k = \text{cov}(X_t, X_{t+k})$$

and spectral density

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k \, e^{-ik\omega}, \qquad -\pi < \omega < \pi.$$

Then, long-range dependence holds if the spectral density is infinite at frequency $\omega = 0$, while short-range dependence holds if $f(0)$ is finite. We shall write LRD(SD) and SRD(SD) for these definitions, the SD referring to spectral density.

The LRD(SD) definition is inflexible in providing for extensions that allow for departure from stationarity and, indeed, infinite variances. Long-range dependence is a phenomenon which is widely observed in nature and it is characterized by sample paths displaying apparent trends and cycles, although these are ephemeral and are eventually replaced by equally convincing looking ones of quite different apparent direction and

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period. Models in which variances may be infinite are needed for the analysis of a variety of such phenomena (e.g. in finance [5], geology [6], [7], [8] and hydrology [11]). For example, in [11] (e.g. Table 1 therein) the one-dimensional distributions required are of type $\boldsymbol{P}(X>x)\sim Cx^{-\beta}$ for large values of x, where β ranges from 1.10 to 3.22, while the self-similarity parameter of the process is close to 2/3. In [6], [7], [8], the permeability and porosity of sedimentary rock formations are analyzed and strong evidence is provided for fractional Lévy stable models with index α, $1<\alpha<2$, and self-similarity parameter $H<1/\alpha$.

Now consider the block means process $\{X_t^{(m)}\}$ given by

$$X_t^{(m)}=(X_{tm-m+1}+\cdots+X_{tm})/m$$

and the Allen variance

$$V_m=\operatorname{var}(X_t^{(m)}).$$

Then, as has been noted by various authors (e.g. [2]), a nearly equivalent formulation of the dichotomy between SRD(SD) and LRD(SD) is given by $mV_m\to$ finite constant as $m\to\infty$ (SRD(AV)) and $mV_m\to\infty$ as $m\to\infty$ (LRD(AV)). Here (AV) refers to definitions with respect to the Allen variance. Indeed, we see that

$$mV_m=\sum_{|j|\leqq m}\left(1-\frac{|j|}{m}\right)\gamma_j,$$

so that, using the Kronecker lemma,

$$mV_m\to\sum_{j=-\infty}^{\infty}\gamma_j=\gamma_0+2\sum_{j=1}^{\infty}\gamma_j$$

as $m\to\infty$ if the right-hand side is finite. Also, if $\Sigma_{|j|\leqq m}\gamma_j$ tends to $\pm\infty$ as $m\to\infty$ then so does mV_m. However, it is not always assured that $mV_m\to\sigma^2$, say, $0<\sigma^2<\infty$, will imply that $f(0)$ is finite.

2. Extensions

It turns out that the LRD(AV) definition is readily amenable to extension whereas that for the spectral density lacks similar flexibility. To deal with the possibility of non-stationarity we define a new block means process

$$Y_t^{(m)}=\frac{\Sigma_{tm-m+1}^{tm}X_j}{\Sigma_{tm-m+1}^{tm}\boldsymbol{E}X_j^2},$$

for zero mean variables, and we shall say that the process is LRD(AV) if

$$\left(\sum_{j=tm-m+1}^{tm}\boldsymbol{E}X_j^2\right)\operatorname{var}Y_t^{(m)}\to\infty$$

as $m\to\infty$. This conveniently allows for departure from stationarity, certainly including the periodic effects associated with seasonality, and it reduces to the original LRD(AV) definition in the stationary case.

An example of a non-stationary process which is LRD(AV) is the random walk $\{X_j = \Sigma_{k=1}^{j} Z_k\}$, the increments $\{Z_k\}$ being independent and identically distributed with zero mean and finite variance. This seems an entirely appropriate interpretation as the random walk is well known to produce long leads and rare sign changes despite being 'fair'.

To deal with the situation in which variances may be infinite we suggest replacing the LRD(AV) definition by an empirical version. Suppose that the X_j are centered to have zero mean when the mean is finite. We shall say that the process is LRD(SAV), SAV denoting sample Allen variance, if

$$\frac{(\Sigma_{j=tm-m+1}^{tm} X_j)^2}{\Sigma_{j=tm-m+1}^{tm} X_j^2} \xrightarrow{\text{P}} \infty$$

as $m \to \infty$. The particular value of this criterion is that it provides a ready basis for statistical testing without *a priori* knowledge about the variances.

It should be noted that if LRD(SAV) holds for X_t stationary with finite variance then

$$\frac{(\Sigma_{j=tm-m+1}^{tm} X_j)^2}{\Sigma_{j=tm-m+1}^{tm} X_j^2} \xrightarrow{\text{P}} \infty \Leftrightarrow m(X_t^{(m)})^2 \xrightarrow{\text{P}} \infty \Rightarrow mV_m \to \infty,$$

so that LRD(AV) holds. Also, LRD(SAV) holds for the random walk example described above, as can be seen by applying Donsker's invariance principle for the central limit theorem (e.g. [1], pp. 137, 142) separately to numerator and denominator in the definition.

To underscore the value of the LRD(SAV) definition for dealing with infinite variance problems we illustrate by considering the collection of fractional stable noise models discussed by Samorodnitsky and Taqqu [10], Section 7.10. Properties such as covariation ([10], Section 2.7) and codifference ([10], Section 2.10), which are surrogates for the covariance when variances are not finite, do not provide a satisfactory basis for an extension of the standard definition of LRD, although an asymptotic analysis of the codifference for linear fractional stable noise is provided in [10]. In Section 7.10 the authors resort to analogy with the finite variance case to call the process LRD if $H > 1/\alpha$, $0 < \alpha < 2$. They note that the codifference does not even tend to zero as the lag increases for the other models described, namely increments of the real harmonizable stable noise process and the sub-Gaussian fractional stable noise. For fractional stable Lévy noise, Painter [7] refers to the cases $H > 1/\alpha$, $H < 1/\alpha$ as persistent and anti-persistent (descriptions typically associated with LRD and SRD respectively), again by analogy with the fractional Brownian case.

The models discussed in [10], which essentially form the natural paradigm models for long-range dependence in the infinite variance case are as follows.

(a) The linear fractional stable noise process has increments

$$X_j = \int_{-\infty}^{\infty} (a[(j+1-x)_+^{H-1/\alpha} - (j-x)_+^{H-1/\alpha}]$$
$$+ b[(j+1-x)_-^{H-1/\alpha} - (j-x)_-^{H-1/\alpha}])M(dx),$$

where a, b are real constants, $|a|+|b|>0$, $0<\alpha<2$, $0<H<1$, $H\neq 1/\alpha$ and M is an α-stable random measure with Lebesgue control measure. Here $a_+=\max(0,a)$, $a_-=-\min(0,a)$.

(b) The real harmonizable fractional stable noise process has increments

$$X_j=\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha j}\,\frac{\mathrm{e}^{\mathrm{i}x}-1}{\mathrm{i}x}\,|x|^{-H+1-1/\alpha}\tilde{M}(dx),$$

where $0<\alpha\leqq 2$, $0<H<1$ and where $\tilde{M}$ is a complete isotropic symmetric α-stable random measure with Lebesgue control measure.

(c) The sub-Gaussian fractional stable noise process has increments

$$X_j=A^{1/2}(B_H(j+1)-B_H(j)),$$

where B_H is fractional Brownian motion of index H, $0<H<1$, and A is an independent $\alpha/2$ stable random variable with Laplace transform $\boldsymbol{E}\mathrm{e}^{-tA}=\mathrm{e}^{-t^{\alpha/2}}$, $t>0$. We shall discuss the use of the LRD(SAV) criterion on these models.

In each case it is easily checked that we have self-similarity with parameter H, so that

$$\sum_{tm-m+1}^{tm} X_j \overset{\mathrm{d}}{=} m^H X_1. \tag{1}$$

We shall henceforth confine attention to processes X_j which have this self-similarity property.

Our basic result on LRD is given in the following theorem.

Theorem 1. *Let X_j be a stationary process for which block sums are self-similar with parameter H (i.e. satisfy* (1)) *and suppose that $\boldsymbol{E}|X_1|^p<\infty$ for some $0<p<2$. Then LRD(SAV) holds if $H>1/p$. In particular, if $\boldsymbol{P}(|X_1|>x)\sim Cx^{-\alpha}$ as $x\to\infty$, $0<\alpha<2$, then LRD(SAV) holds if $H>1/\alpha$.*

The result of this theorem follows from the use of the relationship (1) to deal with the numerator in the LRD(SAV) criterion and the stability theorem E of Loève [4], p. 387, to deal with the denominator. The stability theorem gives $n^{-2/p}\sum_{j=1}^n X_j^2 \overset{\text{a.s.}}{\to} 0$ as $n\to\infty$. The particular case is established by noting that $\boldsymbol{E}|X_1|^p<\infty$ for all $p<\alpha$.

To explore the LRD/SRD relationship further, we need to examine L_p variants of the SAV criterion. Write

$$F_m(p)=\frac{|\sum_{tm-m+1}^{tm} X_j|^p}{\sum_{tm-m+1}^{tm}|X_j|^p},\qquad 0<p\leqq 2.$$

Suppose that X_j is stationary, ergodic, satisfies the self-similarity condition (1) and $\boldsymbol{E}|X_1|^p<\infty$ for $0<p<\alpha<2$. Then, we have that $F_m(p)\overset{\text{a.s.}}{\to}\infty$ as $m\to\infty$ if $pH>1$, $p<\alpha$, i.e. $1/H<p<\alpha$. This follows from the use of self-similarity on the numerator and the ergodic theorem on the denominator of $F_m(p)$. The result corresponds to the case of long-range dependence as identified by the LRD(SAV) criterion, namely as holding for $H<1/\alpha$.

On the other hand, if $E|X_1|^p = \infty$ and if $\alpha < p < 2, pH < 1$, so that $\alpha < p < \min(2, 1/H)$, we have $F_m(p) \stackrel{\text{a.s.}}{\to} 0$ as $m \to \infty$. To obtain this result we again use self-similarity on the numerator of $F_m(p)$ and, for dealing with the denominator we note that, using the ergodic theorem with $I(\cdot)$ for the indicator function,

$$n^{-1} \sum_{j=1}^{n} |X_j|^p \geqq n^{-1} \sum_{j=1}^{n} |X_j|^p I(|X_j| \leqq k) \stackrel{\text{a.s.}}{\to} E|X_j|^p I(|X_j| \leqq k) \to \infty$$

as $k \to \infty$. The result here corresponds to short-range dependence.

Note that $\alpha < \min(2, 1/H)$ implies $H < 1/\alpha$ (since for $H \geqq 1/2$, $\alpha < \min(2, 1/H) = 1/H$, i.e. $H < 1/\alpha$, while for $H < 1/2$, $\alpha < \min(2, 1/H) = 2$ and hence $1/\alpha > 1/2 > H$). This clarifies the SRD ($H < 1/\alpha$) and LRD ($H > 1/\alpha$) dichotomy. We have chosen, however, not to make the $F_m(p)$, $p < 2$, the basis of a formal definition of LRD/SRD in order to avoid the uncertainty associated with p values which would generally be unknown in practice.

It should be observed that, for the sub-Gaussian fractional stable noise process mentioned above, the dichotomy is $H > 1/2$ (LRD) and $H < 1/2$ (SRD). The model here is not ergodic. A single history of the process appears to be from a fractional Brownian motion process and therefore to reflect, as this does, LRD or SRD according to $H > 1/2$ or $H < 1/2$. This can easily be seen via the AV or SAV criteria for LRD/SRD. Note that $H > 1/\alpha$ ensures $H > 1/2$ and the subtlety of ergodic or non-ergodic behaviour does not enter into the LRD consideration of Theorem 1.

Although the formal definition of LRD(SAV) can be used as a practical criterion for classifying behaviour, it must be used with care as it is based on the assumption of zero means, when these are finite. As an exercise we have investigated the time series of daily returns for the S&P 500 market index for the period 2 July 1962 to 31 December 1991, the data set having been supplied by a referee. Daily, weekly and monthly series were examined and there is no evidence of LRD. This assessment contrasts with that of Peters [9], p. 113, who used *R/S* analysis on the returns from the S&P 500 index for the period January 1950 to July 1988 and concluded that the value of the Hurst index H rises steadily from 0.59 for one-day increments to 0.78 for 30-day increments. *R/S* analysis is, however, well-known to be highly sensitive to departures from stationarity (e.g. Heyde and Dai [3]) of the kind that might be expected for financial series.

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