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AN ITERATED LOGARITHM RESULT FOR AUTOCORRELATIONS OF A STATIONARY LINEAR PROCESS

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Let $r(j)$ denote the jth autocorrelation based on a sample of N consecutive observations on a stationary linear stochastic process. Under mild regularity conditions on the process, an iterated logarithm result is given for the convergence of $r(j)$ as $N \to \infty$ to the corresponding process autocorrelation $\rho(j)$.

1. Introduction. In this paper we shall be concerned with a stationary linear stochastic process $\{x(n)\}$ which may be represented in the form

$$x(n) - \mu = \sum_{j=-\infty}^{\infty} \alpha(j)\varepsilon(n-j)\,, \qquad \sum_{j=-\infty}^{\infty} \alpha^2(j) < \infty\,,$$

where the $\varepsilon(n)$ are independent and identically distributed with mean zero and variance σ^2. Most of the standard inferential results for this process (e.g. Hannan [2]) are based on the autocorrelations

$$r(j) = \frac{\sum_{n=1}^{N-j} \{x(n) - \bar{x}\}\{x(n+j) - \bar{x}\}}{\sum_{n=1}^{N} \{x(n) - \bar{x}\}^2}\,, \qquad j \geqq 0\,,$$

$x(1), x(2), \cdots, x(N)$ being a sample of N consecutive observations on the process $\{x(n)\}$ and $\bar{x}$ denoting the sample mean. The prime result is that $r(j)$ converges almost surely (a.s.) to $\rho(j) = \gamma(j)/\gamma(0)$ where $\gamma(j)$ is given by

$$\gamma(j) = \sigma^2 \sum_{n=-\infty}^{\infty} \alpha(n)\alpha(n+j)\,.$$

The most common application of this theory would be in the estimation of parameters in an autoregressive or moving average model or a mixture of both models.

It is our object here to give the following iterated logarithm result which provides information on the rate of a.s. convergence of $r(j)$ to $\rho(j)$.

THEOREM. *Suppose that* $E\varepsilon^4(n) < \infty$ *and* $\sum_{n=1}^{\infty} [\sum_{|r| \geqq n} \alpha^2(r)]^{\frac{1}{2}} < \infty$. *Then, for any fixed* $j \geqq 1$,

$$r(j) = \rho(j) + A(j)\zeta_j(N)N^{-\frac{1}{2}}(2 \log\log N)^{\frac{1}{2}}$$

where $\zeta_j(N)$ *has* a.s. *as its set of limit points the interval* $[-1, 1]$, *in particular* $\limsup_{N\to\infty} \zeta_j(N) = 1$ a.s. *and* $\liminf_{N\to\infty} \zeta_j(N) = -1$ a.s., *while*

$$A(j) = [\{1 + 2\rho^2(j)\} \sum_{s=-\infty}^{\infty} \rho^2(s) + \sum_{s=-\infty}^{\infty} \rho(s+j)\{\rho(s-j) - 4\rho(j)\rho(s)\}]^{\frac{1}{2}}\,.$$

This result provides a detailed supplement to those obtained in [4] for the

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linear combination of autocorrelations which arises in estimation of parameters in an autoregressive model. Here we have an iterated logarithm result for each individual component.

2. Proof of theorem. This rests on use of an invariance principle for the law of the iterated logarithm for processes with stationary increments which was established in [5].

Our setting is that of a probability space $(\Omega, \mathscr{B}, P)$ with ergodic measure preserving transformation T such that $\varepsilon(n)\{\omega\} = \varepsilon(T^n\omega)$, $\omega \in \Omega$ (since the $\varepsilon(n)$ are independent and identically distributed and hence stationary and ergodic). Let $\mathscr{F}_n$ denote the σ-field generated by $\varepsilon(m)$, $m \leqq n$. Then, $\mathscr{F}_k = T^{-k}(\mathscr{F}_0)$ and the framework of Theorem 2 of [5] is applicable.

We shall first establish the following lemma which is of independent interest.

LEMMA. *Suppose that* $\sum_{n=1}^{\infty} [\sum_{|r|\geqq n} \alpha^2(r)]^{\frac{1}{2}} < \infty$. *Then,*

$$\bar{x} = \mu + \zeta(n) \sum_{j=-\infty}^{\infty} \gamma(j) N^{-\frac{1}{2}} (2 \log\log N)^{\frac{1}{2}}$$

where $\zeta(N)$ *has* a.s. *as its set of limit points the interval* $[-1, 1]$.

PROOF. Put $y(n) = x(n) - \mu$. We have

$$N^{-1} \operatorname{Var} \bar{x} = \sum_{i=-(N-1)}^{N-1} (1 - |i| N^{-1}) \operatorname{Cov}\{x(0), x(i)\}$$
$$\to \operatorname{Var} x(0) \sum_{i=-\infty}^{\infty} \rho(i) = \sum_{i=-\infty}^{\infty} \gamma(i)$$

as $N \to \infty$. Also,

$$E\{y(0) \mid \mathscr{F}_{-m}\} = \sum_{r=-\infty}^{\infty} \alpha(r) E\{\varepsilon(-r) \mid \mathscr{F}_{-m}\} = \sum_{r=m}^{\infty} \alpha(r)\varepsilon(-r)$$

and

$$\sum_{m=1}^{\infty} [E\{E(y(0) \mid \mathscr{F}_{-m})\}^2]^{\frac{1}{2}} = \sum_{m=1}^{\infty} [\sum_{r=m}^{\infty} \alpha^2(r)]^{\frac{1}{2}} < \infty .$$

Similarly,

$$\sum_{m=1}^{\infty} [E\{y(0) - E(y(0) \mid \mathscr{F}_m)\}^2]^{\frac{1}{2}} = \sum_{m=1}^{\infty} [\sum_{r=-\infty}^{-m-1} \alpha^2(r)]^{\frac{1}{2}} < \infty .$$

The result of the lemma then follows immediately from Theorem 2 of [5].

We now resume the proof of the theorem. Put $X_n = y(n)y(n+j) - \rho(j)y^2(n)$ and note that $\{X_n\}$ is a stationary sequence with zero mean and finite variance under the conditions of the theorem. Further, writing

$$c^*(j) = N^{-1} \sum_{n=1}^{N-j} y(n)y(n+j) ,$$

it is well known that (e.g. Hannan [1] page 40)

$$N \operatorname{Cov}\{c^*(k), c^*(k+j)\} \to \sum_{s=-\infty}^{\infty} \{\gamma(s)\gamma(s+j) + \gamma(s+j+k)\gamma(s-k)\}$$
$$+ k_4 \gamma(k)\gamma(k+j)$$

as $N \to \infty$ where k_4 is the fourth cumulant of $\varepsilon(n)$. Thus,

$$N^{-1} \operatorname{Var}(\sum_{n=1}^{N} X_n)$$
$$= N^{-1}E[N\{c^*(j) - Ec^*(j) - \rho(j)(c^*(0) - Ec^*(0))\}$$
$$+ \sum_{n=N-j+1}^{N} \{y(n)y(n+j) - Ey(n)y(n+j)\}]^2$$
$$\to \lim_{N\to\infty} NE[\{c^*(j) - Ec^*(j)\} - \rho(j)\{c^*(0) - Ec^*(0)\}]^2$$

$$= \lim_{N\to\infty} N \operatorname{Var} c^*(j) + \rho^2(j) \lim_{N\to\infty} N \operatorname{Var} c^*(0)$$
$$- 2\rho(j) \lim_{N\to\infty} N \operatorname{Cov}(c^*(j), c^*(0))$$
$$= \{1 + 2\rho^2(j)\} \textstyle\sum_{s=-\infty}^{\infty} \gamma^2(s) + \sum_{s=-\infty}^{\infty} \gamma(s+j)\{\gamma(s-j) - 4\rho(j)\gamma(s)\}$$
$$= \gamma^2(0) A^2(j) ,$$

using the fact that $\operatorname{Var}[\sum_{n=N-j+1}^{N} y(n)y(n+j)]$ is independent of N.

In order to apply Theorem 2 of [5] we need to show that

$$\textstyle\sum_{m=1}^{\infty} [E\{E(X_0 \mid \mathscr{F}_{-m})\}^2]^{\frac{1}{2}} < \infty \qquad \text{and} \qquad \sum_{m=1}^{\infty} [E\{X_0 - E(X_0 \mid \mathscr{F}_{m})\}^2]^{\frac{1}{2}} < \infty .$$

We have

$$E(X_0 \mid \mathscr{F}_{-m}) = \textstyle\sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \alpha(r)\alpha(s)$$
$$\times \{E(\varepsilon(-r)\varepsilon(j-s) \mid \mathscr{F}_{-m}) - \rho(j)E(\varepsilon(-r)\varepsilon(-s) \mid \mathscr{F}_{-m})\}$$
$$= \textstyle\sum_{r=m}^{\infty} \sum_{s\neq j+r, s=m+j}^{\infty} \alpha(r)\alpha(s)\varepsilon(-r)\varepsilon(j-s)$$
$$+ \textstyle\sum_{r=m}^{\infty} \alpha(r)\alpha(r+j)\{\varepsilon^2(-r) - \sigma^2\}$$
$$- \rho(j) \textstyle\sum_{r=m}^{\infty} \sum_{s\neq r, s=m}^{\infty} \alpha(r)\alpha(s)\varepsilon(-r)\varepsilon(-s)$$
$$- \rho(j) \textstyle\sum_{r=m}^{\infty} \alpha^2(r)\{\varepsilon^2(-r) - \sigma^2\} ,$$

and after some further algebra

$$E\{E(X_0 \mid \mathscr{F}_{-m})\}^2 = \{E\varepsilon^4(0) - 3\sigma^4\} \textstyle\sum_{r=m}^{\infty} \alpha^2(r)[\alpha(r+j) - \rho(j)\alpha(r)]^2 \tag{1}$$
$$+ \sigma^4 \textstyle\sum_{r=m}^{\infty} \alpha^2(r) \sum_{s=m}^{\infty} [\alpha(s+j) - \rho(j)\alpha(s)]^2$$
$$+ \sigma^4[\textstyle\sum_{r=m}^{\infty} \alpha(r)\{\alpha(r+j) - \rho(j)\alpha(r)\}]^2 .$$

It is then easily checked that $\sum_{m=1}^{\infty} [E\{E(X_0 \mid \mathscr{F}_{-m})\}^2]^{\frac{1}{2}} < \infty$ under the condition

$$\textstyle\sum_{m=1}^{\infty} \sum_{r=m}^{\infty} \alpha^2(r) = \sum_{r=1}^{\infty} r\alpha^2(r) < \infty .$$

Next,

$$E\{X_0 - E(X_0 \mid \mathscr{F}_{m})\}^2 = EX_0^2 - E\{E(X_0 \mid \mathscr{F}_{m})\}^2$$

and a routine calculation gives

$$EX_0^2 = \{E\varepsilon^4(0) - 3\sigma^4\} \textstyle\sum_{r=-\infty}^{\infty} \alpha^2(r)[\alpha(r+j) - \rho(j)\alpha(r)]^2 + \gamma^2(0) - \gamma^2(j) ,$$

so that using (1) (with $-m$ replaced by m) and after some algebra,

$$E\{X_0 - E(X_0 \mid \mathscr{F}_{m})\}^2 = \{E\varepsilon^4(0) - 3\sigma^4\} \textstyle\sum_{r=-\infty}^{-m-1} \alpha^2(r)[\alpha(r+j) - \rho(j)\alpha(r)]^2$$
$$- \sigma^4[\textstyle\sum_{r=-\infty}^{-m-1} \alpha(r)\{\alpha(r+j) - \rho(j)\alpha(r)\}]^2$$
$$+ \sigma^2\gamma(0) \textstyle\sum_{r=-\infty}^{-m-1} [\alpha(r+j) - \rho(j)\alpha(r)]^2$$
$$+ \sigma^2 \textstyle\sum_{r=-\infty}^{-m-1} \alpha^2(r)\{\gamma(0)(1 - \rho^2(j))$$
$$- \sigma^2 \textstyle\sum_{s=-\infty}^{-m-1} [\alpha(s+j) - \rho(j)\alpha(s)]^2\} .$$

Thus, $\sum_{m=1}^{\infty} [E\{X_0 - E(X_0 \mid \mathscr{F}_{m})\}^2]^{\frac{1}{2}} < \infty$ if the following four conditions hold:

(i) $\sum_{m=1}^{\infty} [\sum_{r=-\infty}^{-m-1} \alpha^2(r)\{\alpha(r+j) - \rho(j)\alpha(r)\}^2]^{\frac{1}{2}} < \infty$,
(ii) $\sum_{m=1}^{\infty} |\sum_{r=-\infty}^{-m-1} \alpha(r)\{\alpha(r+j) - \rho(j)\alpha(r)\}| < \infty$,
(iii) $\sum_{m=1}^{\infty} [\sum_{r=-\infty}^{-m-1} \{\alpha(r+j) - \rho(j)\alpha(r)\}^2]^{\frac{1}{2}} < \infty$,
(iv) $\sum_{m=1}^{\infty} [\sum_{r=-\infty}^{-m-1} \alpha^2(r)]^{\frac{1}{2}} < \infty$.

Furthermore, we readily find that (i) and (ii) are satisfied for $\sum_{m=1}^{\infty} \sum_{r=-\infty}^{-m-1} \alpha^2(r) < \infty$. On the other hand, (iii) and (iv) are satisfied when $\sum_{m=1}^{\infty} [\sum_{r=-\infty}^{-m-1} \alpha^2(r)]^{\frac{1}{2}} < \infty$.

Consequently, the conditions of Theorem 2 of [5] are satisfied and

$$(2N \log \log N)^{-\frac{1}{2}} \sum_{n=1}^{N} \{y(n)y(n+j) - \rho(j)y^2(n)\}$$

has a.s. as its set of limit points the interval $[-\gamma(0)A(j), \gamma(0)A(j)]$. But, the sequence $\{y(n)y(n+j)\}$ is stationary with finite variance so a simple application of the Borel–Cantelli lemma gives

$$\lim_{N\to\infty} (N \log \log N)^{-\frac{1}{2}} \sum_{n=N-j+1}^{N} y(n)y(n+j) = 0 \quad \text{a.s.}$$

and thus

$$N^{\frac{1}{2}}(2 \log \log N)^{-\frac{1}{2}}\{c^*(j) - \rho(j)c^*(0)\}$$

has a.s. as its set of limit points the interval $[-\gamma(0)A(j), \gamma(0)A(j)]$.

Now, if

$$c(j) = N^{-1} \sum_{n=1}^{N-j} \{x(n) - \bar{x}\}\{x(n+j) - \bar{x}\}, \qquad j \geqq 0,$$

we have

$$c^*(j) - c(j) = (\bar{x} - \mu)^2 - N^{-1}\{\sum_{n=N-j+1}^{N} x(n) + \sum_{n=1}^{j-1} x(n) - j(\bar{x} + \mu)\}(\bar{x} - \mu),$$

and with the aid of the lemma it is easily seen that for any fixed $k \geqq 0$,

$$\lim_{N\to\infty} N^{\frac{1}{2}} (\log \log N)^{-\frac{1}{2}}\{c^*(k) - c(k)\} = 0 \quad \text{a.s.}$$

The result of the theorem then follows upon noting that, from the ergodic theorem, $c(0) \to_{\text{a.s.}} \gamma(0)$ as $N \to \infty$.

3. Remarks.

1. Since the result of the theorem does not explicitly involve $E\varepsilon^4(n)$ it seems possible that it will continue to hold in the absence of $E\varepsilon^4(n) < \infty$. This is certainly the case with the corresponding central limit result.

2. It is easy to see that similar methods can be applied to investigate the rate of convergence of $c(j)$ to $\gamma(j)$. In this case Theorem 2 of [5] would be applied to the stationary ergodic sequence $\{Y_n\}$ where $Y_n = y(n)y(n+j) - \gamma(j)$. The result that emerges is, again under $E\varepsilon^4(n) < \infty$ and $\sum_{n=1}^{\infty} [\sum_{|r| \geqq n} \alpha^2(r)]^{\frac{1}{2}} < \infty$,

$$c(j) = \gamma(j) + B(j)\eta_j(N)N^{-\frac{1}{2}}(2 \log \log N)^{\frac{1}{2}}.$$

Here $\eta_j(N)$ has a.s. as its set of limit points the interval $[-1, 1]$, while $B(j)$ is given by

$$B(j) = [\sum_{s=-\infty}^{\infty} \{\gamma^2(s) + \gamma(s+j)\gamma(s-j)\} + k_4\gamma^2(j)]^{\frac{1}{2}},$$

k_4 being the fourth cumulant of $\varepsilon(n)$. In this case we explicitly need the finite fourth moment condition.

3. If a restriction is made to the case where the process $\{x(n)\}$ is purely nondeterministic ($\alpha(n) = 0$, $n < 0$), the assumption that the $\varepsilon(n)$ are independent and identically distributed can be replaced by a more general martingale condition along the lines of [3] and [4]. However, it is then necessary to make

certain unattractive assumptions on third and fourth moments and also to impose slightly stronger series conditions on the α's than that used here.

4. From an examination of the proof of the theorem it seems likely that, at least in the purely nondeterministic case, the result will continue to hold under only $\sum_{n=1}^{\infty} \sum_{|r| \geqq n} \alpha^2(r) = \sum_{r=-\infty}^{\infty} |r| \alpha^2(r) < \infty$. A convergence rate result for $\bar{x} - \mu$ considerably weaker than that provided by the lemma is all that is required. For example, $N^{\frac{1}{4}}(\bar{x} - \mu) \rightarrow_{\text{a.s.}} 0$ would suffice.

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