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INVARIANCE PRINCIPLES FOR THE LAW OF THE ITERATED LOGARITHM FOR MARTINGALES AND PROCESSES WITH STATIONARY INCREMENTS

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The main result in this paper is an invariance principle for the law of the iterated logarithm for square integrable martingales subject to fairly mild regularity conditions on the increments. When specialized to the case of identically distributed increments the result contains that of Stout [16] as well as the invariance principle for independent random variables of Strassen [17]. The martingale result is also used to obtain an invariance principle for the iterated logarithm law for a wide class of stationary ergodic sequences and a corollary is given which extends recent results of Oodaira and Yoshihara [10] on ϕ-mixing processes.

1. Introduction. This paper is basically divided into two parts. In the first part a general invariance principle for the law of the iterated logarithm for martingales is obtained. This result (Theorem 1) specializes, in the case of identically distributed increments, to a result (Corollary 2) which contains the invariance principle of Strassen [17] for sums of independent and identically distributed random variables. Corollary 2 also contains Stout's [16] result giving a classical Hartman–Wintner form of iterated logarithm law for stationary ergodic martingales and its minor generalization by Heyde [6]. Our approach to Theorem 1 follows essentially the pattern of that of Theorem 3 of [17] in that it is based on the use of a form of the Skorokhod representation theorem.

In the second part of the paper we make use of Corollary 2 to establish an invariance principle for the law of the iterated logarithm for a certain class of (strictly) stationary processes (Theorem 2). The basic idea is a representation for the increments of the stationary process in terms of the increments of a stationary martingale plus other terms whose sum is negligible under suitable norming. This idea is due to Gordin [4], who has used it to prove a central limit theorem for stationary ergodic sequences and it has been further developed, in that context, by Scott [14]. As a corollary to Theorem 2 we obtain Corollary 3 which extends results of Oodaira and Yoshihara [10], who have given an invariance principle for the law of the iterated logarithm for stationary ϕ-mixing sequences.

Previous work on iterated logarithm results for stationary processes has proceeded via classical methods. Our approach via the Skorokhod representation is

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more direct, gives ready access to the invariance principle, and avoids the necessity of obtaining detailed estimates of probabilities via, for example, rates of convergence to normality. In principle the result of Theorem 2 could be obtained by classical methods by paralleling the work of Chover [2] but a prerequisite would be estimates of rates of convergence to normality along the lines of those of Heyde [5] for sums of independent random variables.

2. Martingale results. Let $\{S_n, \mathscr{F}_n; n \geqq 0\}$ be a martingale on the probability space $(\Omega, \mathscr{A}, p)$ where $\mathscr{F}_0 = \{\phi, \Omega\}$ and $\mathscr{F}_n = \sigma$-field generated by $S_1, S_2, \cdots, S_n$ for $n > 0$. Let $S_0 = X_0 = 0$ almost surely (a.s.) and $S_n = \sum_{k=1}^{n} X_k$ for $n \geqq 1$. Further, let $s_n^2 = ES_n^2 < \infty$.

We consider the metric space (C, ρ) of all real-valued continuous functions on $[0, 1]$ with

$$\rho(x, y) = \sup_{0 \leqq t \leqq 1} |x(t) - y(t)| \qquad \text{for } x, y \in C .$$

Let K be the set of absolutely continuous $x \in C$ such that $x(0) = 0$ and

$$\int_0^1 [\dot{x}(t)]^2 \, dt \leqq 1$$

where $\dot{x}$ denotes the derivative of x determined almost everywhere with respect to Lebesgue measure. Define a real function $g(\cdot)$ on $[0, \infty)$ by

$$g(t) = \sup \{n : s_n^2 \leqq t\} .$$

Let ξ be a standard Wiener process (Brownian motion) on $[0, \infty)$ and define a sequence of real random functions $\xi_n(\cdot)$ on $[0, 1]$, for $n > g(e)$ by

$$\xi_n(t) = [\phi(s_n^2)]^{-1} \xi(s_n^2 t) \qquad t \in [0, 1]$$

where $\phi(\cdot)$ is a real function on $[e, \infty)$ given by

$$\phi(t) = (2t \log \log t)^{\frac{1}{2}} \qquad t \in [e, \infty) .$$

We also define a sequence of real random functions $\eta_n(\cdot)$ on $[0, 1]$, for $n > g(e)$, by

$$\eta_n(t) = [\phi(s_n^2)]^{-1} [S_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} X_{k+1}]$$
$$s_k^2 \leqq s_n^2 t \leqq s_{k+1}^2 , \qquad k = 0, 1, \cdots, n-1 .$$

THEOREM 1. *If* $s_n^2 \to \infty$ *and*

(1) $$\sum_{n=1}^{\infty} s_n^{-4} E\{X_n^4 I(|X_n| < \delta s_n)\} < \infty \qquad \textit{for some } \delta > 0 ,$$

(2) $$\sum_{n=1}^{\infty} s_n^{-1} E\{|X_n| I(|X_n| \geqq \varepsilon s_n)\} < \infty \qquad \textit{for all } \varepsilon > 0 ,$$

(3) $$s_n^{-2} \sum_{k=1}^{n} X_k^2 \to_{\text{a.s.}} 1 \qquad \textit{as } n \to \infty$$

hold, $I(\cdot)$ *denoting the indicator function, then* $\{\eta_n; n > g(e)\}$ *is relatively compact in C and the set of its limit points coincides with K.* ("$\to_{\text{a.s.}}$" means "converges a.s. to.")

COROLLARY 1. *If* $s_n^2 \to \infty$ *as* $n \to \infty$,

(4) $$\sum_{n=1}^{\infty} s_n^{-4} E X_n^4 < \infty$$

and (3) *holds then* $\{\eta_n; n > g(e)\}$ *is relatively compact in* C *and the set of its limit points coincides with* K.

COROLLARY 2. *If the* X_n *are identically distributed and satisfy* (3), *then* $\{\eta_n; n > e[E(X_1^2)]^{-1}\}$ *is relatively compact in* C *and the set of its limit points coincides with* K.

The condition (3) is, unfortunately, often difficult to check. It holds, for example, if $\{X_n\}$ is a stationary ergodic sequence.

The result of Theorem 1 appears to be new even in the case of independent random variables. Other iterated logarithm type results for martingales have been obtained by Stout [15] and Strassen [18] but they involve random norming and do not seem to be directly comparable with the present results.

3. **Martingale proofs.** The first step is to define a truncated martingale sequence $\{S_n^*, \mathscr{F}_n^*; n \geqq 0\}$. For $\delta > 0$, let $X_0^* = 0$ a.s., $\mathscr{F}_0^* = \{\phi, \Omega\}$ and for $k > 0$,

$$X_k^* = X_k I(|X_k| < \delta s_k) - E\{X_k I(|X_k| < \delta s_k) \,|\, \mathscr{F}_{k-1}^*\},$$

where $\mathscr{F}_k^* = \sigma$-field generated by $X_1^*, \cdots, X_k^*$. Then we let

$$S_n^* = \textstyle\sum_{k=0}^n X_k^* \qquad \text{for } n \geqq 0,$$

and $s_n^{*2} = ES_n^{*2} \leqq s_n^2 < \infty$. For $n > g(e)$ we define a sequence of real random functions $\eta_n^*(\cdot)$ on $[0, 1]$ by

$$\eta_n^*(t) = [\phi(s_n^2)]^{-1}[S_k^* + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} X_{k+1}^*]$$
$$s_k^2 \leqq t s_n^2 \leqq s_{k+1}^2,\ k = 0, 1, \cdots, n-1.$$

We first observe that

$$\text{(5)} \qquad \sup_{0 \leqq t \leqq 1} |\eta_n^*(t) - \eta_n(t)| \to_{\text{a.s.}} 0 \qquad \text{as } n \to \infty$$

under the conditions of Theorem 1. This will of course be true if

$$\text{(6)} \qquad s_n^{-1} \textstyle\sum_{k=1}^n |X_k| I(|X_k| \geqq \delta s_k) \to_{\text{a.s.}} 0 \qquad \text{as } n \to \infty$$

and

$$\text{(7)} \qquad s_n^{-1} \textstyle\sum_{k=1}^n |E\{X_k I(|X_k| < \delta s_k) \,|\, \mathscr{F}_{k-1}^*\}| \to_{\text{a.s.}} 0 \qquad \text{as } n \to \infty,$$

and these results follow straightforwardly from (2) via the Kronecker lemma and the monotone convergence theorem.

Now we introduce the Skorokhod representation. From Theorem 4.3 of Strassen [18] we may conclude that there exists a sequence $T_0 = 0, T_1, T_2, \cdots$ of nonnegative random variables such that

$$(\bar{S}_1, \bar{S}_2, \cdots, \bar{S}_n) \equiv (\xi(T_1), \xi(T_1 + T_2), \cdots, \xi(\textstyle\sum_{k=1}^n T_k))$$

is distributed as

$$\text{(8)} \qquad (S_1^*, S_2^*, \cdots, S_n^*)$$

for each $n \geqq 1$. Further, if $\bar{S}_0 = 0$ a.s., $\bar{\mathscr{F}}_0 = \{\phi, \Omega\}$, $\bar{\mathscr{F}}_n$ for $n \geqq 1$ is the σ-field generated by $\bar{S}_1, \cdots, \bar{S}_n$, $\mathscr{G}_0 = \mathscr{F}_0$, $\mathscr{G}_n$ is the σ-field generated by $\xi(t)$ for

$0 \leqq t \leqq \sum_{k=1}^n T_k$ for $n \geqq 1$ and $\bar{S}_n = \sum_{k=0}^n \bar{X}_k$ for $n \geqq 0$, then

$$E\{T_n \mid \mathscr{G}_{n-1}\} = E\{\bar{X}_n^2 \mid \bar{\mathscr{F}}_{n-1}\} \quad \text{a.s.} \qquad \text{for } n \geqq 1 . \tag{9}$$

Also, for $r > 1$,

$$E\{T_n^r \mid \mathscr{G}_{n-1}\} \leqq L_r E\{\bar{X}_n^{2r} \mid \bar{\mathscr{F}}_{n-1}\} \quad \text{a.s.} \qquad \text{for } n \geqq 1 , \tag{10}$$

where L_r is a constant depending only on r.

Define $\bar{\eta}_n(\cdot)$ on $[0, 1]$, for $n > g(e)$, by

$$\bar{\eta}_n(t) = [\phi(s_n^2)]^{-1}[\bar{S}_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1}\bar{X}_{k+1}]$$
$$s_k^2 \leqq s_n^2 t \leqq s_{k+1}^2,\ k = 0, 1, \cdots, n-1 ,$$

and $\eta(\cdot)$ on $[0, \infty)$ by

$$\eta(t) = \bar{S}_n + \bar{X}_{n+1}(t - s_n^2)(s_{n+1}^2 - s_n^2)^{-1} \qquad s_n^2 \leqq t \leqq s_{n+1}^2 .$$

Then

$$\eta(t) = \phi(s_n^2)\bar{\eta}_n(s_n^{-2}t) \qquad \text{for } t \in [0, s_n^2]$$

for any $n > g(e)$.

Our proof follows the work of Strassen in [17]. In fact, in view of (5) and (8) above and the results of [17], Theorem 1 will follow provided we can show that $s_{n+1}^2/s_n^2 \to 1$ and $\rho(\bar{\eta}_n, \xi_n) \to_{\text{a.s.}} 0$ and this last condition is implied by

$$P(\lim_{t\to\infty} [\phi(t)]^{-1} \sup_{\tau \leqq t} |\eta(\tau) - \xi(\tau)| = 0) = 1 . \tag{11}$$

In order to prove that $s_{n+1}^2/s_n^2 \to 1$, we first note that (2) gives, via an application of the Kronecker Lemma,

$$s_n^{-1} \sup_{k\leqq n} |X_k| I(|X_k| \geqq \varepsilon s_k) \to_{\text{a.s.}} 0 , \qquad \text{for any } \varepsilon > 0 ,$$

so that upon squaring we have *a fortiori*

$$s_n^{-2} \sup_{k\leqq n} X_k^2 I(|X_k| \geqq \varepsilon s_n) \to_{\text{a.s.}} 0 .$$

Thus

$$s_n^{-2} \sup_{k\leqq n} X_k^2 \leqq \varepsilon^2 + s_n^{-2} \sup_{k\leqq n} X_k^2 I(|X_k| \geqq \varepsilon s_n) ,$$

and since ε may be chosen arbitrarily small,

$$s_n^{-2} \sup_{k\leqq n} X_k^2 \to_{\text{a.s.}} 0 . \tag{12}$$

From (3) and Theorm 1 of Pratt [12] we then deduce that

$$\lim_{n\to\infty} s_n^{-2} \sup_{k\leqq n} EX_k^2 \leqq \lim_{n\to\infty} Es_n^{-2} \sup_{k\leqq n} X_k^2 = 0 , \tag{13}$$

and $s_{n+1}^2/s_n^2 \to 1$ follows.

We next proceed to the proof of (11) for which we need to know more about the behavior of the sequence $T_1, T_2, \cdots$.

LEMMA 1. *Under the conditions of Theorem* 1,

$$s_n^{-2} \sum_{k=1}^n E\{T_k \mid \mathscr{G}_{k-1}\} \to_{\text{a.s.}} 1 \qquad \textit{as } n \to \infty .$$

PROOF. Using (8) and (9), we need only show

$$s_n^{-2} \sum_{k=1}^n E\{X_k^{*2} \mid \mathscr{F}_{k-1}^*\} \to_{\text{a.s.}} 1 \qquad \text{as } n \to \infty . \tag{14}$$

Now

(15) $$s_n^{-2}\sum_{k=1}^n E\{X_k^{*2}\mid \mathscr{I}_{k-1}^*\} = s_n^{-2}\sum_{k=1}^n [E\{X_k^2 I(|X_k|<\delta s_k)\mid \mathscr{I}_{k-1}^*\} - (E\{X_k I(|X_k|<\delta s_k)\mid \mathscr{I}_{k-1}^*\})^2]$$

and

(16) $$s_n^{-2}\sum_{k=1}^n (E\{X_k I(|X_k|<\delta s_k)\mid \mathscr{I}_{k-1}^*\})^2 \leqq s_n^{-1}\delta\sum_{k=1}^n E\{|X_k| I(|X_k|\geqq \delta s_n)\mid \mathscr{I}_{k-1}^*\} \to_{\text{a.s.}} 0 \qquad \text{as } n\to\infty .$$

Also, using Proposition IV. 6.1, page 147 of Neveu [8],

(17) $$s_n^{-2}\sum_{k=1}^n (X_k^2 I(|X_k|<\delta s_k) - E\{X_k^2 I(|X_k|<\delta s_k)\mid \mathscr{I}_{k-1}^*\}) \to_{\text{a.s.}} 0 \qquad \text{as } n\to\infty$$

since

$$\sum_{n=1}^\infty s_n^{-4} E[X_n^2 I(|X_n|<\delta s_n) - E\{X_n^2 I(|X_n|<\delta s_n)\mid \mathscr{I}_{n-1}^*\}]^2 \leqq \sum_{n=1}^\infty s_n^{-4} E[X_n^4 I(|X_n|<\delta s_n)] < \infty$$

using (1). Consequently, from (15), (16) and (17), it suffices to show that

$$s_n^{-2}\sum_{k=1}^n X_k^2 I(|X_k|<\delta s_k) \to_{\text{a.s.}} 1 \qquad \text{as } n\to\infty$$

or equivalently, in view of (3),

(18) $$s_n^{-2}\sum_{k=1}^n X_k^2 I(|X_k|\geqq \delta s_k) \to_{\text{a.s.}} 0 .$$

However,

$$s_n^{-2}\sum_{k=1}^n X_k^2 I(|X_k|\geqq \delta s_k) \leqq (s_n^{-1}\sup_{k\leqq n}|X_k|) s_n^{-1}\sum_{k=1}^n |X_k| I(|X_k|\geqq \delta s_k)$$

and (18) follows from (2) and (12).

LEMMA 2. *Under the conditions of Theorem* 1,

$$s_n^{-2}\sum_{k=1}^n (T_k - E\{T_k\mid \mathscr{G}_{k-1}\}) \to_{\text{a.s.}} 0 \qquad \textit{as } n\to\infty .$$

PROOF. Again using Proposition IV. 6.1, page 147 of [8], the result will follow if

$$\sum_{n=1}^\infty s_n^{-4} E(T_n - E\{T_n\mid \mathscr{G}_{n-1}\})^2 < \infty .$$

Now

$$\begin{aligned}
&\sum_{n=1}^\infty s_n^{-4} E(T_n - E\{T_n\mid \mathscr{G}_{n-1}\})^2 \\
&\quad\leqq \sum_{n=1}^\infty s_n^{-4} E T_n^2 \\
&\quad\leqq L_2\sum_{n=1}^\infty s_n^{-4} E X_n^{*4} \qquad \text{(by (10))} \\
&\quad= L_2\sum_{n=1}^\infty s_n^{-4} E(X_n I(|X_n|<\delta s_n) - E\{X_n I(|X_n|<\delta s_n)\mid \mathscr{I}_{n-1}^*\})^4 \\
&\quad\leqq L_2\sum_{n=1}^\infty s_n^{-4}[E(X_n^4 I(|X_n|<\delta s_n)) \\
&\qquad\quad + 15\delta^3 s_n^3 E|E\{X_n I(|X_n|<\delta s_n)\mid \mathscr{I}_{n-1}^*\}|] \\
&\quad\leqq L_2\sum_{n=1}^\infty s_n^{-4}[E(X_n^4 I(|X_n|<\delta s_n)) + 15\delta^3 s_n^3 E(|X_n| I(|X_n|\geqq \delta s_n))] \\
&\quad< \infty
\end{aligned}$$

by (1) and (2).

We are now in a position to complete the proof of Theorem 1. Since $EX_k^2 < \infty$ for any k,

$$g(t) \to \infty \quad \text{as} \quad t \to \infty . \tag{19}$$

Then also

$$\begin{aligned} |ts_{g(t)}^{-2} - 1| &\leqq |s_{g(t)}^{-2} EX_{g(t)+1}^2| \\ &= |s_{g(t)+1}^{-2} EX_{g(t)+1}^2| |s_{g(t)+1}^2 s_{g(t)}^{-2}| \\ &\to 0 \qquad \text{as } t \to \infty \end{aligned} \tag{20}$$

by (13) and (19). Similarly,

$$ts_{g(t)+1}^{-2} \to 1 \qquad \text{as } t \to \infty . \tag{21}$$

From (19), (20), (21) and Lemmas 1 and 2 we obtain

$$t^{-1} \textstyle\sum_{k=1}^{g(t)} T_k \to_{\text{a.s.}} 1 \qquad \text{as } t \to \infty$$

and

$$t^{-1} \textstyle\sum_{k=1}^{g(t)+1} T_k \to_{\text{a.s.}} 1 \qquad \text{as } t \to \infty .$$

Finally, we note that

$$|\eta(t) - \xi(t)| \leqq \max \{|\xi(\textstyle\sum_{k=1}^{g(t)} T_k) - \xi(t)|, |\xi(\sum_{k=1}^{g(t)+1} T_k) - \xi(t)|\} .$$

From this point, it is obvious that Strassen's proof on page 217 of [17] may be followed to obtain (11), which completes the proof of Theorem 1.

Corollaries 1 and 2 follow easily from Theorem 1 using standard arguments of the kind of, for example, Neveu [8], pages 153, 154.

4. Stationary process results. We consider a probability space $(\Omega, \mathscr{A}, P)$ with an ergodic automorphism T. Let $L_2(P)$ be the Hilbert space of random variables with finite second moment, and write $(EX^2)^{\frac{1}{2}}$ as $||X||$. Define U on $L_2(P)$ by $UX(\omega) = X(T\omega)$ for $X \in L_2(P)$, $\omega \in \Omega$. Then U is a unitary operator. If $\mathscr{M}_0$ is a σ-field such that $\mathscr{M}_0 \subset \mathscr{A}$ and $\mathscr{M}_0 \subset T^{-1}(\mathscr{M}_0)$, define $\mathscr{M}_k = T^{-k}(\mathscr{M}_0)$, $\mathscr{M}_{-\infty} = \bigcap_{k=-\infty}^{\infty} \mathscr{M}_k$ and $\mathscr{M}_{+\infty} = \sigma$-field generated by $\bigcup_{k=-\infty}^{\infty} \mathscr{M}_k$.

We consider a particular random variable $X_0 \in L_2(P)$. Defining $X_k = U^k X_0$, we see that $\cdots, X_{-1}, X_0, X_1, \cdots$ is a doubly infinite stationary ergodic sequence. Put $X^{(0)} = 0$ and $X^{(n)} = \sum_{k=0}^{n-1} X_k$ for $n \geqq 1$ and define a sequence of random functions $\theta_n(\cdot)$ on $[0, 1]$ by

$$\theta_n(t) = [\phi(||X^{(n)}||^2)]^{-1}(X^{(k)} + (nt - k)X_k)$$
$$k \leqq nt \leqq k + 1, k = 0, 1, \cdots, n - 1 .$$

Also define

$$g = \sup (n : ||X^{(n)}||^2 \leqq e\} .$$

THEOREM 2. *If*

$$\textstyle\sum_{m=0}^{\infty} (||E\{X_0 \mid \mathscr{M}_{-m}\}|| + ||X_0 - E\{X_0 \mid \mathscr{M}_m\}||) < \infty \tag{22}$$

then

$$\lim_{n\to\infty} ||X^{(n)}||/n^{\frac{1}{2}} = \sigma$$

exists for $0 \leqq \sigma < \infty$. *If* $\sigma > 0$ *then* $g < \infty$, $\{\theta_n; n > g\}$ *is relatively compact and the set of its limit points coincides with* K.

PROOF. The key to the proof is a representation

$$X_0 = Y_0 + UZ_0 - Z_0 \tag{23}$$

where Y_0, Z_0 belong to $L_2(P)$ and, writing $Y_k = U^k Y_0$, $Y^{(0)} = 0$ and $Y^{(n)} = \sum_{k=1}^{n-1} Y_k$ for $k \geqq 1$, $\{Y^{(n)}, \mathscr{M}_{n-1}; n \geqq 0\}$ is a stationary ergodic zero-mean martingale. This representation, which is due to Gordin, holds under the condition (22); see Scott [14] for full details.

Setting $||Y_0|| = \sigma$ we have $||Y^{(n)}|| = \sigma n^{\frac{1}{2}}$. Also, from Minkowski's inequality and (23),

$$|||X^{(n)}|| - \sigma n^{\frac{1}{2}}|/n^{\frac{1}{2}} \leqq 2||Z_0||/n^{\frac{1}{2}} \to 0$$

since $Z_0 \in L_2(P)$ and thus

$$\lim_{n\to\infty} ||X^{(n)}||/n^{\frac{1}{2}} = \sigma .$$

We henceforth assume $\sigma > 0$. Then

$$\lim_{n\to\infty} ||X^{(n)}||/\sigma n^{\frac{1}{2}} = 1 \tag{24}$$

and $g < \infty$. Also, from (24) we readily deduce that

$$\phi(||X^{(n)}||^2)/\phi(n\sigma^2) \to 1 . \tag{25}$$

Now define a sequence of random functions $\zeta_n(\cdot)$ on $[0, 1]$ by

$$\zeta_n(t) = [\phi(n\sigma^2)]^{-1}[Y^{(k)} + (nt - k)Y_k]$$
$$k \leqq nt \leqq k + 1, k = 0, 1, \cdots, n - 1 .$$

Then by Corollary 2 we know that $\{\zeta_n; n > e/\sigma^2\}$ is relatively compact and the set of its limit points coincides with K. Thus, the proof will be complete if we can show that

$$\sup_{0\leqq t\leqq 1} |\theta_n(t) - \zeta_n(t)| \to_{\text{a.s.}} 0 . \tag{26}$$

We first consider

$$\sup_{0\leqq t\leqq 1} |\zeta_n(t) - \phi(n\sigma^2)[\phi(||X^{(n)}||^2)]^{-1}\zeta_n(t)| \tag{27}$$
$$\leqq |1 - \phi(n\sigma^2)[\phi(||X^{(n)}||^2)]^{-1}| \sup_{0\leqq t\leqq 1} |\zeta_n(t)| .$$

Now

$$\limsup_{n\to\infty} \sup_{0\leqq t\leqq 1} |\zeta_n(t)| = \sup_{x\in K} \sup_{0\leqq t\leqq 1} |x(t)| \quad \text{a.s.}$$
$$\leqq 1 \quad \text{a.s.}$$

From this and (25) and (27) we conclude that

$$\sup_{0\leqq t\leqq 1} |\zeta_n(t) - \phi(n\sigma^2)[\phi(||X^{(n)}||^2)]^{-1}\zeta_n(t)| \to_{\text{a.s.}} 0 . \tag{28}$$

Now, writing $Z_k = U^k Z_0$, consider

$$\sup_{0\leqq t\leqq 1} |\theta_n(t) - \phi(n\sigma^2)[\phi(||X^{(n)}||^2)]^{-1}\zeta_n(t)|$$
$$= \sup_{1\leqq k\leqq n} [\phi(||X^{(n)}||^2)]^{-1}|Z_k - Z_0| \qquad \text{(by (23))}$$
$$\leqq 2 \sup_{0\leqq k\leqq n} [\phi(||X^{(n)}||^2)]^{-1}|Z_k|$$

which goes almost surely to zero provided

$$\sup_{0\leq k\leq n}[\phi(n\sigma^2)]^{-1}|Z_k| = (n\sigma^2 \log\log n\sigma^2)^{-\frac{1}{2}} \sup_{0\leq k\leq n}|Z_k| \rightarrow_{\text{a.s.}} 0$$

(by (25)) and thus if

$$(n \log\log n)^{-\frac{1}{2}}|Z_n| \rightarrow_{\text{a.s.}} 0 .$$

But, this last result follows from the Borel–Cantelli Lemma since $Z_0 \in L_2(P)$. Thus

$$\sup_{0\leq t\leq 1}|\theta_n(t) - \phi(n\sigma^2)[\phi(||X^{(n)}||^2)]^{-1}\zeta_n(t)| \rightarrow_{\text{a.s.}} 0$$

and taking this with (28), (26) follows and the proof is complete.

As an application of Theorem 2, we shall see how it extends some recent results of Oodaira and Yoshihara [9], [10]. The papers [9], [10] in turn improve results of Iosifescu [7] and Reznik [13] as well as the (strictly) stationary case of results of Philipp [11].

Let $\{X_j, -\infty < j < \infty\}$ be a stationary process defined on a probability space $(\Omega, \mathscr{A}, P)$. Write $\mathscr{M}_k$ for the σ-field generated by $\cdots, X_{k-1}, X_k$; $\mathscr{M}_\infty$ for the σ-field generated by $\cdots, X_{-1}, X_0, X_1, \cdots$ and $\mathscr{M}_k^\infty$ for the σ-field generated by $X_k, X_{k+1}, \cdots$. The sequence $\{X_j]$ is called *ϕ-mixing* if for each $k(-\infty < k < \infty)$ and each $n(n \geqq 1)$,

$$\sup_{B\in\mathscr{M}_{k+n}^\infty} \text{ess sup}\,|P(B\,|\,\mathscr{M}_k) - P(B)| \leqq \phi_n \downarrow 0 \qquad \text{as } n \rightarrow \infty .$$

COROLLARY 3. *If $\{X_n\}$ is a stationary ϕ-mixing process with $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$ some $\delta \geqq 0$ and $\sum_{n=1}^\infty [\phi_n]^{(1+\delta)/(2+\delta)} < \infty$, then*

$$\lim_{n\rightarrow\infty} ||X^{(n)}||^2/n = EX_0^2 + 2\sum_{i=1}^\infty EX_0X_i = \sigma^2 \geqq 0 .$$

If $\sigma > 0$ then $g < \infty$ and $\{\theta_n; n > g\}$ is relatively compact and the set of its limit points coincides with K.

PROOF. First note that $\{X_n\}$ is stationary and *ϕ-mixing* and hence ergodic. Further, there exists an ergodic transformation T on $\mathscr{M}_\infty$ such that $X_k(\omega) = X_0(T^k\omega)$, $\omega \in \Omega$ (e.g. Doob [3], Chapter X) and, since $\mathscr{M}_0$ is the σ-field generated by $\cdots, X_{-1}, X_0$ our present situation fits within the framework of Theorem 2.

Next we need to check (22). This is a simple matter using Lemma 1, page 170 of Billingsley [1]. We have

$$\sum_{m=0}^\infty (||E\{X_0\,|\,\mathscr{M}_{-m}\}|| + ||X_0 - E\{X_0\,|\,\mathscr{M}_m\}||) = \sum_{m=0}^\infty ||E\{X_0\,|\,\mathscr{M}_{-m}\}||$$

and

$$\begin{aligned} E[E\{X_0\,|\,\mathscr{M}_{-m}\}]^2 &= E[X_0E\{X_0\,|\,\mathscr{M}_{-m}\}] \\ &\leqq 2\phi_m^{(1+\delta)/(2+\delta)}[E|E\{X_0\,|\,\mathscr{M}_{-m}\}|^{(2+\delta)/(1+\delta)}]^{(1+\delta)/(2+\delta)}[E|X_0|^{2+\delta}]^{1/(2+\delta)} \\ &\leqq 2\phi_m^{(1+\delta)/(2+\delta)}[E[E\{X_0\,|\,\mathscr{M}_{-m}\}]^2]^{\frac{1}{2}}[E|X_0|^{2+\delta}]^{1/(2+\delta)} \end{aligned}$$

so that

$$||E\{X_0\,|\,\mathscr{M}_{-m}\}|| \leqq 2\phi_m^{(1+\delta)/(2+\delta)}[E|X_0|^{2+\delta}]^{1/(2+\delta)}$$

and (22) holds under the conditions of the corollary.

Finally, from Lemma 3, page 172 of [1],

$$\lim_{n\to\infty} ||X^{(n)}||^2/n = EX_0^2 + 2 \sum_{i=1}^{\infty} E(X_0X_i)$$

and the corollary follows from Theorem 2.

In Theorem 1 of Oodaira and Yoshihara [10], an invariance principle for the law of the iterated logarithm is given under the conditions of Corollary 3 plus the requirements that $\int_{|x|\geqq N} x^2 dP(X_0 \leqq x) = O((\log N)^{-5})$ as $N \to \infty$ if $\delta = 0$ or $\phi_n = O(n^{-1-\varepsilon})$ for some $\varepsilon > (1+\delta)^{-1}$ if $\delta > 0$.

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