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ON LARGE DEVIATION PROBABILITIES IN THE CASE OF ATTRACTION TO A NON-NORMAL STABLE LAW

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SUMMARY. In this paper, the precise asymptotic behaviour of the large deviation probability is found in the case where the random variables are attracted to a non-normal stable law. This extends previous work of the same author in which only the order of magnitude of the large deviation probability was found.

1. Introduction

Let X_i, $i = 1, 2, 3, \ldots$ be a sequence of independent and identically distributed random variables with law $\mathcal{L}(X)$ and write $S_n = \sum_{i=1}^{n} X_i$. Let y_n, $n = 1, 2, 3, \ldots$ be a monotone sequence of positive numbers with $y_n \to \infty$ as $n \to \infty$ such that $y_n^{-1} S_n \xrightarrow{P} 0$. ($P$ stands for convergence in probability). The probabilities $\Pr(|S_n| > y_n)$, or either of the one sided components, are called large deviation probabilities.

In the papers Heyde, (1967a and 1967b), the asymptotic behaviour of $\Pr(|S_n| > y_n)$ has been investigated for various types X which are not attracted to the normal law. The results contained therein, however, have a shortcoming in that only the order of magnitude of the larger deviation probability is found and not the precise behaviour. It is the object of this paper to remedy this deficiency in the case of the (1967a) paper where the X_i belong to the domain of attraction of a non-normal stable law. The context of the (1967b) paper, namely with the X_i not belonging to the domain of partial attraction of the normal distribution, is manifestly too general to allow for a corresponding complete answer.

2. Details

The formulation of the problem follows the same lines as that of Heyde (1967a) to which we refer for background details. Thus, the X_i belong, with normalizing constants B_n, to the domain of attraction of the stable law with characteristic function

$$\exp\left\{-a|t|^{\alpha}\left(1+i\frac{t}{|t|}\,\frac{c_1-c_2}{c_1+c_2}\tan\frac{\pi\alpha}{2}\right)\right\}, \qquad \ldots \quad (1)$$

$0 < \alpha < 2, \alpha \neq 1$, $a > 0$, $c_1 > 0, c_2 > 0$. Certain rather exceptional cases (c_1 or c_2 zero, $\alpha = 1$) are excluded as a matter of convenience. If $\alpha > 1$, we suppose that the X_i are located so that $EX_i = 0$. Consequently, we have $B_n^{-1} S_n$ converging in law to a stable distribution with characteristic function (1) and $x_n^{-1} B_n^{-1} S_n \xrightarrow{P} 0$ if x_n, $n = 1, 2, 3, \ldots$ is a monotone sequence of positive numbers with $x_n \to \infty$ as $n \to \infty$.

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Under the above conditions we may write for $x > 0$, using Gnedenko and Kolmogorov (1954, Theorem 2, 175)

$$\Pr(|X| > x) = \frac{M(x)}{x^{\alpha}}, \qquad \ldots \quad (2)$$

where $M(x)$ is a slowly varying function which satisfies the relation

$$\frac{nM(B_n)}{B_n^{\alpha}} \to (c_1+c_2) \quad \text{as } n \to \infty. \qquad \ldots \quad (3)$$

We shall establish the following theorem concerning the two sided large deviation probability. Corresponding one sided versions can probably be obtained in much the same manner but there are some technical difficulties to overcome.

Theorem : *Let $\{x_n\}$ be a monotone sequence of positive numbers with $x_n \to \infty$ as $n \to \infty$. Then,*

$$\lim_{n \to \infty} \frac{Pr(|S_n| > x_n B_n)}{n\, Pr(|X| > x_n B_n)} = 1, \qquad \ldots \quad (4)$$

or equivalently,

$$\lim_{n \to \infty} \frac{Pr(|S_n| > x_n B_n)}{Pr\left(\max\limits_{1 \leqslant k \leqslant n} |X_k| > x_n B_n\right)} = 1. \qquad \ldots \quad (5)$$

This result extends that of Heyde (1967a) where it is shown, in essence, that

$$0 < \varliminf_{n \to \infty} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} \leqslant \varlimsup_{n \to \infty} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} < \infty.$$

Proof : Firstly we note that the analysis of the theorem of Heyde (1967b) can be used and we deduce immediately that

$$\varliminf_{n \to \infty} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} \geqslant 1.$$

It therefore just remains to show that

$$\varlimsup_{n \to \infty} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} \leqslant 1$$

and the result (4) will follow. The equivalence of the forms (4) and (5) is deduced precisely as in Heyde (1967b) and will not be repeated here.

For $1 \leqslant k \leqslant n$ and $n = 1, 2, 3, \ldots$, define random variables

$$X_{kn} = \begin{cases} X_k & \text{if } |X_k| \leqslant z_n B_n \\ 0 & \text{if } |X_k| > z_n B_n, \end{cases}$$

where $z_n = x_n^\gamma$, $1 > \gamma > 1/2$, and write $S_{nn} = \sum_{k=1}^{n} X_{kn}$. Take $0 < \varepsilon < 1$ and define the events

$$E_n = \{|X_k| > (1-\varepsilon)x_n B_n \text{ for at least one } k \leqslant n\},$$

$$F_n = \{|X_k| > z_n B_n \qquad \text{ for at least two } k\text{'s} \leqslant n\},$$

$$G_n = \{|S_{nn}| > \varepsilon x_n B_n\}.$$

Then, we have

$$\{|S_n| > x_n B_n\} \subset E_n U F_n U G_n,$$

so that

$$\begin{aligned}\Pr(|S_n| > x_n B_n) &\leqslant \Pr(E_n)+\Pr(F_n)+\Pr(G_n)\\ &\leqslant n \Pr(|X| > (1-\varepsilon)x_n B_n)+n^2[\Pr(|X| > z_n B_n)]^2\\ &\qquad +\Pr(|S_{nn}| > \varepsilon x_n B_n). \quad \dots \quad (6)\end{aligned}$$

We shall deal separately with each of the three terms on the right hand side of (6).

From (2), we have

$$\frac{n\Pr(|X| > (1-\varepsilon)x_n B_n)}{n \Pr(|X| > x_n B_n)} = \frac{M[(1-\varepsilon)x_n B_n]}{M[x_n B_n]} \cdot \frac{x_n^\alpha B_n^\alpha}{x_n^\alpha B_n^\alpha (1-\varepsilon)^\alpha} \to \frac{1}{(1-\varepsilon)^\alpha} \text{ as } n \to \infty. \quad \dots \quad (7)$$

Also,

$$\begin{aligned}\frac{n^2[\Pr(|X| > z_n B_n)]^2}{n \Pr(|X| > x_n B_n)} &= \frac{nM(B_n)}{B_n^\alpha} \cdot \frac{x_n^\alpha}{z_n^{2\alpha}} \frac{[M(z_n B_n)]^2}{M(B_n)M(x_n B_n)}\\ &\leqslant C \frac{x_n^\alpha}{z_n^{2\alpha}} \left(\frac{x_n}{z_n}\right)^{\eta_1} z_n^{\eta_2}\\ &= C\, x_n^{\alpha+\eta_1-\gamma(2\alpha+\eta_1-\eta_2)}\end{aligned}$$

for n sufficiently large using estimates obtained in Heyde (1967a), where C is a positive constant and η_1, η_2 are arbitrarily small and positive. Since $1 > \gamma > 1/2$, we can choose η_1 and η_2 such that

$$\alpha+\eta_1-\gamma(2\alpha+\eta_1-\eta_2) < 0,$$

and then

$$\frac{n[\Pr(|X| > z_n B_n)]^2}{\Pr(|X| > x_n B_n)} \to 0 \text{ as } n \to \infty. \quad \dots \quad (8)$$

Finally, it remains to consider $\Pr(|S_{nn}| > \varepsilon x_n B_n)$. We have, using Chebyshev's inequality,

$$\begin{aligned}\Pr(|S_{nn}| > \varepsilon x_n B_n) &\leqslant \varepsilon^{-2} x_n^{-2} B_n^{-2} E(S_{nn}^2)\\ &= \varepsilon^{-2} x_n^{-2} B_n^{-2} [n EX_{kn}^2 + \tfrac{1}{2} n(n-1)(EX_{kn})^2]. \quad \dots \quad (9)\end{aligned}$$

Then,

$$\frac{EX_{kn}^2}{x_n^2B_n^2 \Pr(|X| > x_nB_n)} = \frac{\int\limits_{|x| \leqslant z_n B_n} x^2 d\Pr(X \leqslant x)}{x_n^2B_n^2 \Pr(|X| > x_nB_n)}$$

$$\leqslant \frac{2\int\limits_0^{z_nB_n} x\Pr(|X| > x)dx}{x_n^2B_n^2 \Pr(|X| > x_nB_n)}$$

$$= \frac{2\int\limits_0^{z_nB_n} x\Pr(|X| > x)dx}{z_n^2B_n^2 \Pr(|X| > z_nB_n)} \cdot \frac{z_n^2 \Pr(|X| > z_nB_n)}{x_n^2\Pr(|X| > x_nB_n)}$$

where we have used integration by parts to obtain the second last line. Furthermore, using standard properties of slowly varying functions (Feller, 1966, 273), we have

$$\frac{2\int\limits_0^{z_nB_n} x\Pr(|X| > x)dx}{z_n^2B_n^2 \Pr(|X| > z_nB_n)} \to \frac{2}{2-\alpha} \text{ as } n \to \infty,$$

while

$$\frac{z_n^2 \Pr(|X| > z_nB_n)}{x_n^2 \Pr(|X| > x_nB_n)} = \frac{z_n^{2-\alpha}}{x_n^{2-\alpha}} \frac{M(z_nB_n)}{M(x_nB_n)} \to 0 \text{ as } n \to \infty,$$

and therefore,

$$\frac{EX_{kn}^2}{x_n^2B_n^2 \Pr(|X| > x_nB_n)} \to 0 \text{ as } n \to \infty. \quad \ldots (10)$$

Also, if $\alpha < 1$,

$$|EX_{kn}| \leqslant E|X_{kn}| = \int\limits_{|x| \leqslant z_nB_n} |x| \, d\Pr(X \leqslant x)$$

$$\leqslant \int\limits_0^{z_nB_n} \Pr(|X| > x)dx,$$

using integration by parts, so that

$$\frac{n(EX_{kn})^2}{x_n^2B_n^2\Pr(|X| > x_nB_n)} \leqslant \left[\frac{\int\limits_0^{z_nB_n} \Pr(|X| > x)dx}{z_nB_n \Pr(|X| > z_nB_n)}\right]^2 \frac{nz_n^2[\Pr(|X| > z_nB_n)]^2}{x_n^2 \Pr(|X| > x_nB_n)}.$$

In this case,

$$\frac{\int\limits_0^{z_nB_n} \Pr(|X| > x)dx}{z_nB_n \Pr(|X| > z_nB_n)} \to \frac{1}{1-\alpha},$$

while

$$\frac{nz_n^2[\Pr(|X| > z_nB_n)]^2}{x_n^2 \Pr(|X| > x_nB_n)} < \frac{n[\Pr(|X| > z_nB_n)]^2}{\Pr(|X| > x_nB_n)} \to 0 \text{ as } n \to \infty,$$

according to (8) above, so that for $\alpha < 1.$,

$$\frac{n(EX_{kn})^2}{x_n^2 B_n^2 \Pr(|X| > x_n B_n)} \to 0 \text{ as } n \to \infty. \quad \ldots \quad (11)$$

If, on the other hand, $1 < \alpha < 2$, we have since $EX = 0$,

$$|EX_{kn}| = \Big| \int_{|x| \leqslant z_n B_n} x d \Pr(X \leqslant x) \Big|$$

$$= \Big| \int_{|x| > z_n B_n} x d \Pr(X \leqslant x) \Big|$$

$$\leqslant \int_{|x| > z_n B_n} |x| d \Pr(X \leqslant x)$$

$$= - \int_{z_n B_n}^{\infty} x d \Pr(|X| > x)$$

$$= \int_{z_n B_n}^{\infty} \Pr(|X| > x) dx + z_n B_n \Pr(|X| > z_n B_n),$$

and

$$\frac{\int_{z_n B_n}^{\infty} \Pr(|X| > x) dx}{z_n B_n \Pr(|X| > z_n B_n)} \to \frac{1}{\alpha - 1},$$

so that

$$\frac{n(EX_{kn})^2}{x_n^2 B_n^2 \Pr(|X| > x_n B_n)} \leqslant \left[\frac{\int_{z_n B_n}^{\infty} \Pr(|X| > x) dx}{z_n B_n \Pr(|X| > z_n B_n)} + 1 \right]^2 \cdot \frac{n z_n^2 [\Pr(|X| > z_n B_n)]^2}{x_n^2 \Pr(|X| > x_n B_n)}$$

$$\to 0 \text{ as } n \to \infty, \quad \ldots \quad (12)$$

again using (8). Then, from (9), (10), (11), and (12), we see that

$$\frac{\Pr(|S_{nn}| > \varepsilon x_n B_n)}{n \Pr(|X| > x_n B_n)} \to 0 \text{ as } n \to \infty. \quad \ldots \quad (13)$$

Finally, using (7), (8) and (13) in (6), we see that

$$\overline{\lim_{n \to \infty}} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} \leqslant \frac{1}{(1-\varepsilon)^2}$$

and since ε, $0 < \varepsilon < 1$ is arbitrary,

$$\overline{\lim_{n \to \infty}} \frac{\Pr(|S_n| > x_n B_n)}{n \Pr(|X| > x_n B_n)} \leqslant 1.$$

This furnishes the proof of the theorem.

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