



# Correction to: A Course in Multivariable Calculus and Analysis

## Correction to:

**S. R. Ghorpade, B. V. Limaye, *A Course in Multivariable Calculus and Analysis*, Undergraduate Texts in Mathematics,**  
<https://doi.org/10.1007/978-1-4419-1621-1>

After initial publication of the book, various errors were identified that needed correction. All corrections listed below have been updated within the current version. Note that the correction to Chapter 7 starting p. 422 changes the pagination of all subsequent material, starting from “Absolute Convergence and Conditional Convergence”, previously p. 425, now p. 427.

In the following, **p. i, +j** means the  $j$ th line from the top on page  $i$ , whereas **p. i, -j** means the  $j$ th line from the bottom on page  $i$ . Here page  $i$  refers to the  $i$ th page in the original edition of the book published in 2010. The text to be changed and the corresponding corrected version usually appear in quotes.

## Chapter 1

**p. 5, +16:** Change “least upper bound” to “greatest lower bound”

**p. 10, -12:** Change “ $f(x, y) \geq 0$ ” to “ $f(x, y) > 0$ ”

**p. 34, -6:** Change “handled them with” to “handled with”

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The updated online version for these chapters can be found at

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## Chapter 2

p. 45, +2: Change “If” to “Let  $a_n \rightarrow a$ . If”

## Chapter 3

p. 88, +5: Change “ $x_0 \in [c, d]$ ” to “ $x_0 \in [a, b]$ ”

p. 97, -13: Change “ $u : D \rightarrow \mathbb{R}^2$  are” to “ $u : D \rightarrow \mathbb{R}$  are”

p. 97, -8: Change “ $f$ ” to “ $g$ ”

p. 118, -10: Change “ $f(x, y)$ ” to “ $f(x_1, y_1)$ ”

p. 118, -10: Change at two places “ $k \frac{\partial}{\partial x}$ ” to “ $k \frac{\partial}{\partial y}$ ”

p. 154, +9: Change “ $\sqrt{h^2 + k^2}$ ” to “ $|h| + |k|$ ”

p. 156, -2: Change “ $(y - b)(z - c)$ ” to “ $(y - c)(z - p)$ ”

## Chapter 4

p. 166, +22: Change “ $\mu h(x, y, z)$ ” to “ $\mu \nabla h(x, y, z)$ ”

## Chapter 5

p. 213, +11: Change “ $R$ ” to “ $\mathbb{R}$ ”

p. 213, -6: Change “5.19” to “5.23”

p. 219, -6: Change “ $0 \leq a < b$  and  $0 \leq c < d$ ” to “ $0 < a < b$  and  $0 < c < d$ ”

p. 220, -13: Change “Moreover, by Proposition 5.28, we have” to “Moreover,”

p. 224, +3: Change “ $[a, b] \times [c, d]$ ” to “ $[a, b] \times [c, d]$ ”

p. 238, -19: Change “iterated integral” to “integral”

p. 245, +14: Change “and  $D_1 \cap D_2$  are” to “and  $D_1 \cap D_2$  is”

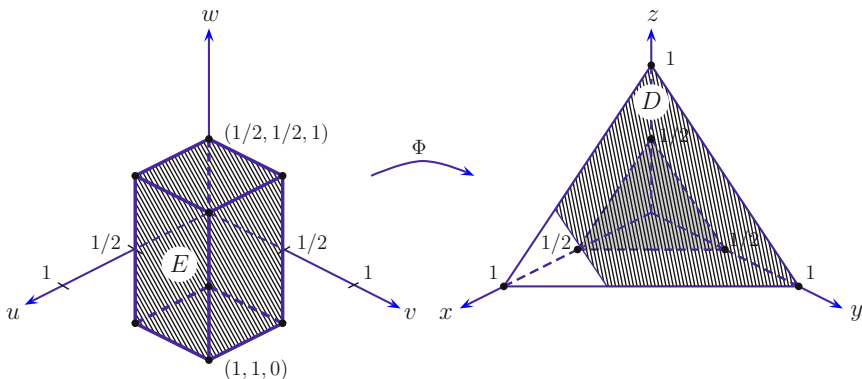
p. 268, +18: Change “Exercise 43” to “Exercise 43 of Chapter 3”

p. 272, -10: Change “ $d(y, z)$ ” to “ $d(x, y)$ ”

p. 272, -9: Change “ $f(x, y, z)$ ” to “ $\int_{\phi_1(x, y)}^{\phi_2(x, y)} f(x, y, z) dz$ ”

p. 275, +16: Change “ $f$ ” to “ $f \circ \Phi$ ”

p. 278: Change Figure 5.26 to the revised figure provided below.



**p. 282, -3:** Change “ $x \leq y$ ” to “ $y \leq x$ ”

**p. 285, +19:** Change “49/192” to “49/576”

## Chapter 6

**p. 302, -16:** Change “ $D$ ,  $yz$ -plane” to “ $D$  by the  $yz$ -plane”

**p. 303, +11:** Change “ $[-\pi, \pi] \times [f_1(x), f_2(x)]$ ” to “ $[f_1(x), f_2(x)] \times [-\pi, \pi]$ ”

## Chapter 7

**p. 422, -6 to p. 425, -11:** Change the entire text from the statement of the Integral Test until the beginning of the next subsection to the revised text provided below.

**Proposition 7.57 (Integral Test).** *Let  $f : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$  be a nonnegative monotonically decreasing function. If  $\iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t)$  is convergent, then the double series  $\sum \sum_{(k, \ell) \geq (2, 2)} f(k, \ell)$  is convergent, and*

$$\sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} f(k, \ell) \leq \iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t).$$

*Conversely, if the full double series  $\sum \sum_{(k, \ell) \geq (1, 1)} f(k, \ell)$  is convergent, then the improper double integral  $\iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t)$  is convergent, and*

$$\iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t) \leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} f(k, \ell),$$

*and moreover, the improper integrals  $\int_{[1, \infty)} f(s, 1) ds$  and  $\int_{[1, \infty)} f(1, t) dt$  are convergent, and*

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} f(k, \ell) \leq f(1, 1) + \int_1^{\infty} f(s, 1) ds + \int_1^{\infty} f(1, t) dt + \iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t).$$

*On the other hand, if the improper double integral  $\iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t)$  diverges to  $\infty$ , then the full double series  $\sum \sum_{(k, \ell) \geq (1, 1)} f(k, \ell)$  diverges to  $\infty$ , whereas if the double series  $\sum \sum_{(k, \ell) \geq (2, 2)} f(k, \ell)$  diverges to  $\infty$ , then the improper double integral  $\iint_{[1, \infty) \times [1, \infty)} f(s, t) d(s, t)$  diverges to  $\infty$ .*

*Proof.* Since  $f$  is monotonic, by part (i) of Proposition 5.12,  $f$  is integrable on  $[1, x] \times [1, y]$  for every  $(x, y) \geq (1, 1)$ . Define  $F : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$  by  $F(x, y) := \iint_{[1, x] \times [1, y]} f(s, t) d(s, t)$ . Since  $f$  is nonnegative, by Corollary 5.10,

the function  $F$  is monotonically increasing. Hence Proposition 7.55 implies that the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  is convergent if and only if the set  $\{F(m,n) : (m,n) \in \mathbb{N}^2\}$  is bounded above, and in this case

$$\begin{aligned} \iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t) &= \sup \{F(x,y) : (x,y) \in [1,\infty) \times [1,\infty)\} \\ &= \sup \{F(m,n) : (m,n) \in \mathbb{N}^2\} \\ &= \lim_{(m,n) \rightarrow (\infty,\infty)} F(m,n). \end{aligned}$$

Here the penultimate equality follows since  $F$  is monotonically increasing, and the last equality follows from part (i) of Proposition 7.4. Similarly,

$$\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t) \text{ diverges to } \infty \iff F(m,n) \rightarrow \infty \text{ as } m,n \rightarrow \infty.$$

Now let us define

$$a_{k,\ell} := \iint_{[k,k+1]\times[\ell,\ell+1]} f(s,t)d(s,t) \quad \text{for } (k,\ell) \in \mathbb{N}^2$$

and

$$A_{m,n} := \sum_{k=1}^m \sum_{\ell=1}^n a_{k,\ell} \quad \text{for } (m,n) \in \mathbb{N}^2.$$

By the Domain Additivity of Double Integrals on Rectangles (Proposition 5.9),

$$A_{m,n} = F(m+1, n+1) \quad \text{for all } (m,n) \in \mathbb{N}^2.$$

Further, since  $a_{k,\ell} \geq 0$  for all  $(k,\ell) \in \mathbb{N}^2$ , it follows from Proposition 7.14 that the double series  $\sum \sum_{(k,\ell) \geq (1,1)} a_{k,\ell}$  is convergent if and only if the double sequence  $(F(m,n))$  is bounded above, that is, the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  is convergent, and in this case, the sum of the double series is equal to the value of the improper double integral. Similarly, the double series  $\sum \sum_{(k,\ell) \geq (1,1)} a_{k,\ell}$  diverges to  $\infty$  if and only if the double sequence  $(F(m,n))$  is not bounded above, that is, the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  diverges to  $\infty$ .

To relate the convergence of the double series  $\sum \sum_{(k,\ell) \geq (1,1)} a_{k,\ell}$  to that of  $\sum \sum_{(k,\ell) \geq (1,1)} f(k,\ell)$ , observe that since  $f$  is monotonically decreasing,

$$f(k+1, \ell+1) \leq a_{k,\ell} \leq f(k,\ell) \quad \text{for all } (k,\ell) \in \mathbb{N}^2.$$

The first inequality above together with what is shown earlier and the Comparison Test for Double Series (Proposition 7.25) implies that if the improper dou-

ble integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  is convergent, then  $\sum \sum_{(k,\ell)\geq(2,2)} f(k,\ell)$  is convergent, and

$$\sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} f(k,\ell) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} f(k+1,\ell+1) \leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} a_{k,\ell} = \iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t).$$

Similarly, if the full double series  $\sum \sum_{(k,\ell)\geq(1,1)} f(k,\ell)$  is convergent, then the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  is convergent, and

$$\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} a_{k,\ell} \leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} f(k,\ell),$$

and moreover, since  $f$  is nonnegative, the series  $\sum_{k\geq 1} f(k,1)$  and  $\sum_{\ell\geq 1} f(1,\ell)$  are convergent as well. Since  $f$  is also monotonically decreasing, by the Integral Test for functions of one variable (Proposition 9.39 of ACICARA), the improper integrals  $\int_{[1,\infty)} f(s,1)ds$  and  $\int_{[1,\infty)} f(1,t)dt$  are convergent, and

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} f(k,\ell) &= f(1,1) + \sum_{k=2}^{\infty} f(k,1) + \sum_{\ell=2}^{\infty} f(1,\ell) + \sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} f(k,\ell) \\ &\leq f(1,1) + \int_1^{\infty} f(s,1)ds + \int_1^{\infty} f(1,t)dt \\ &\quad + \iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t). \end{aligned}$$

Finally, if the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  diverges to  $\infty$ , then as seen above, the double series  $\sum \sum_{(k,\ell)\geq(1,1)} a_{k,\ell}$  diverges to  $\infty$ , and so by the Comparison Test for Double Series (Proposition 7.25), the full double series  $\sum \sum_{(k,\ell)\geq(1,1)} f(k,\ell)$  diverges to  $\infty$ . Similarly, if the double series  $\sum \sum_{(k,\ell)\geq(2,2)} f(k,\ell)$  diverges to  $\infty$ , then the improper double integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  diverges to  $\infty$ .  $\square$

The above result is not completely analogous to the Integral Test for functions of one variable, which says that if  $f : [1,\infty) \rightarrow \mathbb{R}$  is a nonnegative monotonically decreasing function, then the improper integral  $\int_{[1,\infty)} f(t)dt$  is convergent if and only if the series  $\sum_{k\geq 1} f(k)$  is convergent. (See, for example, [22, Proposition 9.39]. In fact, if we define  $f, g : [1,\infty) \times [1,\infty) \rightarrow \mathbb{R}$  by

$$f(s,t) := \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(s,t) := \begin{cases} 1 & \text{if } s \in [1,2), \\ 0 & \text{otherwise,} \end{cases}$$

then we easily see that the double improper integral  $\iint_{[1,\infty)\times[1,\infty)} f(s,t)d(s,t)$  is convergent, but the double series  $\sum \sum_{(k,\ell)\geq(1,1)} f(k,\ell)$  diverges to  $\infty$ , while

the double series  $\sum \sum_{(k,\ell) \geq (2,2)} g(k, \ell)$  is convergent, but the double improper integral  $\iint_{[1,\infty) \times [1,\infty)} g(s, t) d(s, t)$  diverges to  $\infty$ .

Let now  $f : [1, \infty) \times [1, \infty) \rightarrow \mathbb{R}$  be a nonnegative monotonically decreasing function. Since  $\int_1^n f(s, 1) ds \leq \sum_{k=1}^n f(k, 1)$  and  $\int_1^n f(1, t) dt \leq \sum_{\ell=1}^n f(1, \ell)$  for all  $n \in \mathbb{N}$ , Proposition 7.57 yields the following characterization.

*The double series  $\sum \sum_{(k,\ell) \geq (1,1)} f(k, \ell)$  is convergent if and only if the improper double integral  $\iint_{[1,\infty) \times [1,\infty)} f(s, t) d(s, t)$  as well as the improper integrals  $\int_1^\infty f(s, 1) ds$  and  $\int_1^\infty f(1, t) dt$  are convergent.*

The above results are useful in determining whether a double series or an improper double integral is convergent, and in that case, to obtain lower bounds and upper bounds for them. This is illustrated in the example below.

**Example 7.58.** Let  $f(s, t) := 1/(s + t)^p$  for  $(s, t) \in [1, \infty) \times [1, \infty)$ , where  $p \in \mathbb{R}$  with  $p > 0$ . Then  $f$  is a nonnegative monotonically decreasing function. If  $p > 2$ , then the double series  $\sum \sum_{(k,\ell) \geq (1,1)} f(k, \ell)$  is convergent as we have seen in Example 7.17 (i), and so the improper double integral  $\iint_{[1,\infty) \times [1,\infty)} f(s, t) d(s, t)$  is convergent by Proposition 7.57. Conversely, suppose the improper double integral  $\iint_{[1,\infty) \times [1,\infty)} f(s, t) d(s, t)$  is convergent. Then the double series  $\sum \sum_{(k,\ell) \geq (2,2)} f(k, \ell)$  is convergent by Proposition 7.57. Hence proceeding as in Example 7.17 (i) and considering the diagonal series  $\sum_{j \geq 2} c_j$ , where  $c_j := (j - 1)/(j + 2)^p$ , we obtain  $p > 2$ . Thus the improper double integral  $\iint_{[1,\infty) \times [1,\infty)} f(s, t) d(s, t)$  is convergent if and only if  $p > 2$ .

Alternatively, we can directly show that  $\iint_{[1,\infty) \times [1,\infty)} f(s, t) d(s, t)$  is convergent if and only if  $p > 2$  as follows. Indeed, let  $(x, y) \geq (1, 1)$ , and let  $F(x, y) := \iint_{[1,x] \times [1,y]} d(s, t)/(s + t)^p$ . Suppose  $p \leq 2$ . Then

$$\begin{aligned} F(x, y) &\geq \iint_{[1,x] \times [1,y]} \frac{d(s, t)}{(s + t)^2} = \int_1^x \left( \int_1^y \frac{dt}{(s + t)^2} \right) ds \\ &= \int_1^x \left( \frac{1}{s + 1} - \frac{1}{s + y} \right) ds = \ln(x + 1) - \ln 2 - \ln(x + y) + \ln(1 + y) \\ &= \ln \frac{(x + 1)(y + 1)}{x + y} - \ln 2 \geq -\ln \left( \frac{1}{x + 1} + \frac{1}{y + 1} \right) - \ln 2. \end{aligned}$$

Hence  $\iint_{[1,\infty) \times [1,\infty)} d(s, t)/(s + t)^p$  diverges to  $\infty$ . Next, suppose  $p > 2$ . Then

$$\begin{aligned} F(x, y) &= \frac{1}{p - 1} \int_1^x \left[ \frac{1}{(s + 1)^{p-1}} - \frac{1}{(s + y)^{p-1}} \right] ds \\ &= \frac{1}{(p - 1)(p - 2)} \left[ \frac{1}{2^{p-2}} - \frac{1}{(x + 1)^{p-2}} - \frac{1}{(1 + y)^{p-2}} + \frac{1}{(x + y)^{p-2}} \right]. \end{aligned}$$

Hence

$$\iint_{[1,\infty)\times[1,\infty)} \frac{1}{(s+t)^p} d(s,t) = \frac{1}{(p-1)(p-2)2^{p-2}} \quad \text{if } p > 2.$$

Now we can obtain an alternative proof of the fact that the double series  $\sum \sum_{(k,\ell)\geq(1,1)} f(k,\ell)$  is convergent if and only if  $p > 2$  by invoking Proposition 7.57 and by observing that the improper integrals  $\int_1^\infty ds/(s+1)^p$  and  $\int_1^\infty dt/(1+t)^p$  are convergent when  $p > 2$ .

When  $p > 2$ , Proposition 7.57 allows us to estimate the double sum  $\sum \sum_{(k,\ell)\geq(1,1)} 1/(k+\ell)^p$ . First we note that if  $p > 1$ , then

$$\int_1^x \frac{1}{(s+1)^p} ds = \frac{1}{p-1} \left( \frac{1}{2^{p-1}} - \frac{1}{(x+1)^{p-1}} \right) \leq \frac{1}{(p-1)2^{p-1}} \quad \text{for all } x \geq 1.$$

Now

$$\begin{aligned} \frac{1}{(p-1)(p-2)2^{p-2}} &\leq \sum_{(k,\ell)\geq(1,1)} \sum_{(k,\ell)\geq(1,1)} \frac{1}{(k+\ell)^p} \\ &\leq \frac{1}{2^p} + \frac{1}{(p-1)2^{p-1}} + \frac{1}{(p-1)2^{p-1}} + \frac{1}{(p-1)(p-2)2^{p-2}} \\ &= \frac{p^2 + p - 2}{2^p(p-1)(p-2)}. \end{aligned}$$

For example,

$$\frac{1}{4} \leq \sum_{(k,\ell)\geq(1,1)} \sum_{(k,\ell)\geq(1,1)} \frac{1}{(k+\ell)^3} \leq \frac{5}{8} \quad \text{and} \quad \frac{1}{24} \leq \sum_{(k,\ell)\geq(1,1)} \sum_{(k,\ell)\geq(1,1)} \frac{1}{(k+\ell)^4} \leq \frac{3}{16}.$$

by letting  $p = 3$  and  $p = 4$ . Finally, we remark that if we denote by  $\zeta(s)$  the sum of the convergent series  $\sum_{n\geq 1} 1/n^s$ , where  $s \in \mathbb{R}$  with  $s > 1$ , then for  $p > 2$ , the sum  $\sum_{k=1}^\infty \sum_{\ell=1}^\infty 1/(k+\ell)^p$  can be expressed as  $\zeta(p-1) - \zeta(p)$ . Indeed,

$$\sum_{k=1}^\infty \sum_{\ell=1}^\infty \frac{1}{(k+\ell)^p} = \sum_{n=2}^\infty \sum_{\substack{k+\ell=n \\ (k,\ell)\geq(1,1)}} \frac{1}{n^p} = \sum_{n=2}^\infty \frac{(n-1)}{n^p} = (\zeta(p-1)-1) - (\zeta(p)-1).$$

This indicates that it is not easy to find an exact value of the sum of the above double series for  $p > 2$ , even if  $p$  is an integer. Indeed, while values of  $\zeta(n)$  are known when  $n$  is an even positive integer (Theorem 5.6.3 of Hijab's book [30]), values of  $\zeta(n)$  when  $n$  is an odd integer  $> 1$  remain mostly a mystery. For instance, it is not known whether or not  $\zeta(5)$  is irrational.  $\diamond$