ON APPROXIMATE ROBUST COUNTERPARTS OF UNCERTAIN SEMIDEFINITE AND CONIC QUADRATIC PROGRAMS

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Abstract We present efficiently verifiable sufficient conditions for the validity of specific NP-hard semi-infinite systems of semidefinite and conic quadratic constraints arising in the framework of Robust Convex Programming and demonstrate that these conditions are "tight" up to an absolute constant factor. We discuss applications in Control on the construction of a quadratic Lyapunov function for linear dynamic system under interval uncertainty.

1. Introduction

The subject of this paper are "tractable approximations" of intractable semi-infinite convex optimization programs arising as *robust counterparts* of *uncertain* conic quadratic and semidefinite problems. We start with specifying the relevant notions. Let $\mathbf{K}$ be a cone in $\mathbf{R}^N$ (closed, pointed, convex and with a nonempty interior). A *conic program* asso-

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ciated with $\mathbf{K}$ is an optimization program of the form

$$\min_x \{f^T x \mid Ax - b \in \mathbf{K}\}; \tag{CP}$$

here $x \in \mathbf{R}^n$. An *uncertain* conic problem is a family

$$\left\{\min_x \left\{f^T x \mid Ax - b \in \mathbf{K}\right\} |(f, A, b) \in \mathcal{U}\right\} \tag{UCP}$$

of conic problems with common $\mathbf{K}$ and data (f, A, B) running through a given *uncertainty set* $\mathcal{U}$. In fact, we always can get rid of uncertainty in f and assume that f is "certain", i.e., common for all data from $\mathcal{U}$); indeed, we always can rewrite the problems of the family as

$$\min_{t,x} \left\{t \mid \begin{bmatrix} Ax - b \\ t - f^T x \end{bmatrix} \in \bar{\mathbf{K}} = \mathbf{K} \times \mathbf{R}_+ \right\}.$$

Thus, we lose nothing (and gain a lot, as far as notation is concerned) when assuming from now on that f is certain, so that $n, \mathbf{K}, f$ form the common "structure" of problems from the family, while A, b are the data of particular problems ("instances") from the family.

The Robust Optimization methodology developed in [1, 2, 3, 5, 8, 9] associates with (UCP) its *Robust Counterpart* (RC)

$$\min_x \left\{f^T x \mid Ax - b \in \mathbf{K} \quad \forall (c, A, b) \in \mathcal{U}\right\}. \tag{R}$$

Feasible/optimal solutions of (R) are called *robust feasible*, resp., *robust optimal* solutions of the uncertain problem (UCP); the importance of these solutions is motivated and illustrated in [1, 2, 3, 5, 8, 9].

Accepting the concept of robust feasibile/optimal solutions, the crucial question is how to build these solutions. Note that (R) is a *semi-infinite* conic program and as such can be computationally intractable. In this respect, there are "good cases", where the RC is equivalent to an explicit computationally tractable convex optimization program, as well as "bad cases", where the RC is NP-hard (see [3, 5] for "generic examples" of both types). In "bad cases", the Robust Optimization methodology recommends replacing the computationally intractable robust counterpart by its tractable approximation. An *approximate robust counterpart* of (UCP) is a conic problem

$$\min_{x,u} \left\{f^T x \mid Px + Qu + r \in \widehat{\mathbf{K}}\right\} \tag{AR}$$

such that the projection $\mathcal{X}$(AR) of the feasible set of (AR) onto the plane of x-variables is contained in the feasible set of (R); thus, (AR) is "more conservative" than (R). An immediate question is how to measure the "conservativeness" of (AR), with the ultimate goal to use a "moderately conservative" computationally tractable approximate RCs instead of the "true" (intractable) RCs. A natural way to measure the quality of an approximate RC is as follows. Assume that the uncertainty set $\mathcal{U}$ is of the form

$$\mathcal{U} = \{(A, b) = (A^{\mathrm{n}}, b^{\mathrm{n}}) + \mathcal{V}\},$$

where $(A^{\mathrm{n}}, b^{\mathrm{n}})$ is the "nominal data" and $\mathcal{V}$ is the *perturbation set* which we assume from now on to be a convex compact set symmetric w.r.t. the origin. Under our assumptions, (UCP) can be treated as a member of the *parametric family*

$$\left\{\min_x \left\{f^T x \mid Ax - b \in \mathbf{K}\right\} : (A, b) \in \mathcal{U}_\rho = \{(A, b) = (A^{\mathrm{n}}, b^{\mathrm{n}}) + \rho\mathcal{V}\}\right\} \tag{UCP$_\rho$}$$

of uncertain conic problems, where $\rho \geq 0$ can be viewed as the "level of uncertainty". Observing that the robust feasible set $\mathcal{X}_\rho$ of (UCP$_\rho$) shrinks as ρ increases and that (AR) is an approximation of (R) if and only if $\mathcal{X}(\mathrm{AR}) \subset \mathcal{X}_1$, a natural way to measure the quality of (AR) is to look at the quantity

$$\rho(\mathrm{AR{:}R}) = \inf\{\rho \geq 1 : \mathcal{X}(\mathrm{AR}) \supset \mathcal{X}_\rho\},$$

which we call the *conservativeness* of the approximation (AR) of (R). Thus, the fact that (AR) is an approximation of (R) with the conservativeness $< \alpha$ means that

(i) If x can be extended to a feasible solution of (AR), then x is a robust feasible solution of (UCP);

(ii) If x cannot be extended to a feasible solution of (AR), then x is *not* robust feasible for the uncertain problem (UCP$_\alpha$) obtained from (UCP)$\equiv$(UCP$_1$) by increasing the level of uncertainty by the factor α.

Note that in real-world applications the level of uncertainty normally is known "up to a factor of order of 1"; thus, we have basically the same reasons to use the "true" robust counterpart as to use its approximation with $\rho(\mathrm{AR{:}R})$ of order of 1.

The goal of this paper is to overview recent results on tractable approximate robust counterparts with "$O(1)$-conservativeness", specifically, the results on semidefinite problem affected by box uncertainty and on conic quadratic problem affected by ellipsoidal uncertainty. We present the approximation schemes, discuss their quality, illustrate the results by some applications (specifically, in Lyapunov Stability Analysis/Synthesis for uncertain linear dynamic systems with interval uncertainty) and establish links of some of the results with recent developments in the area of semidefinite relaxations of difficult optimization problems.

2. Uncertain SDP with box uncertainty

Let $\mathbf{S}^m$ be space of real symmetric $m \times m$ matrices and $\mathbf{S}^m_+$ be the cone of positive semidefinite $m \times m$ matrices, and let $A^\ell[x] = A^{\ell 0} + \sum_{j=1}^{n} x_j A^{\ell j} : \mathbf{R}^n \to \mathbf{S}^m$ be affine mappings, $\ell = 0, ..., L$. Let also $C[x]$ be a symmetric matrix affinely depending on x. Consider the uncertain semidefinite program

$$\left\{ \min_x \left\{ f^T x : \begin{array}{c} C[x] \succeq 0 \\ A[x] \succeq 0 \end{array} \right\} \middle| \exists (u : \| u \|_\infty \leq \rho) : A[x] = A^0[x] + \sum_{\ell=1}^{L} u_\ell A^\ell[x] \right\} \tag{USD[ρ]}$$

here an in what follows, for $A, B \in \mathbf{S}^m$ the relation $A \succeq B$ means that $A - B \in \mathbf{S}^m_+$. Note that (USD[ρ]) is the general form of an uncertain semidefinite program affected by "box" uncertainty (one where the uncertainty set is an affine image of a multi-dimensional cube). Note also that the Linear Matrix Inequality (LMI) $C[x] \succeq 0$ represents the part of the constraints which are "certain" – not affected by the uncertainty.

The robust counterpart of (USD[ρ]) is the semi-infinite semidefinite program

$$\min_x \left\{ f^T x : \begin{array}{l} C[X] \succeq 0 \\ A^0[x] + \sum_{\ell=1}^{L} u_\ell A^\ell[x] \succeq 0 \quad \forall (u : \| u \|_\infty \leq \rho) \end{array} \right\}. \tag{R[ρ]}$$

It is known (see, e.g., [11]) that in general (R[ρ]) is NP-hard; this is so already for the associated analysis problem "Given x, check whether it is feasible for (R[ρ])", and even in the case when all the "edge matrices" $\mathcal{A}^\ell[x]$, $\ell = 1, ..., L$, are of rank 2. At the same time, we can easily point out a tractable approximation of (R[ρ]), namely, the semidefinite program

$$\min_{x, \{X_\ell\}} \left\{ f^T x : \begin{array}{l} C[x] \succeq 0 \\ X^\ell \succeq \pm A^\ell[x],\ \ell = 1, ..., L, \\ A^0[x] - \rho \sum_{\ell=1}^{L} X^\ell \succeq 0, \end{array} \right\}. \tag{AR[ρ]}$$

This indeed is an approximation – the x-component of a feasible solution to (AR[ρ]) clearly is feasible for (R[ρ]). Surprisingly enough, this fairly simplistic approximation turns out to be pretty tight, provided that the edge matrices $\mathcal{A}^\ell[x]$, $\ell \geq 1$, are of small rank:

Theorem 1 [6] *Let $A_0, ..., A_L$ be $m \times m$ symmetric matrices. Consider the following two predicates:*

$$\begin{array}{l} (I[\rho]) : A_0 + \sum\limits_{\ell=1}^{L} u_\ell A_\ell \succeq 0 \quad \forall (u : \| u \|_\infty \leq \rho); \\ (II[\rho]) : \exists X_1, ..., X_L : X_\ell \succeq \pm A_\ell,\ \ell = 1, ..., L,\ \rho \sum\limits_{\ell=1}^{L} X_\ell \preceq A_0; \end{array} \tag{1}$$

here $\rho \geq 0$ is a parameter.

Then

(i) *If $(II[\rho])$ is valid, so is $(I[\rho])$;*

(ii) *If $(II[\rho])$ is not valid, so is $(I[\vartheta(\mu)\rho])$, where*

$$\mu = \max_{1 \leq \ell \leq L} \mathrm{Rank}(A_\ell)$$

(note $1 \leq \ell$ in the max*) and $\vartheta(\mu)$ is a universal function of μ given by*

$$\frac{1}{\vartheta(k)} = \min_\alpha \left\{ \int\limits_{\mathbf{R}^k} \left| \sum_{i=1}^{k} \alpha_1 u_i^2 \right| (2\pi)^{-k/2} \exp\left\{ -\frac{u^T u}{2} \right\} du \left| \sum_{i=1}^{k} |\alpha_i| = 1 \right. \right\}. \tag{2}$$

Note that

$$\vartheta(1) = 1, \vartheta(2) = \frac{\pi}{2} \approx 1.57..., \vartheta(3) = 1.73..., \vartheta(4) = 2;\ \vartheta(\mu) \leq \frac{\pi\sqrt{\mu}}{2} \quad \forall \mu. \tag{3}$$

Corollary 2.1 *Consider the robust counterpart* $(\mathrm{R}[\rho])$ *of an uncertain SDP affected by interval uncertainty along with the approximated robust counterpart* $(\mathrm{AR}[\rho])$ *of the problem, and let*

$$\mu = \max_{1 \leq \ell \leq L} \max_x \mathrm{Rank}(\mathcal{A}^\ell[x])$$

(note $1 \leq \ell$ in the max*). Then* $(\mathrm{AR}[\rho])$ *is at most $\vartheta(\mu)$-conservative approximation of* $(\mathrm{R}[\rho])$*, where ϑ is given by* (2)*. In particular,*

- *The suprema $\rho^\star$ and $\widehat{\rho}$ of those $\rho \geq 0$ for which* $(\mathrm{R}[\rho])$*, respectively,* $(\mathrm{AR}[\rho])$ *is feasible, are linked by the relation*

$$\widehat{\rho} \leq \rho^\star \leq \vartheta(\mu)\widehat{\rho};$$

- *The optimal values* $f^\star(\rho)$, $\widehat{f}(\rho)$ *of* (R[ρ]), *respectively,* (AR[ρ]) *are linked by the relation*

$$f^\star(\rho) \le \widehat{f}(\rho) \le f^\star(\vartheta(\mu)\rho), \quad \rho \ge 0.$$

The essence of the matter is that the quality of the approximate robust counterpart as stated by Corollary 2.1 depends solely on the ranks of the "basic perturbation matrices" $A^\ell[x]$, $\ell \ge 1$, and is independent of any other sizes of the problem. Fortunately, there are many applications where the ranks of the basic perturbations are small, so that the quality of the approximation is not too bad. As an important example, consider the Lyapunov Stability Analysis problem.

Lyapunov Stability Analysis. Consider an uncertain time-varying linear dynamic system

$$\dot{z}(t) = A(t)z(t) \tag{4}$$

where all we know about the matrix $A(t)$ of the system is that it is a measurable function of t taking values in a given compact set $\mathcal{U}$ which, for the sake of simplicity, is assumed to be an interval matrix:

$$A(t) \in \mathcal{U} = \mathcal{U}_\rho = \{A \in \mathbf{R}^{n\times n} : |A_{ij} - \mathbf{A}_{ij}| \le \rho \mathbf{D}_{ij},\ i,j = 1,...,n\}; \tag{5}$$

here $\mathbf{A}$ corresponds to the "nominal" time-invariant system, and $\mathbf{D}$ is a given "scale matrix" with nonnegative entries.

In applications, the very first question about (4) is whether the system is *stable*, i.e., whether it is true that whatever is a measurable function $A(\cdot)$ taking values in $\mathcal{U}_\rho$, every trajectory $z(t)$ of (4) converges to 0 as $t \to \infty$. The standard *sufficient* condition for the stability of (4) – (5) is the existence of a common *quadratic Lyapunov stability certificate* for all matrices $A \in \mathcal{U}_\rho$, i.e., the existence of an $n \times n$ matrix $X \succ 0$ such that

$$A^T X + XA \prec 0 \quad \forall A \in \mathcal{U}_\rho.$$

Indeed, if such a certificate X exists, then

$$A^T X + XA \preceq -\alpha X$$

for certain $\alpha > 0$ and all $A \in \mathcal{U}_\rho$. As an immediate computation demonstrates, the latter inequality implies that

$$\frac{d}{dt}(z^T(t)Xz(t)) \le -\alpha z^T(t)Xz(t)$$

for all t and all trajectories of (4), provided that $A(t) \in \mathcal{U}_\rho$ for all t. The resulting differential inequality, in turn, implies that $z^T(t)Xz(t) \le$

$\exp\{-\alpha t\}(z^T(0)Xz(0)) \to 0$ as $t \to +\infty$; since $X \succ 0$, it follows that $z(t) \to 0$ as $t \to \infty$.

Note that by homogeneity reasons a stability certificate, if it exists, always can be normalized by the requirements

$$\begin{array}{lrcl} (a) & X & \succeq & I, \\ (b) & A^T X + XA & \preceq & -I \quad \forall A \in \mathcal{U}_\rho. \end{array} \tag{L[ρ]}$$

Thus, whenever (L[ρ]) is solvable, we can be sure that system (4) – (5) is stable. Although this condition is in general only sufficient and not necessary for stability (it is necessary only when $\rho = 0$, i.e., in the case of a certain time-invariant system), it is commonly used to certify stability. This, however, is not always easy to check the condition itself, since (L[ρ]) is a semi-infinite system of LMIs. Of course, since the LMI (L[ρ].b) is linear in A, this semi-infinite system of LMIs is equivalent to the usual – finite – system of LMIs

$$\begin{array}{lrcl} (a) & X & \succeq & I, \\ (b) & A_v^T X + XA_v & \preceq & -I \quad \forall v = 1, ..., 2^N, \end{array} \tag{6}$$

where N is the number of uncertain entries in the matrix A (i.e., the number of pairs ij such that $\mathbf{D}_{ij} \neq 0$) and $A_1, ..., A_{2^N}$ are the vertices of $\mathcal{U}_\rho$. However, the size of (6) is not polynomial in n, except for the (unrealistic) case when N is once for ever fixed or logarithmically slow grows with n. In general, it is NP-hard to check whether (6), or, which is the same, (L[ρ]) is feasible [11].

Now note that with the interval uncertainty (5), the troublemaking semi-infinite LMI (L[ρ].b) is nothing but the robust counterpart

$$\left[I - \mathbf{A}^T X - X\mathbf{A}\right] + \sum_{i,j=1}^{n} u_{ij}\mathbf{D}_{ij}\left[e_j(Xe_i)^T + (Xe_i)e_j^T\right] \succeq 0 \tag{7}$$
$$\forall (u = \{u_{ij}\} : -\rho \leq u_{ij} \leq \rho)$$

of the uncertain LMI

$$\left\{\left\{A^T X + XA \preceq -I\right\} : A \in \mathcal{U}_\rho\right\};$$

here e_i are the standard basic orths in $\mathbf{R}^n$. Consequently, we can approximate (L[ρ]) with a tractable system of LMIs

$$\begin{array}{ll} (a) & X \succeq I \\ (b_1) & X^{ij} \succeq \pm \underbrace{\mathbf{D}_{ij}\left[e_j(Xe_i)^T + (Xe_i)e_j^T\right]}_{A^{ij}[X]}, \ \forall (i,j : \mathbf{D}_{ij} > 0) \\ (b_2) & \underbrace{\left[I - \mathbf{A}^T X - X\mathbf{A}\right]}_{A^0[X]} - \rho \sum\limits_{i,j:\mathbf{D}_{ij}>0} X^{ij} \succeq 0 \end{array} \tag{AL[ρ]}$$

in matrix variables $X, \{X^{ij}\}_{i,j:\mathbf{D}_{ij}>0}$.

Invoking Corollary 2.1, we see that the relations between (AL[ρ]) and (L[ρ]) are as follows:

1 Whenever X can be extended to a feasible solution of the system (AL[ρ]), X is feasible for (L[ρ]) and thus certifies the stability of the uncertain dynamic system (4), the uncertainty set for the system being $\mathcal{U}_\rho$;

2 If X cannot be extended to a feasible solution of the system (AL[ρ]), then X does not solve the system (L[$\frac{\pi}{2}\rho$]) (note that the ranks of the basic perturbation matrices $A^{ij}[X]$ are at most 2, and $\vartheta(2) = \frac{\pi}{2}$).

It follows that the supremum $\widehat{\rho}$ of those $\rho \geq 0$ for which (AL[ρ]) is solvable is a lower bound for the *Lyapunov stability radius* of uncertain system (4), i.e., for the supremum $\rho^\star$ of those $\rho \geq 0$ for which all matrices from $\mathcal{U}_\rho$ share a common Lyapunov stability certificate, and that this lower bound is tight up to factor $\frac{\pi}{2}$:

$$1 \leq \frac{\rho^\star}{\widehat{\rho}} \leq \frac{\pi}{2},$$

provided, of course, that $\mathbf{A}$ is stable (or, which is the same, that $\rho^\star > 0$).

Note that the bound $\widehat{\rho}$ on the Lyapunov stability radius is efficiently computable; this is the optimal value in the Generalized Eigenvalue Problem of maximizing ρ in variables $\rho, X, \{X^{ij}\}$ under the constraints (LA[ρ]).

We have considered a specific application of Theorem 1 in Control. There are many other applications of this theorem to systems of LMIs arising in Control and affected by an interval data uncertainty. Usually the structure of such a system ensures that when perturbing a single data entry, the right hand side of every LMI is perturbed by a matrix of a small rank, which is the favourable case for our approximation scheme.

Simplifying the approximation. A severe computational shortcoming of the approximation (AR[ρ]) is that its sizes, although polynomial in the sizes of the approximated system (R[ρ]) and the uncertainty set, are pretty large, since the approximation has an additional $m \times m$ matrix variable X_ℓ and two $m \times m$ LMIs $X_\ell \succeq \pm A^\ell[x]$ per every one of the basic perturbations. It turns out that under favourable circumstances the sizes of the approximation can be reduced dramatically. This size reduction is based upon the following two facts:

Lemma 2.1 [6] (i) *Let $a \neq 0, b$ be two vectors. A matrix X satisfies the relation*

$$X \succeq \pm[ab^T + ba^T]$$

if and only if there exists $\lambda \geq 0$ such that

$$X \succeq \lambda aa^T + \frac{1}{\lambda} bb^T$$

(when $\lambda = 0$, the left hand side, by definition, is the zero matrix when $b = 0$ and is undefined otherwise).

(ii) *Let S be a symmetric $m \times m$ matrix of rank $k > 0$, so that $S = P^T RP$ with a nonsingular $k \times k$ matrix R and $k \times m$ matrix P of rank k.*

A matrix X satisfies the relation

$$X \succeq \pm S$$

if and only if there exists a $k \times k$ matrix Y such that $Y \succeq \pm R$ and $X \succeq P^T YP$.

The simplification of (AR[ρ]), based on Lemma 2.1, is as follows. Let us partition the basic perturbation matrices $A^\ell[x]$ into two groups: those with $A^\ell[x]$ actually depending on x, and those with $A^\ell[x]$ independent of x. Assume that

(A) *The basic perturbation matrices depending on* x, let them be $A^1[x], ..., A^M[x]$, *are of the form*

$$A^\ell[x] = a_\ell b_\ell^T[x] + b_\ell[x]a_\ell^T, \quad \ell = 1, ..., M, \tag{8}$$

where a_ℓ, $b_\ell[x]$ are vectors and $b_\ell[x]$ are affine in x.

Note that the assumption holds true, e.g., in the case of the Lyapunov Stability Analysis problem under interval uncertainty, see (AL[ρ]).

The basic perturbation matrices A^ℓ with $\ell > M$ are independent of x, and we can represent these matrices as

$$A^\ell = P_\ell^T B_\ell P_\ell, \tag{9}$$

where B_ℓ are nonsingular symmetric $k_\ell \times k_\ell$ matrices and $k_\ell = \text{Rank}(A^\ell)$.

Observe that when speaking about the approximate robust counterpart (AR[ρ]), we are not interested at all in the additional matrix variables X_ℓ; all which matters is the projection of the feasible set of (AR[ρ]) on the plane of the original design variables x. In other words, as far as the approximating properties are concerned, we lose nothing when replacing the constraints in (AR[ρ]) with any other system $\mathcal{S}$ of constraints in variables x and, perhaps, additional variables, provided that the projection of the feasible set of the new system on the plane of x-variables is the same as the one for (AR[ρ]). Invoking Lemma 2.1, we see that the latter property is possessed by the system of constraints

$$\begin{array}{ll} (a) & C[x] \succeq 0 \\ (b) & Y_\ell \succeq \pm B_\ell,\ \ell = M+1, ..., L, \\ (c) & \lambda_\ell \geq 0,\ \ell = 1, ..., M, \\ (d) & A^0[x] - \rho \left[\sum\limits_{\ell=1}^{M} \left[\lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x] \right] + \sum\limits_{\ell=M+1}^{L} P_\ell^T Y_\ell P_\ell \right] \succeq 0, \end{array} \tag{10}$$

in variables $x, \{\lambda_\ell\}, \{Y_\ell\}$. By the Schur Complement Lemma, (10) is equivalent to the system of LMIs

$$\begin{array}{ll} (a) & C[x] \succeq 0 \\ (b) & Y_\ell \succeq \pm B_\ell,\ \ell = M+1, ..., L, \\ (c) & \left[\begin{array}{c|cccc} A^0[x] - \rho \sum\limits_{\ell=1}^{M} \lambda_\ell a_\ell a_\ell^T - \rho \sum\limits_{\ell=M+1}^{L} P_\ell^T Y_\ell P_\ell & b_1[x] & b_2[x] & \cdots & b_M[x] \\ \hline b_1^T[x] & \lambda_1 & & & \\ b_2^T[x] & & \lambda_2 & & \\ \vdots & & & \ddots & \\ b_M^T[x] & & & & \lambda_M \end{array} \right] \succeq 0 \end{array} \tag{11}$$

in variables $x, \{\lambda_\ell \in \mathbf{R}\}_{\ell=1}^M, \{Y_\ell \in \mathbf{S}^{k_\ell}\}_{\ell=M+1}^L$. Consequently, (AR[$\rho$]) is equivalent to the semidefinite program of minimizing the objective $f^T x$ under the constraints (11). Note that the resulting problem (A[ρ]) is typically much better suited for numerical processing than (AR[ρ]). Indeed, the first M of the $m \times m$ matrix variables X_ℓ arising in the original problem are now replaced with M *scalar* variables λ_ℓ, while the remaining $L - M$ of X_ℓ's are replaced with $k_\ell \times k_\ell$ matrix variables Y_ℓ; normally, the ranks k_ℓ of the basic perturbation matrices A^ℓ are much smaller than the sizes m of these matrices, so that this transformation reduces quite significantly the design dimension of the problem. As about LMIs, the $2L$ "large" (of the size $m \times m$) LMIs $X_\ell \succeq \pm A^\ell[x]$ of the original problem are now replaced with $2(L - M)$ "small" (of the sizes $k_\ell \times k_\ell$) LMIs (11.b) and a single "very large" – of the size $(m + M) \times (m + M)$ – LMI (11.c). Note that the latter LMI, although large, is of a very simple "arrow" structure and is extremely sparse.

Links with quadratic maximization over the unit cube. It turns out that Theorem 1 has direct links with the problem of maximizing a positive definite quadratic form over the unit cube. The link is given by the following simple observation:

Proposition 2.1 *Let Q be a positive definite $m \times m$ matrix. Then the reciprocal $\rho(Q)$ of the quantity*

$$\omega(Q) = \max_x \{x^T Q x : \| x \|_\infty \leq 1\}$$

equals to the maximum of those $\rho > 0$ for which all matrices from the matrix box

$$\mathcal{C}_\rho = Q^{-1} + \{A = A^T : |A_{ij}| \leq \rho\}$$

are positive semidefinite.

Proof. $\omega(Q)$ is the minimum of those ω for which the ellipsoid $\{x : x^T Q x \leq \omega\}$ contains the unit cube, or, which is the same, the minimum of those ω for which the polar of the ellipsoid (which is the ellipsoid $\{\xi : \xi^T Q^{-1} \xi \leq \omega^{-1}\}$) is contained in the polar of the cube (which is the set $\{\xi : \| \xi \|_1 \leq 1\}$). In other words,

$$\rho(Q) \equiv \omega^{-1}(Q) = \max\left\{\rho : \xi^T Q^{-1} \xi \geq \rho \| \xi \|_1^2 \ \forall \xi\right\}.$$

Observing that by evident reasons

$$\| \xi \|_1^2 = \max_A \left\{\xi^T A \xi : A = A^T, |A_{ij}| \leq 1, \, i,j = 1,...,m\right\},$$

we conclude that

$$\rho(Q) = \max\{\rho : Q^{-1} \succeq \rho A \quad \forall (A = A^T : |A_{ij}| \leq 1, \, i,j = 1,...,m)\},$$

as claimed. ■

Since the "edge matrices" of the matrix box $\mathcal{C}_\rho$ are of ranks 1 or 2 (these are the basic symmetric matrices $E^{ij} = \begin{cases} e_i e_i^T, & i = j \\ e_i e_j^T + e_j e_i^T, & i < j \end{cases}$, $1 \leq i \leq j \leq m$, where e_i are the standard basic orths), Theorem 1 says that the efficiently computable quantity

$$\widehat{\rho} = \sup\left\{\rho : \exists \{X^{ij}\} : \begin{array}{c} X^{ij} \succeq \pm E^{ij}, \, 1 \leq i \leq j \leq m \\ Q^{-1} - \rho \sum\limits_{1 \leq i \leq j \leq m} X^{ij} \succeq 0 \end{array}\right\}$$

is a lower bound, tight within the factor $\vartheta(2) = \frac{\pi}{2}$, for the quantity $\rho(Q)$, and consequently the quantity $\widehat{\omega}(Q) = \frac{1}{\widehat{\rho}(Q)}$ is an upper bound, tight within the same factor $\frac{\pi}{2}$, for the maximum $\omega(Q)$ of the quadratic form $x^T Q x$ over the unit cube:

$$\omega(Q) \leq \widehat{\omega}(Q) \leq \frac{\pi}{2} \omega(Q). \tag{12}$$

On a closest inspection (see [6]), it turns out that $\widehat{\omega}(Q)$ is nothing but the standard semidefinite bound

$$\begin{array}{rcl}\widehat{\omega}(Q) & = & \max\{\mathrm{Tr}(QX) : X \succeq 0, X_{ii} \leq 1,\, i = 1, ..., m\} \\ & = & \min\left\{\sum\limits_i \lambda_i : \mathrm{Diag}\{\lambda\} \succeq Q\right\}\end{array} \tag{13}$$

on $\omega(Q)$. The fact that bound (13) satisfies (12) was originally established by Yu. Nesterov [13] via a completely different construction which heavily exploits the famous MAXCUT-related "random hyperplane" technique of Goemans and Williamson [10]. Surprisingly, the re-derivation of (12) we have presented, although uses randomization arguments, seems to have nothing in common with the random hyperplane technique.

Theorem 1: Sketch of the proof.. We believe that not only the statement, but also the proof of Theorem 1 is rather instructive, this is why we sketch the proof here. We intend to focus on the most nontrivial part of the Theorem, which is the claim that when $(II[\rho])$ is not valid, so is $(I[\vartheta(\mu)\rho])$ (as about the remaining statements of Theorem 1, note that (i) is evident, while the claim that the function (2) satisfies (3) is more or less straightforward). Thus, assume that $(II[\rho])$ is not valid; we should prove that then $(I[\vartheta(\mu)\rho])$ also is not valid, where $\vartheta(\cdot)$ is given by (2).

The fact that $(II[\rho])$ is not valid means exactly that the optimal value in the semidefinite program

$$\min_{t,\{X_\ell\}}\left\{t : X_\ell \succeq \pm A_\ell,\ \ell = 1, ..., L;\ \rho\sum_{\ell=1}^{L} X_\ell \preceq A_0 + tI\right\}$$

is positive. Applying semidefinite duality (which is a completely mechanical process) we, after simple manipulations, conclude that in this case there exists a matrix $W \succeq 0$, $\mathrm{Tr}(W) = 1$, such that

$$\sum_{\ell=1}^{L} \| \lambda(W^{1/2}A_\ell W^{1/2}) \|_1 > \mathrm{Tr}(W^{1/2}A_0W^{1/2}), \tag{14}$$

where $\lambda(B)$ is the vector of eigenvalues of a symmetric matrix B. Now observe that if B is a symmetric $m \times m$ matrix and ξ is an m-dimensional Gaussian random vector with zero mean and unit covariance matrix, then

$$\mathbf{E}\left\{|\xi^T B\xi|\right\} \geq \frac{1}{\vartheta(\mathrm{Rank}(B))} \| \lambda(B) \|_1 . \tag{15}$$

Indeed, in the case when B is diagonal, this relation is a direct consequence of the definition of $\vartheta(\cdot)$; the general case can be reduced immediately to the case of diagonal B due to the rotational invariance of the distribution of ξ.

Since the matrices $W^{1/2}A_\ell W^{1/2}$ are of ranks not exceeding $\mu = \max\limits_{1\le\ell\le L} \mathrm{Rank}(A_\ell)$, (14) combines with (15) to imply that

$$[\rho\vartheta(\mu)]\sum_{\ell=1}^{L}\mathbf{E}\{|\xi^T W^{1/2}A_\ell W^{1/2}\xi|\} > \mathrm{Tr}(W^{1/2}A_0W^{1/2}) = \\ = \mathbf{E}\{\xi^T W^{1/2}A_0W^{1/2}\xi\}.$$

It follows that there exists $\zeta = W^{1/2}\xi$ such that

$$[\rho\vartheta(\mu)]\sum_{\ell=1}^{L}|\zeta^T A_\ell\zeta| > |\zeta^T A_0\zeta|;$$

setting $\epsilon_\ell = \mathrm{sign}(\zeta^T A_\ell\zeta)$, we can rewrite the latter relation as

$$\zeta^T\left[\sum_{\ell=1}^{L}\underbrace{(\rho\vartheta(\mu)\epsilon_\ell)}_{u_\ell}A_\ell\right]\zeta > \zeta^T A_0\zeta,$$

and we see that the matrix $A_0 - \sum\limits_\ell u_\ell A_\ell$ is not positive definite, while by construction $|u_\ell| \le \vartheta(\mu)\rho$. Thus, $(I[\vartheta(\mu)\rho])$ indeed is not valid.

3. Approximate robust counterparts of uncertain convex quadratic problems

Recall that a generic convex quadratically constrained problem is

$$\min_x\left\{f^T x : x^T A_i^T A_i x \le 2b_i^T x + c_i,\ i = 1,...,m\right\} \qquad \text{(QP)}$$

here $x \in \mathbf{R}^n$, and A_i are $m_i \times n$ matrices. The data of an instance is $(c, \{A_i, b_i, c_i\}_{i=1}^m)$. When speaking about uncertain problems of this type, we may focus on the robust counterpart of the system of constraints (since we have agreed to treat the objective as certain). In fact we can restrict ourselves with building an (approximate) robust counterpart of a *single* convex quadratic constraint, since the robust counterpart is a "constraint-wise" construction. Thus, we intend to focus on building an approximate robust counterpart of a single constraint

$$x^T A^T Ax \le 2b^T x + c \qquad (16)$$

with the data (A, b, c). We assume that the uncertainty set is "parameterized" by a vector of perturbations:

$$\mathcal{U} \equiv \mathcal{U}_\rho = \left\{ (A, b, c) = (A^{\rm n}, b^{\rm n}, c^{\rm n}) + \sum_{\ell=1}^{L} \zeta_\ell (A^\ell, b^\ell, c^\ell) : \zeta \in \rho \mathcal{V} \right\}, \quad (17)$$

here $\mathcal{V}$ is a convex compact symmetric w.r.t. the origin set in $\mathbf{R}^L$ ("the set of standard perturbations") and $\rho \geq 0$ is the "perturbation level".

In what follows, we shall focus on the case when $\mathcal{V}$ is given as an intersection of ellipsoids centered at the origin:

$$\mathcal{V} = \left\{ \zeta \in \mathbf{R}^L \mid \zeta^T Q_i \zeta \leq 1, \ i = 1, ..., k \right\}, \quad (18)$$

where $Q_i \succeq 0$ and $\sum_{i=1}^{k} Q_i \succ 0$. We will be interested also in two particular cases of general ellipsoidal uncertainty (18), namely, in the cases of

- *simple ellipsoidal uncertainty* $k = 1$;
- *box uncertainty:* $k = L$ and $\zeta^T Q_i \zeta \equiv \zeta_i^2$, $i = 1, ..., L$.

Note that the ellipsoidal robust counterpart of an uncertain quadratic constraint affected by uncertainty (18) is, in general, NP-hard. Indeed, already in the case of box uncertainty to verify robust feasibility of a given candidate solution is not easier than to maximize a convex quadratic form over the unit cube, which is known to be NP-hard. Thus, all we can hope for in the case of uncertainty (18) is a "computationally tractable" approximate robust counterpart with a moderate level of conservativeness, and we are about to build such a counterpart.

3.1 Building the robust counterpart of (16) – (18)

We build an approximate robust counterpart via the standard *semidefinite relaxation scheme.* For $x \in \mathbf{R}^n$, let

$$a[x] = A^{\rm n} x, \ A[x] = \rho \left[A^1 x, A^2 x, ..., A^L x \right], \ b[x] = \rho \begin{pmatrix} x^T b^1 \\ \vdots \\ x^T b^L \end{pmatrix}, \qquad d = \frac{\rho}{2} \begin{pmatrix} c^1 \\ \vdots \\ c^L \end{pmatrix}, \ e[x] = 2 x^T b^{\rm n} + c^{\rm n}, \quad (19)$$

so that for all ζ one has

$$x^T \left[A^{\rm n} + \rho \sum_\ell \zeta_\ell A^\ell \right]^T \left[A^{\rm n} + \rho \sum_\ell \zeta_\ell A^\ell \right] x = = \left(a[x] + A[x]\zeta \right)^T \left(a[x] + A[x]\zeta \right),$$

$$2x^T\left[b^{\rm n}+\rho\sum_{\ell=1}^{L}\zeta_\ell b^\ell\right]+\left[c^{\rm n}+\rho\sum_{\ell=1}^{L}\zeta_\ell c^\ell\right]=2\,(b[x]+d)^T\zeta+e[x].$$

From these relations it follows that

$$\begin{array}{ll}
(a) & x \text{ is robust feasible for (16) – (18)}\\
& \Updownarrow\\
& (a[x]+A[x]\zeta)^T(a[x]+A[x]\zeta)\le 2\,(b[x]+d)^T\zeta+e[x]\\
& \qquad\forall(\zeta:\zeta^TQ_i\zeta\le 1, i=1,...,k)\\
& \Updownarrow\\
& \zeta^TA^T[x]A[x]\zeta+2\zeta^T\left[A[x]a[x]-b[x]-d\right]\le e[x]-a^T[x]a[x]\\
& \qquad\forall(\zeta:\zeta^TQ_i\zeta\le 1, i=1,...,k)\\
& \Updownarrow\\
& \zeta^TA^T[x]A[x]\zeta+2t\zeta^T\left[A[x]a[x]-b[x]-d\right]\le e[x]-a^T[x]a[x]\\
& \qquad\forall(\zeta,t:\zeta^TQ_i\zeta\le 1, i=1,...,k, t^2=1)\\
& \Updownarrow\\
(b) & \zeta^TA^T[x]A[x]\zeta+2t\zeta^T\left[A[x]a[x]-b[x]-d\right]\le e[x]-a^T[x]a[x]\\
& \qquad\forall(\zeta,t:\zeta^TQ_i\zeta\le 1, i=1,...,k, t^2\le 1).
\end{array}\tag{20}$$

Thus, x *is robust feasible if and only if the quadratic inequality* (20.b) *in variables* ζ, t *is a consequence of the system of quadratic inequalities* $\zeta^TQ_i\zeta\le 1, i=1,...,k, t^2\le 1$. An evident *sufficient condition* for (20.b) to hold true is the possibility to majorate the left hand side of (20.b) *for all* ζ, t by a sum $\sum_i \lambda_i\zeta^TQ_i\zeta+\mu t^2$ with nonnegative weights λ_i, μ satisfying the relation $\sum_i\lambda_i+\mu\le e[x]-a^T[x]a[x]$. Thus, we come to the implication

$$\begin{array}{ll}
(a) & \exists(\mu\ge 0,\{\lambda_i\ge 0\}): \quad \begin{array}{l}\sum_{i=1}^k\lambda_i+\mu\le e[x]-a^T[x]a[x],\\ \sum_{i=1}^k\lambda_i\zeta^TQ_i\zeta+\mu t^2\ge\zeta^TA^T[x]A[x]\zeta\\ +2t\zeta^T\left[A[x]a[x]-b[x]-d\right]\ \forall(\zeta,t)\end{array}\\
& \Downarrow\\
& \zeta^TA^T[x]A[x]\zeta+2t\zeta^T\left[A[x]a[x]-b[x]-d\right]\le e[x]-a^T[x]a[x]\\
& \qquad\forall(\zeta,t:\zeta^TQ_i\zeta\le 1, i=1,...,k, t^2\le 1)\\
& \Downarrow\\
(b) & x \text{ is robust feasible}
\end{array}\tag{21}$$

A routine processing of condition (21.a) which we skip here demonstrates that the condition is equivalent to the solvability of the system of LMIs

$$\begin{pmatrix} e[x] - \sum_{i=1}^{k} \lambda_i & [-b[x]-d]^T & a^T[x] \\ [-b[x]-d] & \sum_{i=1}^{k} \lambda_i Q_i & -A^T[x] \\ a[x] & -A[x] & I \end{pmatrix} \succeq 0, \tag{22}$$
$$\lambda_i \geq 0,\ i = 1, ..., k,$$

in variables x, λ. We arrive at the following

Proposition 3.1 *The system of LMIs* (22) *is an approximate robust counterpart of the uncertain convex quadratic constraint* (16) – (18).

The level of conservativeness Ω of (22) can be bounded as follows:

Theorem 2 [7]
(i) *In the case of a general-type ellipsoidal uncertainty* (18), *one has*

$$\Omega \leq \widetilde{\Omega} \equiv \sqrt{3.6 + 2\ln\left(\sum_{i=1}^{k} \mathrm{Rank}(Q_i)\right)}. \tag{23}$$

Note that the right hand side in (23) *is* < 6, *provided that*

$$\sum_{i=1}^{k} \mathrm{Rank}(Q_i) \leq 10,853,519.$$

(ii) *In the case of box uncertainty:*

$$\zeta^T Q_i \zeta = \zeta_i^2, 1 \leq i \leq k = L \equiv \dim \zeta,\ \ \Omega \leq \tfrac{\pi}{2}$$

(iii) *In the case of simple* ($k = 1$) *ellipsoidal uncertainty* (18), $\Omega = 1$ (22) *is equivalent to the robust counterpart of* (16) – (18).

An instrumental role in the proof is played by the following fact which seems to be interesting by its own right:

Theorem 3 [7] *Let* $R, R_0, R_1, ..., R_k$ *be symmetric* $n \times n$ *matrices such that* $R_1, ..., R_k \succeq 0$ *and there exist nonnegative weights* λ_i *such that* $\sum_{i=0}^{k} \lambda_i R_i \succ 0$. *Consider the optimization program*

$$\mathrm{OPT} = \max_y \left\{ y^T R y : y^T R_i y \leq 1,\ i = 0, ..., k \right\} \tag{24}$$

along with the semidefinite program

$$\mathrm{SDP} = \min_{\mu} \left\{ \sum_{i=0}^{k} \mu_i : \sum_{i=0}^{k} \mu_i R_i \succeq R, \ \mu \geq 0 \right\}. \tag{25}$$

Then (25) *is solvable, its optimal value majorates the one of* (24), *and there exists a vector* y_* *such that*

$$y_*^T R y_* = \mathrm{SDP}; \quad y_*^T R_0 y_* \leq 1; \quad y_*^T R_i y_* \leq \widetilde{\Omega}^2, \, i = 1, ..., k,$$

$$\widetilde{\Omega} = \begin{cases} \sqrt{3.6 + 2\ln\left(\sum\limits_{i=1}^{k} \mathrm{Rank}(Q_i)\right)}, & R_0 = q^T q \text{ is dyadic} \\ \sqrt{8\ln 2 + 4\ln n + 2\ln\left(\sum\limits_{i=1}^{k} \mathrm{Rank}(Q_i)\right)}, & \text{otherwise} \end{cases} \tag{26}$$

In particular,

$$\mathrm{OPT} \leq \mathrm{SDP} \leq \widetilde{\Omega}^2 \cdot \mathrm{OPT}.$$

4. Approximate robust counterparts of uncertain conic quadratic problems

The constructions and results of the previous section can be extended to the essentially more general case of *conic quadratic* problems. Recall that a generic conic quadratic problem (another name: SOCP – Second Order Cone Problem) is

$$\min_{x} \left\{ f^T x : \| A_i x + b_i \|_2 \leq \alpha_i^T x + \beta_i, \, i = 1, ..., m \right\} \tag{CQP}$$

here $x \in \mathbf{R}^n$, and A_i are $m_i \times n$ matrices; the data of (CQP) is the collection $(f, \{A_i, b_i, \alpha_i, \beta_i\}_{i=1}^{m})$. As always, we assume the objective to be "certain" and thus may restrict ourselves with building an approximate robust counterpart of a *single* conic quadratic constraint

$$\| Ax + b \|_2 \leq \alpha^T x + \beta \tag{27}$$

with data (A, b, α, β).

We assume that the uncertainty set is parameterized by a vector of perturbations and *that the uncertainty is "side-wise"*: the perturbations affecting the left- and the right hand side data of (27) run independently of each other through the respective uncertainty sets:

$$\begin{array}{l} \mathcal{U} \equiv \mathcal{U}_\rho^{\rm left} \times \mathcal{U}_\rho^{\rm right}, \\ \mathcal{U}_\rho^{\rm left} = \left\{ (A,b) = (A^{\rm n}, b^{\rm n}) + \sum\limits_{\ell=1}^{L} \zeta_\ell (A^\ell, b^\ell) \middle| \zeta \in \rho \mathcal{V}^{\rm left} \right\}, \\ \mathcal{U}_\rho^{\rm right} = \left\{ (\alpha,\beta) = (\alpha^{\rm n}, \beta^{\rm n}) + \sum\limits_{r=1}^{R} \eta_r (\alpha^\ell, \beta^\ell) \middle| \eta \in \rho \mathcal{V}^{\rm right} \right\}. \end{array} \tag{28}$$

In what follows, we focus on the case when $\mathcal{V}^{\rm left}$ is given as an intersection of ellipsoids centered at the origin:

$$\mathcal{V}^{\rm left} = \left\{ \zeta \in \mathbf{R}^L \mid \zeta^T Q_i \zeta \le 1,\ i = 1, ..., k \right\}, \tag{29}$$

where $Q_i \succeq 0$ and $\sum\limits_{i=1}^{k} Q_i \succ 0$. We will be interested also in two particular cases of general ellipsoidal uncertainty (29), namely, in the cases of

- *simple ellipsoidal uncertainty* $k = 1$;
- *box uncertainty:* $k = L$ and $\zeta^T Q_i \zeta \equiv \zeta_i^2$, $i = 1, ..., L$.

As about the "right hand side" perturbation set $\mathcal{V}^{\rm right}$, we allow for a much more general geometry, namely, we assume only that $\mathcal{V}^{\rm right}$ is bounded, contains 0 and is *semidefinite-representable*:

$$\eta \in \mathcal{V}^{\rm right} \Leftrightarrow \exists u :\ P(\eta) + Q(u) - S \succeq 0, \tag{30}$$

where $P(\eta)$, $Q(u)$ are symmetric matrices linearly depending on η, u, respectively. We assume also that the LMI in (30) is strictly feasible, i.e., that

$$P(\bar{\eta}) + Q(\bar{u}) - S \succ 0$$

for appropriately chosen $\bar{\eta}$, $\bar{u}$.

4.1 Building approximate robust counterpart of (27) – (30)

For $x \in \mathbf{R}^n$, let

$$a[x] = A^{\rm n} x + b^{\rm n},\ A[x] = \rho \left[A^1 x + b^1, A^2 x + b^2, ..., A^L x + b^L \right], \tag{31}$$

so that for all ζ one has

$$\left[A^{\rm n} + \rho \sum_\ell \zeta_\ell A^\ell \right] x + \left[b^{\rm n} + \rho \sum_\ell \zeta_\ell b^\ell \right] = a[x] + A[x]\zeta.$$

Since the uncertainty is side-wise, x is robust feasible for (27) – (30) if and only if there exists τ such that the left hand side in (27), evaluated at

x, is at most τ for all left-hand-side data from $\mathcal{U}_\rho^{\text{left}}$, while the right hand side, evaluated at x, is at least τ for all right-hand-side data from $\mathcal{U}_\rho^{\text{right}}$. The latter condition can be processed straightforwardly via *semidefinite duality*, with the following result:

Proposition 4.1 *A pair (x, τ) is such that $\tau \leq \alpha^T x + \beta$ for all $(\alpha, \beta) \in \mathcal{U}_\rho^{\text{right}}$ if and only if it can be extended to a solution of the system of LMIs*

$$\begin{array}{c} \tau \leq x^T \alpha^{\text{n}} + \beta^{\text{n}} + \text{Tr}(SV), \quad P^*(V) = \gamma[x] \equiv \begin{pmatrix} \rho[x^T \alpha^1 + \beta^1] \\ \vdots \\ \rho[x^T \alpha^R + \beta^R] \end{pmatrix} \\ Q^*(V) = 0, \quad V \succeq 0 \end{array} \tag{32}$$

in variables x, τ, V. Here for a linear mapping $A(z) = \sum_{i=1}^{k} z_k A_k : \mathbf{R}^k \to \mathbf{S}^m$ taking values in the space $\mathbf{S}^m$ of $m \times m$ symmetric matrices, $A^*(Z) = \begin{pmatrix} \text{Tr}(ZA_1) \\ \vdots \\ \text{Tr}(ZA_1) \end{pmatrix} : \mathbf{S}^m \to \mathbf{R}^k$ is the mapping conjugate to $A(\cdot)$.

In view of the above observations, we have

$$\begin{array}{ll} (a) & x \text{ is robust feasible for (27) – (30)} \\ & \Updownarrow \\ & \exists(\tau, V) : \left\{ \begin{array}{c} (x, \tau, V) \text{ solves (32)} \\ \| a[x] + A[x]\zeta \|_2 \leq \tau \quad \forall(\zeta : \zeta^T Q_i \zeta \leq 1, i = 1, ..., k) \end{array} \right. \\ & \Updownarrow \\ & \exists(\tau, V) : \left\{ \begin{array}{c} (x, \tau, V) \text{ solves (32)} \\ \| \pm a[x] + A[x]\zeta \|_2 \leq \tau \quad \forall(\zeta : \zeta^T Q_i \zeta \leq 1, i = 1, ..., k) \end{array} \right. \\ & \Updownarrow \\ & \exists(\tau, V) : \left\{ \begin{array}{l} (x, \tau, V) \text{ solves (32)} \\ \| ta[x] + A[x]\zeta \|_2 \leq \tau \\ \quad \forall(\zeta, t : \zeta^T Q_i \zeta \leq 1, i = 1, ..., k, t^2 \leq 1) \end{array} \right. \\ & \Updownarrow \\ (b) & \exists(\tau, V) : \left\{ \begin{array}{ll} (1) & (x, \tau, V) \text{ solves (32)} \\ (2) & \tau \geq 0 \\ (3) & \begin{array}{l} \| ta[x] + A[x]\zeta \|_2^2 \leq \tau^2 \\ \forall(\zeta, t : \zeta^T Q_i \zeta \leq 1, i = 1, ..., k, t^2 \leq 1) \end{array} \end{array} \right. \end{array} \tag{33}$$

Thus, x is robust feasible *if and only if* $(33.b)$ holds true. Now observe that

$$\boxed{\begin{array}{ll} \exists(\mu,\lambda_1,...,\lambda_k): & \\ (a) & \mu\geq 0, \lambda_i\geq 0,\ i=1,...,k,\\ (b) & \mu+\sum\limits_{i=1}^k \lambda_i \leq \tau,\\ (c) & \tau\left(\mu t^2+\sum\limits_{i=1}^k \lambda_i\zeta^TQ_i\zeta\right)\geq \| ta[x]+A[x]\zeta\|_2^2 \quad \forall(t,\zeta)\end{array}} \qquad (34)$$

$$\Downarrow$$

$$\| ta[x]+A[x]\zeta\|_2^2\leq \tau^2 \quad \forall(\zeta,t:\zeta^TQ_i\zeta\leq 1, i=1,...,k, t^2\leq 1).$$

Via Schur complement, for nonnegative $\tau,\mu,\{\lambda_i\}$ the inequality

$$\tau\left(\mu t^2+\sum_{i=1}^k \lambda_i\zeta^TQ_i\zeta\right)\geq \| ta[x]+A[x]\zeta\|_2^2$$

holds true for a given pair (t,ζ) if and only if

$$\begin{array}{rcl} 0 & \preceq & \left(\begin{array}{c|c} \begin{bmatrix} t & \zeta^T\end{bmatrix}\left[\begin{array}{c|c}\mu & \\ \hline & \sum\limits_i\lambda_iQ_i\end{array}\right]\begin{bmatrix} t\\ \zeta\end{bmatrix} & \begin{bmatrix} t & \zeta^T\end{bmatrix}\begin{bmatrix} a^T[x]\\ A^T[x]\end{bmatrix} \\ \hline \begin{bmatrix} a[x] & A[x]\end{bmatrix}\begin{bmatrix} t\\ \zeta\end{bmatrix} & \tau I\end{array}\right) \\ & = & \left(\begin{array}{c|c|c}\tau & \zeta^T & \\ \hline & & I\end{array}\right)\left(\begin{array}{c|c|c}\mu & & a^T[x]\\ \hline & \sum\limits_{i=1}^k\lambda_iQ_i & A^T[x]\\ \hline a[x] & A[x] & \tau I\end{array}\right)\left(\begin{array}{c|c|c}\tau & \zeta^T & \\ \hline & & I\end{array}\right)^T. \end{array}$$

In view of this observation combined with the fact that the union, over all τ,ζ, of the image spaces of the matrices $\left(\begin{array}{c|c} & \tau\\ \hline & \zeta\\ \hline I & \end{array}\right)\in\mathbf{R}^{(1+L+m)\times(m+1)}$ is the entire $\mathbf{R}^{1+L+m}$, we conclude that in the case of nonnegative $\tau,\mu,\{\lambda_i\}$ the relation $(34.c)$ is equivalent to

$$\left(\begin{array}{c|c|c}\mu & & a^T[x]\\ \hline & \sum\limits_{i=1}^k\lambda_iQ_i & A^T[x]\\ \hline a[x] & A[x] & \tau I\end{array}\right)\succeq 0.$$

Looking at this relation, (34) and (33), we see that the following implication is valid:

$$\boxed{\begin{array}{lc} \exists(\tau, V, \mu, \lambda_1, ..., \lambda_k): & \\ (a) & \mu \geq 0, \lambda_i \geq 0,\ i = 1, ..., k, \\ (b) & \sum\limits_{i=1}^{k} \lambda_i + \mu \leq \tau, \\ (c) & \left(\begin{array}{c|c|c} \mu & & a^T[x] \\ \hline & \sum\limits_{i=1}^{k} \lambda_i Q_i & A^T[x] \\ \hline a[x] & A[x] & \tau I \end{array}\right) \succeq 0, \\ (d) & (x, \tau, V) \text{ solves } (32) \end{array}} \quad (35)$$

$$\Downarrow$$

$$x \text{ is robust feasible for (27) – (30).}$$

We can immediately eliminate μ from the premise of (35), thus arriving at the following result:

Proposition 4.2 *The system of LMIs in variables* $x, \tau, V, \{\lambda_i\}_{i=1}^k$

$$\left(\begin{array}{c|c|c} \tau - \sum\limits_i \lambda_i & & a^T[x] \\ \hline & \sum\limits_{i=1}^{k} \lambda_i Q_i & A^T[x] \\ \hline a[x] & A[x] & \tau I \end{array}\right) \succeq 0, \quad P^*(V) = \begin{pmatrix} \rho[x^T \alpha^1 + \beta^1] \\ \vdots \\ \rho[x^T \alpha^R + \beta^R] \end{pmatrix},$$
$$\lambda_i \geq 0,\ i = 1, ..., k, \quad \tau \leq x^T \alpha^{\mathrm{n}} + \beta^{\mathrm{n}} + \mathrm{Tr}(SV), \quad Q^*(V) = 0, \quad V \succeq 0, \tag{36}$$

where $a[x], A[x]$ *are given by* (31), *is an approximate robust counterpart of the uncertain conic quadratic constraint* (27) – (30).

The level of conservativeness Ω of (36) can be bounded as follows:

Theorem 4 [7] (i) *In the case of a general-type ellipsoidal uncertainty* (29), *one has*

$$\Omega \leq \widetilde{\Omega} \equiv \sqrt{3.6 + 2 \ln \left(\sum_{i=1}^{k} \mathrm{Rank}(Q_i) \right)}. \tag{37}$$

(ii) *In the case of box uncertainty:* $\zeta^T Q_i \zeta = \zeta_i^2$, $1 \leq i \leq k = L \equiv \dim \zeta$, $\Omega \leq \frac{\pi}{2}$.

(iii) *In the case of simple* ($k = 1$) *ellipsoidal uncertainty* (29), $\Omega = 1$ – *The system* (36) *is equivalent to the robust counterpart of* (27) – (30).

References

[1] A. Ben-Tal, A. Nemirovski, "Stable Truss Topology Design via Semidefinite Programming" – *SIAM Journal on Optimization* **7** (1997), 991-1016.

[2] A. Ben-Tal, A. Nemirovski, "Robust solutions to uncertain linear programs" – *OR Letters* **25** (1999), 1-13.

[3] A. Ben-Tal, A. Nemirovski, "Robust Convex Optimization" – *Mathematics of Operations Research* **23** (1998).

[4] S. Boyd, L. El Ghaoui, F. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory* – volume 15 of *Studies in Applied Mathematics*, SIAM, Philadelphia, 1994.

[5] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, "Robust Semidefinite Programming" – in: R. Saigal, H. Wolkowicz, L. Vandenberghe, Eds. *Handbook on Semidefinite Programming*, Kluwer Academis Publishers, 2000, 139-162.

[6] A. Ben-Tal, A. Nemirovski, "On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty" – *SIAM J. on Optimization*, 2001, to appear.

[7] A. Ben-Tal, A. Nemirovski, C. Roos, "Robust solutions of uncertain quadratic and conic quadratic problems" – Research Report #2/01, Minerva Optimization Center, Technion – Israel Institute of Technology, Technion City, Haifa 32000, Israel. http://iew3.technion.ac.il:8080/subhome.phtml?/Home/research

[8] L. El-Ghaoui, H. Lebret, "Robust solutions to least-square problems with uncertain data matrices" – *SIAM J. of Matrix Anal. and Appl.* **18** (1997), 1035-1064.

[9] L. El-Ghaoui, F. Oustry, H. Lebret, "Robust solutions to uncertain semidefinite programs" – *SIAM J. on Optimization* **9** (1998), 33-52.

[10] M.X. Goemans, D.P. Williamson, "Improved approximation algorithms for Maximum Cut and Satisfiability problems using semidefinite programming" – *Journal of ACM* **42** (1995), 1115-1145.

[11] A. Nemirovski, "Several NP-hard problems arising in Robust Stability Analysis" – *Math. Control Signals Systems* **6** (1993), 99-105.

[12] A. Nemirovski, C. Roos, T. Terlaky, "On maximization of quadratic form over intersection of ellipsoids with common center" – *Mathematical Programming* **86** (2000), 463-473.

[13] Yu. Nesterov, "Semidefinite relaxation and non-convex quadratic optimization" – *Optimization Methods and Software* **12** (1997), 1-20.

[14] Yu. Nesterov, "Nonconvex quadratic optimization via conic relaxation", in: R. Saigal, H. Wolkowicz, L. Vandenberghe, Eds. *Handbook on Semidefinite Programming*, Kluwer Academis Publishers, 2000, 363-387.

[15] Y. Ye, "Approximating quadratic programming with bounds and quadratic constraints" – *Math. Programming* **47** (1999), 219-226.