UNIFORM STABILITY OF A COUPLED SYSTEM OF HYPERBOLIC/PARABOLIC PDE'S WITH INTERNAL DISSIPATION*

Francesca Bucci
Università degli Studi di Firenze
Dipartimento di Matematica Applicata "G. Sansone"
Via S. Marta 3, I-50139 Firenze, ITALY
fbucci@dma.unifi.it

Abstract We study the uniform stability of a coupled system of hyperbolic/parabolic partial differential equations (PDEs) with nonlinear internal dissipation. We analyze both the case of distributed damping on the entire domain, and the case of damping with localised support. In the corresponding stability results, decay rates of weak solutions to the PDE system under consideration are described via the solutions to appropriate nonlinear ordinary differential equations.

Keywords: coupled partial differential equations, uniform decay rates, saturation, locally distributed damping, multipliers' method.

Introduction

The problem of stabilization of coupled or interconnected systems of Partial Differential Equations (PDEs) has become in the last several years a central topic of mathematical control theory of infinite-dimensional systems [11, 9]. The present paper is focused on uniform stabilization of a coupled system of hyperbolic/parabolic PDEs, which is a nonlinear generalization of a PDE model originated in [6].

The mathematical model studied in [6] consists of a wave equation in a bounded domain Ω of $\mathbb{R}^3$ (the "acoustic chamber"), which is strongly coupled to a linear (abstract) structurally damped plate-like equation acting only on the elastic, flat wall of the chamber (the *interface*). A distinguished feature of this model is that it displays a boundary dissi-

*Research supported by the Italian Ministero dell'Istruzione, dell'Università e della Ricerca.

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35690-7_44

V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

pation term of the wave component which accounts for lack of uniform stability of the overall system, even in the presence of viscous damping on the entire domain (*overdamping* phenomenon). In [6] it has been shown that by introducing a comparable static damping in the boundary condition of the wave component, then the corresponding feedback system is (exponentially) uniformly stable.

In this paper we study the stability properties of that ultimate model when internal damping is subject to *nonlinear* effects, namely system (1.1) described in the next section. More precisely, we aim to estabilish uniform decay rates for the (natural) energy $E(t)$ of the corresponding weak solutions, as $t \to +\infty$. We shall examine both (i) the case of (nonlinear) damping distributed on the entire domain, and (ii) the more challenging case when internal damping is active in a thin layer near the interface.

In the former situation uniform decay rates of the underlying energy are obtained, without assuming *a priori* growth conditions on the nonlinear function F near the origin. Furthermore, we allow maximum polynomial growth of F at infinity (up to power five). Decay rates are described via the solution to an appropriate nonlinear ODE (see Theorem 2.2).

Unlike most literature, our analyis will include the case when the dissipation term is bounded at infinity (*saturation*). In this significant case uniform decay rates are achieved for the solutions with initial data which belong to a slightly smoother function space than the energy space [4]. We stress that the novelty of this second result is that

- we obtain uniform decay rates of the energy of solutions correponding to a bit smoother initial data, rather than of *strong* solutions (i.e. solutions corresponding to initial data in the domain of the dynamic operator), as done, e.g., in [14] in the case of dissipative wave equations;
- in addition, we still do not require any *a priori* assumption on the growth of F near the origin.

Next, if appropriate geometric conditions on Ω are in force (Hypothesis 3.1), by restricting the polynomial growth at infinity on the nonlinearity, we are able to show that a locally distributed damping near the interface is in fact sufficient to stabilize the system at a uniform decay rate (Theorem 3.6), thus completing the analysis of case (ii).

We note that these stability results obtained for the coupled PDE system (1.1) hold, as well, for the single, uncoupled wave equation (even without "overdamping" term). A technical comparison with the vast literature on stabilization of dissipative wave equations can be found in [4].

As a final comment we point out that it would be very interesting to study (infinite-dimensional) Hamilton-Jacobi-Bellman (HJB) Equations associated with optimal control problems for abstract equations in Banach spaces of the form $y' = Ay + Bu + F(y)$, when (a) the control operator B is *unbounded*, and (b) the operator $e^{At}B$ satisfies the so called "singular estimate" (*cf.* [1, 9, 12, 6]). Let us recall that in the special Linear Quadratic (LQ) case, recently it has been shown that in the framework defined by (a)-(b), namely if the singular estimate holds, yet in the absence of analyticity of the underlying semigroup, then well-posedness of associated Riccati equations, along with the soughtafter property that the gain operator is *bounded*, follow as well [12]. The question whether in the more general case of control problems for *semi-linear* equations subject to (a) and (b) (and with possibly non quadratic cost functionals) one can develop a theory of HJB equations closely akin to the one relative to the "analytic" class of systems with unbounded control operator—such as it arises in the context of nonlinear control problems for a parabolic PDE with boundary or point control [7, 8]—is still an open problem.

1. The PDE model

The PDE model under consideration in this paper is a nonlinear generalization of a coupled system of hyperbolic/parabolic PDEs studied in [6]. A brief mathematical description of this system is given below; see [6] for a thorough analysis of the linear problem.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where Γ_0 and Γ_1 are open, connected, disjoint parts, $\Gamma_0 \cap \Gamma_1 = \emptyset$ in $\mathbb{R}^2$, of positive measure. The sub-boundary Γ_0 is flat and is referred to as the elastic (or flexible) wall, while Γ_1 is referred to as the rigid (or hard) wall. At the outset, we assume that either Ω is sufficiently smooth (say, Γ is of class C^2), or else Ω is convex.

The acoustic medium in the chamber Ω is described by the wave equation in the variable z, while v represents the (abstract) deflection of the abstract *structurally damped* plate equation on Γ_0. The interaction between wave and plate takes place on Γ_0 (the *interface*).

Thus, our goal is to investigate the stability properties of the following system of coupled PDEs:

$$\begin{cases} z_{tt} = \Delta z - F(z_t) & \text{in } Q = (0,\infty)\times\Omega \\ \dfrac{\partial z}{\partial \nu} + d_1 z = 0 & \text{on } \Sigma_1 = (0,\infty)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} + D_0 z_t + \beta D_0 z = v_t & \text{on } \Sigma_0 = (0,\infty)\times\Gamma_0 \\ z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1 & \text{in } \Omega \\ v_{tt} + \mathcal{A}v + \rho\mathcal{A}^\alpha v_t + z_t|_{\Gamma_0} = 0 & \text{on } \Sigma_0 = (0,\infty)\times\Gamma_0 \\ v(0,\cdot) = v^0, \quad v_t(0,\cdot) = v^1 & \text{in } \Gamma_0\,. \end{cases} \tag{1.1}$$

Basic assumptions. The assumptions pertaining to the *linear* operators $\mathcal{A}$ and D_0 in (1.1) will be held throughout the paper and will not be mentioned explicitly any more.

(H0) $\mathcal{A}$ (the elastic operator): $L^2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \to L^2(\Gamma_0)$ is a positive, self-adjoint operator. Moreover, $\rho > 0$, $\beta \geq 0$, $d_1 > 0$ and $\frac{1}{2} \leq \alpha \leq 1$ are constants.

(H1) $D_0 : L^2(\Gamma_0) \supset \mathcal{D}(D_0) \to L^2(\Gamma_0)$ is a positive, self-adjoint operator, and there exists a constant r_0, $0 \leq r_0 \leq 1/4$, and positive constants δ_1, δ_2 such that

$$\delta_1 ||z||^2_{\mathcal{D}(\mathcal{A}^{r_0})} \leq (D_0 z, z)_{L^2(\Gamma_0)} \leq \delta_2 ||z||^2_{\mathcal{D}(\mathcal{A}^{r_0})} \qquad \forall\, z \in \mathcal{D}(\mathcal{A}^{r_0}) \subset \mathcal{D}(D_0^{1/2}). \tag{1.2}$$

Moreover, it is assumed that $H^1(\Gamma_0) \subseteq \mathcal{D}(D_0^{1/2})$; that is,

$$D_0^{1/2} : \text{continuous } H^1(\Gamma_0) \to L^2(\Gamma_0).$$

In particular, for simplicity of exposition, we shall examine only the case when D_0 is the realization of a differential operator of order s, with $1 < s \leq 2$. Accordingly, it is assumed that $\beta > 0$, see [6, Sections 1.1–1.2].

Instead, the mathematical features of the dissipation term $F(z_t)$ for the wave component will be made more precise in the different frameworks of model (1.1) that we are going to consider. We recall that a thorough analysis of stability properties of system (1.1) with *full viscous damping* (i.e. with $F(z_t) = z_t$), has been performed in [6]. Here, we shall examine first the case when (1.1) displays nonlinear damping distributed on the entire domain Ω, next the more challenging case of damping with localised support. Specific assumptions on the nonlinear function F and possible geometric constraints on Ω will be introduced correspondingly.

Function spaces. Before going into the heart of the stability issue, we need to introduce the natural state space for system (1.1). It is known from [6] that the function space Y_1 which is needed to describe the wave component of system (1.1) is given by $Y_1 := Z \times L^2(\Omega)$, with

$$Z := \left\{ f \in H^1(\Omega) : \quad f|_{\Gamma_0} \in \mathcal{D}(D_0^{1/2}) \right\}, \tag{1.3}$$

endowed with the norm

$$\|f\|_Z^2 := \|\nabla f\|_{L_2(\Omega)}^2 + \beta \|D_0^{1/2} f|_{\Gamma_0}\|_{L_2(\Gamma_0)}^2 \,. \tag{1.4}$$

Then, the state space Y for problem (1.1) is given by

$$Y := Z \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{1/2}) \times L^2(\Gamma_0) \,. \tag{1.5}$$

Thus, the energy $E(t)$ of weak solutions $[\vec{z}(t,\cdot), \vec{v}(t,\cdot)]$ to the coupled system (1.1) is defined by

$$\begin{aligned} E(t) &= E_z(t) + E_v(t)\,; && (1.6a)\\ E_z(t) &:= \|\nabla z(t)\|_{L_2(\Omega)}^2 + \beta \|D_0^{1/2} z(t)\|_{L_2(\Gamma_0)}^2 + \|z_t(t)\|_{L_2(\Omega)}^2 && (1.6b)\\ E_v(t) &:= \|\mathcal{A}^{1/2} v(t)\|_{L_2(\Gamma_0)}^2 + \|v_t(t)\|_{L_2(\Gamma_0)}^2 \,, && (1.6c) \end{aligned}$$

where $E_z(t)$ and $E_v(t)$ denote the wave and plate energy, respectively.

Remark 1.1. We note that due to space constraints, we shall omit from this article the description of the abstract set-up for problem (1.1) and the analysis of well-posedness of the corresponding nonlinear evolution equation (see [6, 4]).

Notation. In order to simplify the notation, hereafter the norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma_0)}$ will be denoted by $|\cdot|_{s,\Omega}$ and $|\cdot|_{s,\Gamma_0}$, respectively. In particular, the symbols $|\cdot|_{0,\Omega}$ and $|\cdot|_{0,\Gamma_0}$ will represent $\|\cdot\|_{L_2(\Omega)}$ and $\|\cdot\|_{L_2(\Gamma_0)}$, respectively.

2. Stabilization by nonlinear damping on the entire domain

This section is focused on the asymptotic behaviour of solutions to the coupled PDE system (1.1) with dissipation distributed on the entire domain Ω. Regarding the nonlinear function F, we shall assume that

Hypothesis 2.1. *F is a continuous function on the real line such that:*

(i) *F is monotone strictly increasing, $F(0) = 0$;*

(ii) *$ms^2 \leq s\,F(s) \leq Ms^6$ for $|s| \geq 1$, with $0 < m \leq M$.*

We recall that from part (i) of Hypothesis 2.1 it follows that there exists a real valued function $h(x)$ which is defined for $x \geq 0$, it is concave, strictly increasing, with $h(0) = 0$, and it satisfies

$$h(s\,F(s)) \geq s^2 + F^2(s)\,, \quad \text{for } |s| \leq N, \text{ for some } N > 0. \tag{2.1}$$

Such function can always be constructed, thanks to the monotonicity property assumed on F by Hypothesis 2.1, as explained in [10, p. 510]. With this function one first defines

$$\tilde{h}(x) = h\left(\frac{x}{|Q_{0T}|}\right), \quad x \geq 0\,, \tag{2.2}$$

where $|Q_{0T}|$ denotes the measure of $Q_{0T} = (0, T) \times Q$. Note that since $\tilde{h}$ is monotone increasing, $cI + \tilde{h}$ is invertible for any constant $c \geq 0$. Then, with K a positive constant, we set

$$p(x) := \left(cI + \tilde{h}\right)^{-1}(Kx)\,, \tag{2.3}$$

which is readily a continuous, positive, strictly increasing function, with $p(0) = 0$. Finally, let

$$q(x) := x - \left(I + p\right)^{-1}(x), \quad x > 0\,. \tag{2.4}$$

Then, decay rates of the energy $E(t)$ of weak solutions $[\vec{z}(t,\cdot), \vec{v}(t,\cdot)]$ to the PDE model (1.1), as defined by (1.6), are described by the following result.

Theorem 2.2. ([4]) *Assume Hypothesis* 2.1. *Then the energy $E(t)$ of every weak solution to the coupled system* (1.1) *decays uniformly to zero, as $t \to +\infty$. More precisely, there exists a $T_0 > 0$ such that*

$$E(t) \leq s(t/T_0 - 1) \quad \textit{for } t > T_0\,, \tag{2.5}$$

where $\lim_{t\to+\infty} s(t) = 0$ and $s(t)$ is the solution to the Ordinary Differential Equation

$$\begin{cases} s'(t) + q(s(t)) = 0 \\ s(0) = E(0)\,, \end{cases} \tag{2.6}$$

(*where q is as given in* (2.4)). *Here, the constant K in* (2.3) *will depend on $E(0)$ and time T_0, and the constant c (in* (2.3)) *depends on the measure of Q_{0T_0}.*

Remark 2.3. We note that we do not assume *a priori* any kind of growth condition on the nonlinearity F near the origin. Knowledge of the rate growth of the nonlinear function F at the origin allows to obtain more explicit decay rates of the energy (see [10, 9]).

The case of saturated feedback laws. Since the dissipation term $F(z_t)$ in (1.1) can be interpreted as a control in *feedback* form, it is both natural and of significance to include in the present analysis the case of feedback laws F subject to *saturation.* A typical example is given by $F(y) = \min\{1, k/|y|\}\, y$, with k a positive constant. However, in this case the lower growth condition in Hypothesis 2.1(ii) is not fulfilled and needs to be removed. To accomplish this goal, we aim to study the stability properties of the coupled PDE system (1.1) by simply assuming that

Hypothesis 2.4. *F is a continuous function on the real line such that*

(i) *F is (monotone) strictly increasing for $|s| \leq 1$, while it is non-decreasing for $|s| \geq 1$; $F(0) = 0$;*

(ii) *there exists $M > 0$ such that $0 < s\,F(s) \leq M s^6$ for $|s| \geq 1$.*

Under this weak condition we are able to show that decay rates of the energy $E(t)$ are still uniform, provided that initial data belong to a slightly smoother function space $W_\epsilon \subset Y$, rather than the energy space Y (see [4, Theorem 2.8]).

Remark 2.5. Due to space constraints, the precise statement of the aforementioned stability result will not be included in the present article. It would require the introduction of some preliminary material: first of all, the definition of the new function space W_ϵ and the corresponding new energy $E_1(t)$. In short, Theorem 2.8 in [4] estabilishes that if the nonlinear function F fulfils the weaker Hypothesis 2.4, then decays rates can be described as well via the solution $s_1(t)$ to a nonlinear ODE (as (2.6) of Theorem 2.2), depending this time on $E_1(0)$ instead of $E(0)$. The new key ingredient in the proof of this result is an estimate of the norm $|z_t|_{\epsilon,\Omega}$ (of solutions to (1.1) with initial data in W_ϵ), which can be shown by using linear and nonlinear interpolation methods (see [4]).

3. Stabilization by a locally distributed damping

In this section we consider the PDE system (1.1) with damping acting only on an arbitrary small layer around the interface Γ_0. Henceforth, we shall write in this case

$$F(z_t(t,x)) = d(x)\, g(z_t(t,x))\,, \tag{3.1}$$

where d is a nonnegative function on Ω which is active only in a neighbourhood of Γ_0, and g is a real function subject to appropriate conditions (see Hypothesis 3.5 below). Preliminarly, we recall that model (1.1) with *linear* localized damping (i.e. with $g(z_t) = z_t$ in (3.1)) has been studied in [5], in the case when the following geometric assumptions hold true (*cf.* [13]):

Hypothesis 3.1.

- *The domain Ω is convex and the following negatively star shaped condition is satisfied: there exists a point $x_0 \in \mathbb{R}^n$ such that*

$$(x - x_0) \cdot \nu \leq 0 \quad on\ \Gamma_1\,;$$

- $d(x) \equiv 1$ *on* $\tilde{\Omega} \subset \Omega$ *and* $\overline{\tilde{\Omega}} \supset \Gamma_0$.

Under these conditions it has been shown that the energy $E(t)$ of system (1.1), as defined by (1.6), decays exponentially to zero, as $t \to +\infty$. More precisely, the following stability result has been estabilished for system (1.1) with $F(z_t) = d(x)z_t$, which we rewrite here for readers' convenience:

$$\begin{cases} z_{tt} = \Delta z - d(x)z_t & \text{in } Q \\ \dfrac{\partial z}{\partial \nu} + d_1 z = 0 & \text{on } \Sigma_1 \\ \dfrac{\partial z}{\partial \nu} + D_0 z_t + \beta D_0 z = v_t & \text{on } \Sigma_0 \\ z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1 & \text{in } \Omega \\ v_{tt} + \mathcal{A}v + \rho\mathcal{A}^\alpha v_t + z_t|_{\Gamma_0} = 0 & \text{on } \Sigma_0 \\ v(0,\cdot) = v^0, \quad v_t(0,\cdot) = v^1 & \text{in } \Gamma_0\,. \end{cases} \tag{3.2}$$

Theorem 3.2. ([5]) *Assume Hypothesis* 3.1. *Then the energy $E(t)$ of every solution $[\vec{z}, \vec{v}]$ to the coupled system* (3.2) *decays exponentially to zero, as $t \to +\infty$, that is*

$$E(t) \leq Ce^{-\omega t}E(0)\,, \qquad t \geq 0\,, \tag{3.3}$$

for some positive constants C, ω. The constants C and ω do not depend on $E(0)$, but they depend on $\tilde{\Omega}$. More precisely, $C \to +\infty$ as $\beta \to 0$ or the area of the support of d (the "height" of $\tilde{\Omega}$) goes to zero.

Remark 3.3. The proof of Theorem 3.2 is contained in [5]. The exponential decay in (3.3) is achieved by showing the equivalent property that for T sufficiently large one has $E(T) \leq \xi E(0)$, with $0 < \xi < 1$. In turn, this property follows as a consequence of an appropriate integral estimate of the energy functional, whose proof is rather technical and requires several intermediate steps. The integral estimates leading to the final estimate are obtained by applying the well known multipliers' method. Here a key role is played by suitable multipliers which are constructed by using both appropriate cut-off functions and a non-radial vector field $\mathbf{h} \in [C^2(\overline{\Omega})]^n$ such that

$$\mathbf{h} \cdot \nu \leq 0 \quad \text{on } \Gamma_1 , \qquad \text{and} \qquad J(\mathbf{h}) > 0 \quad \text{in } \overline{\Omega} , \tag{3.4}$$

where $J(\mathbf{h})$ denotes the Jacobian matrix of $\mathbf{h}$. Existence of $\mathbf{h}$ with the features in (3.4) is guaranteed by Hypothesis 3.1 (*cf.* [13]).

Our present goal is to extend the previous result to a more general model with *nonlinear* localized damping. More precisely, we shall consider the PDE system

$$\begin{cases} z_{tt} = \Delta z - d(x) g(z_t) & \text{in } Q \\ \dfrac{\partial z}{\partial \nu} + d_1 z = 0 & \text{on } \Sigma_1 \\ \dfrac{\partial z}{\partial \nu} + D_0 z_t + \beta D_0 z = v_t & \text{on } \Sigma_0 \\ z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1 & \text{in } \Omega \\ v_{tt} + \mathcal{A} v + \rho \mathcal{A}^\alpha v_t + z_t|_{\Gamma_0} = 0 & \text{on } \Sigma_0 \\ v(0,\cdot) = v^0, \quad v_t(0,\cdot) = v^1 & \text{in } \Gamma_0 , \end{cases} \tag{3.5}$$

where $d(\cdot)$ is the characteristic function of the neighbourhood $\tilde{\Omega}$ of Γ_0 where the damping is active.

Remark 3.4. In the present case where the model displays *locally distributed* damping, we cannot expect to maintain the polynomial growth of the nonlinear term g allowed by Hypothesis 2.1 (up to power five). In fact, as in the case of similar models with *boundary* dissipation (see [10, 2], [9] and references therein) we need to require linear growth at infinity.

Let us assume that

Hypothesis 3.5. *g is a continuous function on the real line such that:*

(i) *g is monotone strictly increasing, $g(0) = 0$;*

(ii) *$ms^2 \leq s\,g(s) \leq Ms^2$ for $|s| \geq 1$, with $0 < m \leq M$.*

Then, the following generalization of Theorem 3.2 holds true.

Theorem 3.6. *Assume Hypotheses* 3.1 *and* 3.5. *Then the energy* $E(t)$ *of every weak solution to the coupled system* (3.5) *decays uniformly to zero, as* $t \to +\infty$. *Decay rates are estimated via the solution* $s(t)$ *to an appropriate nonlinear ODE, as described by* (2.5) *and* (2.6).

Proof of Theorem 3.6 (sketch). Here we give an outline of the proof of Theorem 3.6 for which complete details will be available in a separate paper.

A preliminary step: well-posedness. As a preliminary step in the proof of Theorem 3.6, one needs to consider system (3.5) as an abstract evolution equation of the form $y' = Ay$ in an appropriate Hilbert space Y, where A denotes the nonlinear dynamics operator. Then, well-posedness of this system can be shown by applying the theory of *nonlinear* semigroups ([3]) as done in [1, 4]. Moreover, the regularity of solutions corresponding to smooth initial data will contribute to justification of the computations which are performed next.

Basic approach. In order to obtain decay rates estimates of (finite energy) solutions to system (3.5), we shall follow once more the powerful method introduced by the authors of [10] in the study of semilinear wave equations with nonlinear boundary velocity feedbacks and applied subsequently to various linear and nonlinear coupled PDE models (see [9], providing numerous references). Following this approach, our final objective will be to achieve the nonlinear *functional* inequality (3.10) below. To accomplish this goal, a major role will be played by the choice of suitable multipliers, as described in Remark 3.3.

Energy identity. The starting point is to derive the usual energy identity which illustrates the fact that the system is dissipative.

With respect to the PDE system (3.5), *the following energy equality holds for all* s *and* T, *with* $0 \leq s \leq T$:

$$E(T)+2\int_0^T \left(|D_0^{1/2} z_t|^2_{0,\Gamma_0}+(d(x)g(z_t), z_t)_\Omega+\rho|\mathcal{A}^{\alpha/2} v_t|^2_{0,\Gamma_0}\right) \mathrm{d}t=E(s) \tag{3.6}$$

In particular, $E(T) \leq E(t)$, $\forall t \leq T$.

Estimate of the energy functional. Next, we seek to obtain an integral estimate of the energy functional on a finite time interval $[0, T]$. Initially, estimates are performed separately on each component of the system. The most difficult step is to obtain an estimate of the *wave* energy functional $\int_0^T E_z(t)\mathrm{d}t$. It is here where Hypothesis 3.1 is used in a crucial way. Then, by combining the integral inequalities pertaining to the plate and wave energy functionals, and by using Hypothesis 3.5, along with monotonicity and concavity properties of $\tilde{h}$ as defined by (2.2), one gets the following soughtafter estimate for the coupled system.

With respect to the total energy $E(t)$ of system (3.5) *as defined in* (1.6)*, the following inequality holds for all $T > 0$:*

$$\begin{aligned}\int_0^T E(t)dt \le C_1\big(E(0)+E(T)\big) + C_2\big[c_1 I + \\ +|Q_{0T}|\tilde{h}\big]\Big(\int_0^T \{|D_0^{1/2} z_t|^2_{0,\Gamma_0} + (d(x)g(z_t), z_t)_\Omega + \\ + \rho|\mathcal{A}^{\alpha/2} v_t|^2_{0,\Gamma_0}\}\, \mathrm{d}t\Big) + C_3\, lot(z,v)\,,\end{aligned} \tag{3.7}$$

where crucially C_1 does not depend on T, while the expression $lot(z,v)$ includes all terms which are below energy level. More precisely, for some constant C we have that

$$lot(z,v) \le C\int_0^T \Big(|z|^2_{0,\Omega} + |z|^2_{\frac12-\delta,\Gamma_0} + |v_t|^2_{-\frac12+\delta,\Gamma_0}\Big)\, \mathrm{d}t, \;\; 0<\delta<\frac12\,. \tag{3.8}$$

Absorption of lower order terms and final estimate. The lower order terms can be absorbed by means of a (by now) standard nonlinear compactness/uniqueness argument (see [9]). Then, by using the energy identity (and dissipativity property), the following estimate of the energy function is estabilished.

For T large enough, the energy $E(t)$ of every solution to system (3.5) *satisfies*

$$\begin{aligned}E(T) \le C_T(E(0))\big[c I + |\tilde{h}\big]\Big(\int_0^T \{|D_0^{1/2} z_t|^2_{0,\Gamma_0} + (d(x)g(z_t), z_t)_\Omega + \\ + \rho|\mathcal{A}^{\alpha/2} v_t|^2_{0,\Gamma_0}\}\, \mathrm{d}t\Big)\end{aligned} \tag{3.9}$$

where the constant $C_T(E(0))$ remains bounded for bounded values of $E(0)$.

Conclusion. By using the energy identity (3.6) in the right hand side of inequality (3.9), we finally attain

$$E(T) + p(E(T)) \le E(0)\,, \tag{3.10}$$

with the function p as defined by (2.3). We recall, in particular, that p is constructed in terms of $\tilde{h}$ (hence, in terms of h), which depends on the growth rate of g near the origin. Thus, conclusion follows by applying the general result given in [10, Lemma 3.3], here with $s_m = E(mT)$ (see [10, p. 532] and [2, p. 304]).

References

[1] Avalos, G. and Lasiecka, I. (1996). Differential Riccati equation for the active control of a problem in structural acoustics. *J. Optim. Theory Appl.*, **91**(3):695–728.

[2] Avalos, G. and Lasiecka, I. (1998). Uniform decay rates for solutions to a structural acoustic model with nonlinear dissipation. *Appl. Math. and Comp. Sci.*, **8**(2):287–312.

[3] Barbu, V. (1976). *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, Leyden.

[4] Bucci, F. Uniform decay rates of solutions to a system of coupled PDEs with nonlinear internal dissipation. *Advances in Differential Equations* (to appear).

[5] Bucci, F. and Lasiecka, I. (2002). Exponential decay rates for structural acoustic model with overdamping on the interface and boundary layer dissipation. *Applicable Analysis*, **81**(4):977–999.

[6] Bucci, F., Lasiecka, I., and Triggiani, Roberto. (2002). Singular estimates and uniform stability of coupled systems of hyperbolic/parabolic PDEs. *Abstr. Appl. Anal.*, **7**(4):169–236.

[7] Cannarsa, P., Gozzi, F., and Soner, H.M. (1993). A dynamic programming approach to nonlinear boundary control problems of parabolic type. *J. Funct. Anal.*, **117**(1):25–61.

[8] Cannarsa, P. and Tessitore, M.E.. (1996). Infinite-dimensional Hamilton-Jacobi equations and Dirichlet boundary control problems of parabolic type. *SIAM J. Control Optim.*, **34**(6):1831–1847.

[9] Lasiecka, I. (2002). *Mathematical Control Theory of Coupled PDEs*, CBMS-NSF Regional Conference Series in Applied Mathematics, **75**, SIAM, Philadelphia.

[10] Lasiecka, I. and Tataru, D. (1993). Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential and Integral Equations*, **6**(3):507–533.

[11] Lasiecka, I. and Triggiani, R. (2000). *Control Theory for Partial Differential Equations: Continuous and Approximation Theories, Vol. I: Abstract Parabolic Systems; Vol. II: Abstract Hyperbolic-like Systems over a Finite Time Horizon.* Encyclopedia of Mathematics and its Applications, Vol. **74-75**, Cambridge University Press.

[12] Lasiecka, I. and Triggiani, R. (2001). Optimal control and Algebraic Riccati Equations under singular estimates for $e^{At}B$ in the absence of analiticity. Part I: The stable case. In *Differential Equations and Control Theory*, Marcel Dekker Lecture Notes in Pure and Applied Mathematics, **225**:193–219.

[13] Lasiecka, I., Triggiani, R., and Zhang, X. (2000). Nonconservative wave equations with unobserved Neumann boundary conditions: global uniqueness and observability in one shot. *Contemporary Mathematics*, **268**:227–325.

[14] Martinez, P. and Vancostenoble, J. (2000). Exponential stability for the wave equation with weak nonmonotone damping, *Port. Math.*, **57**(3):285–310.