SINGULAR PERTURBATIONS OF HYPERBOLIC-PARABOLIC TYPE

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Abstract We study the bihavior of solutions of the problem $\varepsilon u''(t)+u'(t)+Au(t)+Bu(t) = f(t), u(0) = u_0, u'(0) = u_1$ in the Hilbert space H as $\varepsilon \to 0$, where A is linear, symmetric, continuous and strong positive operator and B is nonlinear lipschitzian operator.

Keywords: singular perturbations, hyperbolic equation, parabolic equation.

1. Statment of the problem

Let V and H be the real Hilbert spaces endowed with the norm $||\cdot||$ and $|\cdot|$ respectively such that $V \subset H \subset V'$, where every embeding is densely defined and continuous. By $(,)$ we denote the duality between V and V' and also the scalar product in H. Let $A : V \to V'$ be a linear, continuous, symmetric operator and

$$(Au, u) \geq \omega ||u||^2, \quad \forall u \in V, \, \omega > 0. \tag{1}$$

Let $B : H \to H$ be a nonlinear operator which satisfies the Lipschitz condition

$$|Bu - Bv| \leq L|u - v|, \quad \forall u, v \in H. \tag{2}$$

We shall study the behavior of the solutions of problem

$$\begin{cases} \varepsilon u''(t) + u'(t) + Au(t) + Bu(t) = f(t), \\ u(0) = u_0, \, u'(0) = u_1 \end{cases} \tag{P_ε}$$

as $\varepsilon \to 0$, where ε is a small positive parameter. Our aim is to show that $u \to v$ as $\varepsilon \to 0$, where v is the solution of the problem

$$\begin{cases} v'(t) + Av(t) + Bv(t) = f(t), \\ v(0) = u_0, \end{cases} \tag{P_0}$$

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The main tool in our approach is the relation between the solutions of problem (P_ε) and the coresponding solutions of problem (P_0) in the linear case. This relation will be defined below by formula (3) and it was inspired by the work [1].

2. Notations

For $k \in \mathbb{N}$, $p \in [1,\infty)$ and $(a,b) \subset (-\infty,+\infty)$ by $W^{k,p}(a,b;H)$ denote the usual Sobolev spaces of vectorial distributions $W^{k,p}(a,b;H) = \{f \in D'(a,b;H);\ u^{(l)} \in L^p(a,b;H), l = 0,1,\dots,k\}$ endowed with the norm

$$||f||_{W^{k,p}(a,b;H)} = \left(\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;H)}\right)^{1/p}.$$

For $k \in \mathbb{N}$, $W^{k,\infty}(a,b;H)$ is the Banach space with the norm

$$||f||_{W^{k,\infty}(a,b;H)} = \max_{0\le l\le k} ||f^{(l)}||_{L^\infty(a,b;H)}$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ define the following Banach spaces $W^{k,p}_s(a,b;H) = \{f : (a,b) \to H; e^{-st}f^{(l)} \in L^p(a,b;H), l = 0,1,\dots,k\}$ with norms

$$||f||_{W^{k,p}_s(a,b;H)} = ||e^{-st}f^{(l)}(\cdot)||_{W^{k,p}(a,b;H)}.$$

3. *A priori* estimates for solutions of the problem (P_ε)

At first we give the well-known existence theorems for the solutions of the problems (P_ε) and (P_0) [2].

Theorem A. *Suppose that $f \in W^{1,1}(0,T;H)$, $u_0 \in V, u_1 \in H$ and A and B satisfy conditions* (1) *and* (2), *then there exists a unique solution $u \in C(0,T;H) \cap L^\infty(0,T;V)$ of the problem (P_ε) which satisfies the following condition: $u' \in L^\infty(0,T;H) \cap C(0,T;V'), u'' \in L^\infty(0,T;V')$. If in addition $Au_0 \in H, u_1 \in V$, then $Au \in L^\infty(0,T;H), u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;H)$.*

Theorem B. *If $f \in W^{1,1}(0,T;H)$, $u_0 \in V$ and A and B satisfy the conditions* (1), (2), *then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) and the estimates*

$$\begin{aligned} |v(t)| &\le e^{Lt}(|u_0| + \int_0^t e^{-Lt}(|B0| + |f(\tau)|)d\tau), \\ |v'(t)| &\le Ce^{Lt}(|f(0)| + |Au_0 + Bu_0| + \int_0^t e^{-Lt}|f'(\tau)|d\tau), \end{aligned}$$

are true for $0 \le t \le T$.

Now we shall give the *a priori* estimates of the solutions of the problem (P_ε). Denote by

$$E_1(u,t) = \varepsilon|u'(t)| + |u(t)| + \Big(\varepsilon(Au(t),u(t))\Big)^{1/2} + \\ +\Big(\varepsilon\int_0^t |u'(\tau)|^2 d\tau\Big)^{1/2} + \Big(\int_0^t (Au(\tau),u(\tau))d\tau\Big)^{1/2}.$$

Lemma 1. *Let* $f \in W^{1,1}_{2L}(0,\infty;H)$, $u_0, u_1 \in H$, $Au_0, Au_1 \in H$ *and* A *and* B *satisfy the conditions* (1) *and* (2), *then for every solution of the problem* (P_ε) *the following estimates*

$$E_1(u,t) \leq Ce^{2Lt}\Big(\varepsilon|u_1|+|u_0|+(\varepsilon(Au_0,u_0))^{1/2}+||f||_{L^1_{2L}(0,\infty;H)}\Big),\ t\geq 0,$$

$$E_1(u',t) \leq Ce^{2Lt}\Big(|f(0)| + |u_1| + |Au_0| + (\varepsilon(Au_1,u_1))^{1/2}+ \\ +||f'||_{L^1_{2L}(0,\infty;H)}\Big),\ t \geq 0,$$

are true.

4. The limit of solutions of the problem (P_ε) as $\varepsilon \to 0$

The studying of behavior of solutions of the problem (P_ε) as $\varepsilon \to 0$ is based on two key points. The first is getting the estimates of solutions which was obtained in Lemma 1 and the second is the relation (3) between the solutions of the problems (P_ε) and (P_0). To establish the relation (3), at first we give some properties of the kernel $K(t,\tau)$ of transformation which realises this conection in the linear case, i. e. in the case when $B = 0$.

For $\varepsilon > 0$ denote

$$K(t,\tau) = \frac{1}{2\sqrt{\pi\varepsilon}}\Big(K_1(t,\tau) + 3K_2(t,\tau) - 2K_3(t,\tau)\Big),$$

where

$$K_1(t,\tau) = e^{\frac{3t-2\tau}{4\varepsilon}}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$
$$K_2(t,\tau) = e^{\frac{3t+6\tau}{4\varepsilon}}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$
$$K_3(t,\tau) = e^{\frac{\tau}{\varepsilon}}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

and $\lambda(s) = \int_s^\infty e^{-\eta^2}d\eta$.

Lemma 2. *The functoin $K(t,\tau)$ possesses the following properties:*

(i) $K \in C(\bar{R}_+ \times \bar{R}_+) \cap C^2(R_+ \times R_+)$;

(ii) $K_t(t,\tau) = \varepsilon K_{\tau\tau}(t,\tau) - K_\tau(t,\tau),\ t > 0,\ \tau > 0$;

(iii) $\varepsilon K_\tau(t,0) - K(t,0) = 0,\ t \geq 0$;

(iv) $K(0,\tau) = \frac{1}{2\varepsilon} e^{-\frac{\tau}{2\varepsilon}},\ \tau \geq 0$;

(v) *For every fixed $t \geq 0$ there exist constants $C_1(t,\varepsilon) > 0$, $C_2(t,\varepsilon) > 0$ such that*

$$
\begin{aligned}
|K(t,\tau)| &\leq C_1(t,\varepsilon)e^{-C_2(t)\tau/\varepsilon}, && \tau > 0;\\
|K_t(t,\tau)| &\leq C_1(t,\varepsilon)e^{-C_2(t)\tau/\varepsilon}, && \tau > 0;\\
|K_\tau(t,\tau)| &\leq C_1(t,\varepsilon)e^{-C_2(t)\tau/\varepsilon}, && \tau > 0;\\
|K_{\tau\tau}(t,\tau)| &\leq C_1(t,\varepsilon)e^{-C_2(t)\tau/\varepsilon}, && \tau > 0;
\end{aligned}
$$

(vi) $K(t,\tau) > 0,\ t \geq 0, \tau \geq 0$;

(vii) *For every $C \in \mathbb{R}$ and every $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ and $\varphi \in L^1_C(0,\infty;H)$ the relation*

$$\lim_{t\to 0} \int_0^\infty K(t,\tau)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$

in H is valid;

(viii) $\int_0^\infty K(t,\tau)d\tau = 1,\ t \geq 0$;

(ix) *Let $\rho \in C^1[0,\infty), \rho$ and ρ' be increasing functions and $|\rho(t)| \leq Me^{ct}, |\rho'(t)| \leq Me^{ct}, t \in [0,\infty)$, then there exist positive constants C_1 and C_2 such that*

$$\int_0^\infty K(t,\tau)|\rho(t) - \rho(\tau)|d\tau \leq C_1\sqrt{\varepsilon}e^{C_2 t},\ t > 0;$$

(x) *Let $C \geq 0$ and $f \in W^{1,\infty}_C(0,\infty : H)$, then*

$$\left|f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau\right|_H \leq C_1\sqrt{\varepsilon}e^{C_2 t},\ t \geq 0.$$

Theorem 1. *Let $A : D(A) \subset H \to H$ be a linear closed operator, $f \in L^\infty_C(0,\infty;H)$ (with real C). If u is a solution of the problem (P_ε)*

($B = 0$) such that $u \in W^{2,p}_C(0,\infty;H), 1 \le p \le \infty$, then the function v_0, which is defined by

$$v_0(t) = \int_0^\infty K(t,\tau)u(\tau)d\tau \tag{3}$$

is the solution of the problem

$$\begin{cases} v_0'(t) + Av_0(t) = F(t,\varepsilon), \\ v_0(0) = \varphi_\varepsilon, \end{cases} \tag{P_0}$$

where

$$F(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big(2e^{\frac{3t}{4\varepsilon}}\lambda\Big(\sqrt{t/\varepsilon}\Big) - \lambda\Big(\frac{1}{2}\sqrt{t/\varepsilon}\Big)\Big)u_1 + \int_0^\infty K(t,\tau)f(\tau)d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

From Lemmas 1, 2 and from Theorem 1 the main result follows.

Theorem 2. *Suppose that $f \in W^{1,\infty}_{2L}(0,\infty;H)$, $u_0, u_1 \in V, Au_0$, $Au_1 \in H$ and the operators A and B satisfy conditions* (1) *and* (2). *Then*

$$|u(t) - v(t)| \le C\sqrt{\varepsilon}, \quad 0 \le t \le T,$$

with C depending on $||f||_{W^{1,\infty}_{2L}(0,\infty;H)}, |u_0|, |u_1|, |Au_0|, |Au_1|, \omega$ and T.

Corollary 1. *Let $B = 0$, operator A satisfy condition* (1), *u_0, u_1, $Au_0, Au_1 \in V, A^2u_0, A^2u_1 \in H, f \in W^{2,\infty}_{2L}(0,\infty;H)$ and*

$$f(0) - u_1 - Au_0 = 0. \tag{4}$$

Then

$$|u'(t) - v'(t)| \le C\varepsilon^{1/2}, \quad 0 \le t \le T,$$
$$||u(t) - v(t)|| \le C\varepsilon^{1/2}, \quad 0 \le t \le T.$$

It means that if condition (4) is satisfied, then the problem (P_0) is regularly perturbed, otherwise the problem (P_0) is singularly perturbed.

References

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