RECURSIVE DECONVOLUTION: AN OVERVIEW OF SOME RECENT RESULTS

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Abstract We present some recent results on recursive solution of Volterra integral equations of first kind. The method which is used is suggested by control and game theory.

Keywords: Volterra integral equations, deconvolution, Abel equation.

1. Preliminaries

The deconvolution problem has a long history and many different aspects. We are concerned here with the deconvolution problem for *causal* systems, which in the time invariant case is the solution of

$$y = k * u$$

in the unknown u. Here y, k and u are defined for $t \geq 0$ and the *kernel* k is in general a distribution supported on $t \geq 0$.

Our study of the deconvolution problem uses ideas that arose in control theory. In this context, the problem is as follows: a linear system is given,

$$\dot{x} = Ax + Bu\,, \qquad y = Cx + Du\,, \qquad x(0) = 0\,. \tag{1}$$

We want to know whether a measured output y is produced by a unique input u. If this is the case than the system is called *left invertible* or *ideally observable* in the russian literature.

If the system is left invertible then we want to construct u as the output of a new system

$$\dot{\xi} = \tilde{A}\xi + \tilde{B}y\,, \qquad u = \tilde{C}\xi + \tilde{D}y\,. \tag{2}$$

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System (2) is the *inverse system* to (1).

We observe that the output y to system (1) is given by

$$y = k * u\,, \qquad K(t) = D\delta + Ce^{At}B$$

(δ denotes Dirac's delta). If D is invertible and the system is time invariant, then the construction of the inverse system is trivial, $\tilde{A} = A - BD^{-1}C$, $\tilde{B} = BD^{-1}$, $\tilde{C} = -D^{-1}C$, $\tilde{D} = D^{-1}$. If D is not invertible, and in the most important case $D = 0$, the construction of the inverse system leads to an ill posed problem and its study was most fruitful, since it led to the construction of the "geometric" theory of linear finite dimensional systems in the papers and books of Basile, Marro, Morse and Wonham and, stimulated by the ideas of Krasovski in [7], the construction of an iterative scheme for the approximate inversion by Osipov [8]. We present an overview of our results, obtained in the line of the book [8].

We note that the geometric theory concerns mostly finite dimensional systems. The examples in [14] shows that extensions of the geometric theory to distributed systems can only produce weak results.

We note that in many applications the output is read only on a finite time interval $[0, T]$, at discrete times $\tau_k = k\tau$, $\tau = T/N$ and the measures are corrupted by errors of known tolerance. Hence, available data are $\xi_k = y(k\tau) + \theta_k\,, \qquad |\theta_k| < h$.

We noted that, when $D = 0$, the problem is ill posed. Hence we relay on a penalization approach for the approximate inversion of the system. The penalization approach is performed at each step and, in fact, in the overall it leads to a "shift of the spectrum" of an operator. The method depends on the introduction of an additional parameter α and constructs functions

$$v = v_{\tau,\{\xi\},\alpha}$$

where $\{\xi\}$ is the vector of the measures. We want: **1)** at time t, $v(t)$ only depends on the measures taken at $\tau_k \leq t$; **2)** when τ, h and α converge to zero, while respecting suitable relations, v should converge to u in a suitable topology.

We observe that we cannot hope that v converges to u if τ, h and α converge to zero independently, since the problem is ill posed.

2. The key idea

Finite dimensional systems, both linear and non linear but with $C = I$ and $D = 0$, full state observations (and special cases of $C \neq I$), have been investigated in [8]. The key idea is to associate a "model" to the system:

$$\dot{w} = Aw + Bv\,, \qquad z = Cw\,. \tag{3}$$

and to choose v so to force z to track y. Hopefully, under suitable conditions, v will track u. A general analysis of system (1) in finite dimensional spaces and $C \neq I$ is in [2]. The proposed algorithm is as follows: the input v is piecewise continuous, updated at each step τ_k, $v(t) = v_k(t)$, $t \in [\tau_k, \tau_{k+1})$, defined by

$$v_k = \arg\min \left\{ ||Cw(\tau_{k+1}) - \xi_k||^2 + \alpha \int_{\tau_k}^{\tau_{k+1}} ||v(s)||^2 \, ds \right\}.$$

This same idea will be used to study also the case of Volterra integral equations.

We quote the papers [9, 10] for distributed systems in state space form.

3. Finite dimensional results

The class of left invertible finite dimensional systems (with $D = 0$) is geometrically characterized as those systems whose *maximal controllability subspace in* $\ker C$, denoted $\mathcal{R}_*$, *is* $\{0\}$.

The idea that we describe in sect. 2 was thoroughly analyzed in [2]. We proved:

Theorem 1 *If* $\ker CB = \{0\}$ *then* $\lim_{\alpha \to 0} \left[\lim_{\tau \to 0,\, h \to 0} v_{\tau, \{\xi\}, \alpha} \right] = u$. *More precisely we have the following result (where* $v = v_{\tau,\{\xi\},\alpha}$*).* **1)** *If the unknown input u is square integrable on* $[0, T]$ *then v converges to u in* $L^2(0,T)$*;* **2)** *if the unknown input u is of class* $W^{12}(0,T)$ *then v converges to u uniformly on* $[\sigma, T]$ *for every* $\sigma > 0$*, and it converges uniformly to u on* $[0,T]$ *if, furthermore,* $u(0) = 0$.

A system which satisfies condition $\ker CB = \{0\}$ is a "system of relative degree 1".

The class of left invertible systems is *larger* then the class of those systems for which $\ker CB = \{0\}$. The key instrument for the extension of the above result from the special case $\ker CB = \{0\}$ to the general case $\mathcal{R}_* = \{0\}$ is *Morse canonical form,* see [12]. We do not describe this complicated instrument here. We simply note that the use of this form reduce the problem to the case that the system is a chain of integrators, essentially a system of n_i–th order scalar equations, to which the method above can be applied step by step, as in the next example.

Example 2 Let

$$\dot{x}_1 = x_2\,, \qquad \dot{x}_2 = u\,, \qquad y = x_1\,.$$

In this case $CB = 0$ and Theorem 1 cannot be applied. But, we can associate $w_1 = \hat{x}_2$ to the first component $\dot{x}_1 = x_2$. We observe that $x_2 \in W^{1,2}$ and $x_2(0) = 0$ so that we can give a uniform estimate $\hat{x}_2(t)$ of $x_2(t)$. At time τ_k we can apply the procedure outlined above to the system $\dot{x}_2 = u$ and we can recursively identify the input u. ∎

3.1. An application

The results above have been applied to the reduction of the effect of a disturbance (internally or externally generated), in [1]. We present some simulations taken from this paper concerning the control of a robot motor, against (internally generated) disturbances, due to variations in the transported load.

The block diagram of a robot motor can be found in [6]. It is composed of two blocks: the main block is the motor itself whose output is the track (let it be X) followed by the robot. This is fed-back to an "acceleration controller" which also accept as the input the signal X^{cmd}, which is the track to be followed. It can easily be shown that the difference between the real path and the nominal path is

$$X - X^{cmd} = \frac{1}{M_0} \cdot \frac{1}{s^2 + K_1 s + K_2} v$$

where K_1 and K_2 are constants which enter in the definition of the "acceleration control". These constants are chosen in such a way to have an asymptotically stable system. The motor and its "acceleration controller" are designed on the basis of the choice of the nominal mass M_0 that the robot should carry. In this way, if the disturbance v is equal to 0, path tracking is achieved: the values of the path X (and of its first and second derivatives) coincide with the one of X^{cmd}. Errors on the initial condition, or impulsive disturbances, generate fast transients. A persistent disturbance v however is not canceled by the "acceleration controller".

We use the deconvolution ideas in order to *identify* and then to *cancel* the persistent disturbance due to changing loads to be transported. The results are in the plots below. The plots represent on the left X^{cmd} and X, when we apply our algorithm for disturbance reduction and, on the left, the errors $X^{\mathrm{cmd}} - X$ with and without compensation. in two extreme cases: the case that X^{cmd} is a sinusoidal path $(\sin \pi t)$ and the case that the path has an abrupt change.

Figure 1. Motor. $\tau = 0.01$, $\alpha = 1/85$, $h = 0.1$.

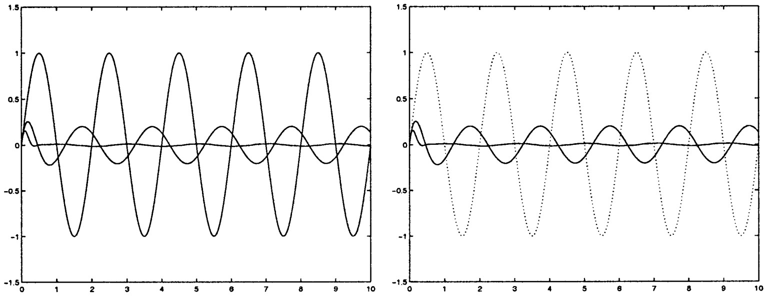
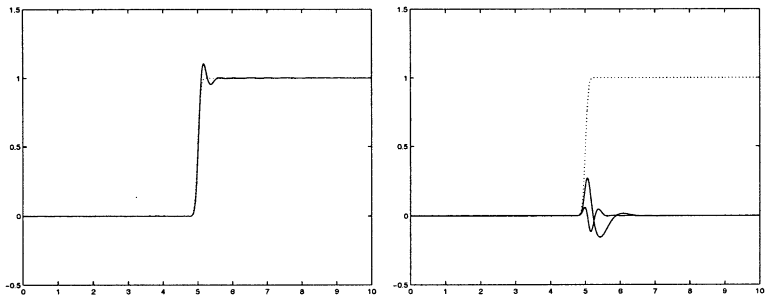


Figure 2. Motor. $\tau = 0.01$, $\alpha = 1/85$, $h = 0.1$.



4. Distributed systems

Applications to distributed systems of the previous ideas have been widely investigated by the Ekaterinburg school. An overview is in [13], see also the paper by Maksimov in these proceedings. As the geometric theory of linear systems seems not be extendible to distributed systems, the state space analysis at the moment gives weak results, which require full state observation, see [9, 10, 15], unless the system has a known special structure. See also the case of systems with delays examined in [11]. A special case of distributed input–output system is examined in [4], in the context of degenerate systems.

It is well known that the general deconvolution problem for distributed systems (in particular for the heath equation) is a very hard problem, due to the fact that the kernel may have a zero for $t \to 0+$, of infinite order. However, this does not happen in important cases. Even more, the kernel may be singular, as in the Abel equation

$$y(t) = \int_0^t \frac{1}{(t-s)^\gamma} u(s)\, \mathrm{d}s\,, 0 \le \gamma < 1$$

which is encountered, for example, in some input–output problem of heat transmission.

The solution of an Abel equation is a classical subject, see [5] and close formulas for the solution exist, which however require the computation of the derivative of an integral (fractional derivative). In spite of the fact that close formulas are always important, numerically these formulas contains redundancies since a part of the computed derivative is killed by the presence of the integral. For this reason in the next section we describe the results that we can obtain when applying the method outlined above to a class of integral equations which includes Abel equations.

We shall distinguish the case of convolution equations, i.e. the case that the kernel depends on the difference $t-s$ from the general case since, in the convolutional case, we can use powerful frequency domain techniques.

5. Volterra integral equations

We consider a Volterra integral equation

$$y(t) = \int_0^t K(t,s)u(s)\, \mathrm{d}s \qquad t \in [0,T]$$

and we want to solve for u, on the basis of observations taken on y at the time instants $\tau_k = kT/N$.

We represent

$$y(\tau_{k+1}) = y(\tau_k) + \int_0^\tau K(\tau_{k+1}, \tau_k + s)u(\tau_k + s)\, \mathrm{d}s + \int_0^{\tau_k} F(\tau_k, s)u(s)\, \mathrm{d}s\,, \tag{4}$$

where

$$F(t,s) = K(t+\tau, s) - K(t,s)$$

We choose now a piecewise constant function v,

$$v(t) = v_k\,, \qquad t \in [\tau_k, \tau_{k+1}).$$

We represent

$$\begin{aligned} w_{k+1} &= w_k + \int_0^\tau K(\tau_{k+1}, \tau_k + s)v(\tau_k + s)\, \mathrm{d}s + \int_0^{\tau_k} F(\tau_k, s)v(s)\, \mathrm{d}s \\ &= w_k + A_k v_k + \sum_{j=0}^{k-1} F_{k,j} v_j\,. \end{aligned} \tag{5}$$

Here

$$A_k = \int_0^\tau K(\tau_{k+1}, \tau_k + s)\, \mathrm{d}s\,, \qquad F_{k,j} = \int_{\tau_j}^{\tau_{j+1}} F(\tau_k, s)\, \mathrm{d}s\,.$$

We want a rule for choosing the constant value v_k at each time τ_k. As suggested by the finite dimensional case, We choose

$$v_k = \arg\min \left\{ ||w_k + A_k v - \xi_{k+1}||^2 + \alpha\tau ||v||^2 \right\}$$

i.e.

$$v_k = -[\alpha\tau I + A_k^* A_k]^{-1} A_k^* [w_k - \xi_{k+1}] \tag{6}$$

(we recall that ξ_k is the observation).

We shall prove that the piecewise constant function $v(t)$ so constructed approximates the unknown input u in the following two cases: the convolution case, under the assumptions described in sect. 7; the case that the kernel $K(t,s)$ is Lipschitz continuous in t, uniformly for $0 \le s \le t \le T$. If the kernel is merely continuous, with $\det K(t,t) \ne 0$ (see subsection 6 for the precise statement) we must replace the piecewise constant function v with the piecewise continuous function v defined by

$$v(t) = -\frac{1}{\alpha}[w(t) - \xi_k]\,, \qquad t \in [\tau_k, \tau_{k+1})\,. \tag{7}$$

With this definition, the candidate approximation v of u is constructed by

$$w(t) = -\frac{1}{\alpha} \int_0^t K(t,s)[w(s) - \xi(s)]\, \mathrm{d}s\,, \qquad v(t) = -\frac{1}{\alpha}[w(t) - \xi(t)]\,,$$

where

$$\xi(t) = \xi_k \qquad t \in [\tau_k, \tau_{k+1})\,. \tag{8}$$

We prove that this input v indeed approximates the unknown input u if τ, α and h converge to zero while respecting suitable conditions.

6. The nonconvolution equation

We state first the assumption on the kernel K:

Assumption 1. The kernel is a square $n \times n$ matrix, continuous for $0 \le s \le t \le T$ and satisfies

$$K(t,t) = I \qquad t \in [0,T]\,.$$

Moreover, $t \to K(t,s)$ is differentiable for *a.e.* $s \in [0,t]$ and $H(t,s) = K_t(t,s)$ satisfies

$$||H(t,s)|| \le L(s),\ 0 \le s \le t \le T\,,\ \lim_{h\to 0} \int_0^t ||K_t(t+h,s) - K_t(t,s)||^p\, \mathrm{d}s = 0 \tag{9}$$

where $L(s) \in L^p(0,T)$, $p > 1$.

If these conditions hold then the solution u is unique and we can show the following result, where $\gamma = (p-1)/p$ if $p < +\infty$, $\gamma = 1$ if $p = \infty$:

Theorem 3 *Let*

$$\tau,\ \alpha \text{ and } h \text{ converge to zero;} \qquad \lim \frac{\tau^\gamma}{\alpha} = 0\,, \quad \lim \frac{h}{\alpha} = 0\,. \tag{10}$$

Then: **1)** *If u is measurable and bounded then the sequence of the functions v converges to u in $L^p(0,T)$ for every $p \in [1,+\infty)$.* **2)** *If $u \in C(a,b)$ and $[a,b] \subseteq (0,T]$ then the convergence is uniform on $[a,b]$.* **3)** *if $u \in C(0,T)$ then for every $\sigma > 0$ the convergence is uniform on $[\sigma,T]$.* **4)** *If $u \in C(0,T)$ and if, furthermore, $u(0) = 0$ then the convergence is uniform on $[0,T]$.*

The first step in the proof of Theorem 3 is the proof that w tracks y. We sketch this part of the proof.

It is clear that, under **Assumption 1,** the output y is Hölder continuous: there exists a number $M_0 > 0$ and $\gamma \in (0,1]$ such that for $t \in [\tau_k, \tau_{k+1})$ we have $||y(t) - y(\tau_k)|| \le M_0\tau^\gamma$. Let us introduce the function $\phi(t) = \xi(t) - y(t)\,, \quad t \in [\tau_k,\tau_{k+1})\,, \qquad ||\phi(t)|| \le M_0(h + \tau^\gamma)$ (the function $\xi(t)$ is the one defined in (8)) so that

$$||w(t) - \xi(t)|| \le ||w(t) - y(t)|| + ||\phi(t)|| \le ||w(t) - y(t)|| + C(\tau^\gamma + h)\,.$$

Hence, v and w solve

$$v(t) = \tfrac{y(t)-w(t)}{\alpha} + \tfrac{\phi(t)}{\alpha}$$

$$y(t) - w(t) = -\tfrac{1}{\alpha}\int_0^t K(t,s)[y(s) - w(s)]\ \mathrm{d}s - \frac{1}{\alpha}\int_0^t K(t,s)\phi(s)\ \mathrm{d}s + y(t).$$

Let

$$e(t) = y(t) - w(t)\,.$$

We prove firstly an estimate for $e(t)$ in terms of the parameters τ, h and α.

The function e is a.e. differentiable, with

$$e'(t) = -\frac{1}{\alpha}e(t) - \frac{1}{\alpha}\int_0^t H(t,s)e(s)\ \mathrm{d}s - \frac{\phi(t)}{\alpha} - \frac{1}{\alpha}\int_0^t H(t,s)\phi(s)\ \mathrm{d}s + y'(t)\,.$$

It follows

$$\begin{aligned} e(t) \ &= \ \int_0^t e^{-(t-s)/\alpha}y'(s)\ \mathrm{d}s - \int_0^t e^{-(t-s)/\alpha}\frac{\phi(s)}{\alpha}\ \mathrm{d}s \\ &\quad - \int_0^t e^{-(t-s)/\alpha}\frac{1}{\alpha}\int_0^s H(s,r)e(r)\ \mathrm{d}r\ \ \mathrm{d}s \\ &\quad - \int_0^t e^{-(t-s)/\alpha}\frac{1}{\alpha}\int_0^s H(s,r)\phi(r)\ \mathrm{d}r\ \ \mathrm{d}s\,. \end{aligned}$$

We use now *boundedness* of u to obtain

$$||e(t)|| \leq \mathcal{C}[\alpha + \tau^\gamma + h] + \int_0^t L(r)||e(r)||\,\mathrm{d}r\,.$$

It follows:

$$0 \leq ||e(t)|| \leq z_\alpha(t) \leq \mathcal{M}[\alpha + \tau^\gamma + h]\,. \tag{11}$$

The constant $\mathcal{M}$ *does not* depend on τ, α and h. this implies that e converges to zero *uniformly* on $[0, T]$: Once that this is known, the proof of Theorem 3 is in two steps: we first prove that v converges weakly to u and then we use the compactness properties of the Volterra operator so to prove norm convergence.

6.1. Explicit convergence estimate

It is not possible to give convergence estimates without "a priori" information on the unknown input u. In order to give convergence estimates, we assume that u is Hölder continuous,

$$||u(t) - u(s)|| \leq \mathcal{C}|t - s|^\eta\,.$$

Furthermore we assume that K is of class C^1 on the triangle $0 \leq s \leq t \leq T$.

We have:

Theorem 4 *Let u be Hölder continuous of exponent η on $[0, T]$ and let K be Lipschitz continuous. There exists a number M such that for every $\sigma > 0$, τ, h, α the following convergence estimate holds:*

$$||v(t) - u(t)|| < M_\sigma \left\{\frac{\tau^\gamma + h}{\alpha} + \alpha^\eta + e^{-\sigma/\alpha}\right\}, \qquad t \in [\sigma, T]\,.$$

7. Convolution equations

We consider now the case that the kernel is a function of the difference of the arguments, $K(t, s) = K(t - s)$. In this case we assume that the kernel is *scalar* and *of class* $L^1(0, T)$. For simplicity of presentation, in order to have a continuous output, in this talk we assume u piecewise continuous. The general case of $u \in L^2(0, T)$ can be found in [3].

Experience with the previous cases suggests that now we can choose $v(t)$ as

$$v(t) = -\frac{w(\tau_k) - \xi_k}{\alpha} \qquad t \in [\tau_k, \tau_{k+1})\,.$$

We study directly the error between v and u. Hence now the *error function* is $e(t) = v(t) - u(t)$. It solves

$$\alpha e = -\alpha u + [K * u - K * v]^\tau + \theta^\tau \tag{12}$$

where $^\tau$ denotes sampling,

$$f^\tau(t) = f(\tau_k) \qquad t \in [\tau_k, \tau_{k+1})\,.$$

We want to compute the Laplace transform of both sides. We recall that we are working on a finite time interval. However, we can extend $K(t)$ to $[0, +\infty)$ so to have a Laplace transform and we can let $e(t)$ be defined by (12) for every $t > 0$. Let ν_C be the abscissa of convergence of the Laplace transform $\hat{K}(\lambda)$ of $K(t)$.

In order to compute the Laplace transform of both sides we need a formula for the transfer function from a *sample data input to the samples of a convolution.* Let $\hat{K}(\tau, \lambda)$ be such transfer function. It turns out that

$$\hat{K}(\tau, \lambda) = e^{-\lambda\tau} \sum_{n=0}^{+\infty} K_n e^{-\lambda\tau n}\,, \quad K_n = \int_{\tau n}^{\tau(n+1)} K(s)\, ds\,.$$

This is the fundamental object of our study.

We list now three conditions on the kernel K:

(HP1) there exist *positive* numbers γ_1, M_1 and $R > \nu_c$ such that $|\hat{K}(\lambda)| \le \frac{M_1}{|\lambda|^{\gamma_1}}$ for $|\lambda| > R$;

(HP2) there exist *positive* numbers γ_2, M_2 and $R > \nu_c$ such that $|\hat{K}(\lambda)| \ge \frac{M_2}{|\lambda|^{\gamma_2}}$. for $|\lambda| > R$.

(HP3) We assume that there is a sector $\mathcal{S}_{r,\theta} = \{\lambda \in \mathbf{C},\ |\lambda| < r,\ |\mathrm{Arg}\,\lambda| > \theta\}$ and a positive number $\nu_S \ge \nu_C$ such that $\Re e\,\lambda > \nu_S \Rightarrow \hat{K}(\lambda) \notin \mathcal{S}_{r,\theta}$.

Remark 5 We observe that the previous assumptions **(HP1)**—**(HP3)** are satisfied by a very large class of kernels, in particular Abel kernels $1/t^\gamma$, $[0 \le \gamma < 1)$ and piecewise regular kernels, as proved in [3]. The sector condition **(HP3)** has been formulated with a condition $\liminf_{t\to 0+} K(t) \ge 0$ in mind. In concrete applications it may require to work with $-K$ instead then with K. ∎

Conditions **(HP1)** and (**HP2**) justify the use of a formula originally due to Poisson, from which we obtain:

$$\hat{K}(\tau, \lambda) = \sum_{n=-\infty}^{n=+\infty} \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau + 2n\pi i}\right] \hat{K}\left(\lambda + \frac{2n\pi i}{\tau}\right)\,. \tag{13}$$

Formula (13) shows that $\hat{K}(\tau,\lambda)$ is *periodic of period* $2\pi i/\tau$ and moreover, on each horizontal strip $[2k\pi i/\tau, 2(k+1)\pi i/\tau)$ one of the term is dominant. In particular, in the strip $[-\pi i/\tau, \pi i/\tau)$ the dominant term is the one of index 0.

Combining the three assumptions on K we have:

Theorem 6 *Assume conditions* **(HP0)**, **(HP1)**, *and* **(HP3)**. *Then, there exist numbers* $\tau_0 > 0$, $L > 0$ *and a sector* $S_{\tilde{\tau},\tilde{\theta}}$ *such that if* $\tau \in (0,\tau_0)$ *and if* λ *is such that* $\Re e\,\lambda > \nu_S$, *then* $\hat{K}(\tau,\lambda) \in S_{\tilde{\tau},\tilde{\theta}} \implies \Re e\,\hat{K}(\tau,\lambda) > -L\tau^{\gamma_1}$.

This is the crucial result needed in the proof of the following consistency result:

Theorem 7 *Assume conditions* **(HP0)**–**(HP3)**. *Then, for every* $\epsilon > 0$, *there exist* α_0, τ_0, h_0 *such that*

$$||e_{\alpha_0,\tau,\xi}||_2 < \epsilon, \qquad \forall\tau < \tau_0\,,\ \forall\xi\ :\ ||\xi||_2 < h_0\,.$$

As usual, convergence estimates can be obtained provided that we have "a priori" informations on the regularity of the unknown input u. For example,

Theorem 8 *Assume conditions* **(HP0)**–**(HP3)** *and let* $u \in W^{1,2}(0,T)$. *Then, there exist* $M \geq 0$, δ_1, $\delta_2 > 0$ *such that*

$$||e_{\alpha,\tau,\xi}||_2 \leq M\left[\alpha^{\frac{1}{1+2\gamma_2}} + \frac{\sqrt{\tau}}{\alpha} + \frac{h}{\alpha}\right]$$

provided that τ, α, ξ : $\tau^{\gamma_1}/\alpha < \delta_1$, $\alpha < \delta_2$, $||\xi|| < h$.

Finally we note that in the proof of the previous results the fact that we are working on a finite time interval is explicitly used, in spite of the frequency domain nature of the method. The analysis of problems on $[0,+\infty)$ is still under study. We state a preliminary result which holds when condition **(HP2)** holds in the following more restrictive form:
(HP2+) There exist *positive* numbers γ_2 and $\tilde{M}_2$ such that

$$|\hat{K}(\lambda)| \geq \frac{\tilde{M}_2}{1+|\lambda|^{\gamma_2}}\,.$$

for any λ such that $\Re e\,\lambda > 0$. We have:

Theorem 9 *Let* $K \in L^1(0,+\infty)$ *and assume conditions* **(HP1)**, **(HP2+)**, *and* **(HP3)**. *Assume moreover,* $\nu_S = 0$. *Then, for every* $\epsilon > 0$, *there exist* α_0, τ_0, h_0 *such that*

$$||e_{\alpha_0,\tau,\xi}||_{L^2(0,+\infty)} < \epsilon, \qquad \forall\tau < \tau_0\,,\ \forall\xi\ :\ ||\xi||_2 < h_0\,.$$

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