# ON SOME CASES OF CAUCHY PROBLEM

Silvia–Otilia Corduneanu
*Department of Mathematics*
*Gh. Asachi Technical University of Iaşi*
*11 Copou Blvd., RO-6600 Iaşi, Romania*
silvcord@math.tuiasi.ro

**Abstract** The study of almost periodic measures depending on the parameter $t \in \mathbb{R}$ suggested to consider some cases of Cauchy problem. The solutions are functions belonging to $\mathcal{C}^1(\mathbb{R}, ap(G))$, where $ap(G)$ is the space of almost periodic measures on a locally compact abelian group $G$. A generalization of these cases is given by replacing $ap(G)$ with a locally convex space.

**Keywords:** Cauchy problem, almost periodic measure, almost periodic function.

## 1. Introduction

Let $X$ be a locally convex space, $Y$ a Banach algebra and $\bullet : Y \times X \to X$ a bilinear mapping. Consider $y \in Y$ and the linear operator $A : X \to X$, where $Ax = y \bullet x,\ x \in X$. In our paper we give hypotheses for solving the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \quad t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u_0 \in X$ and $u \in \mathcal{C}^1(\mathbb{R}, X)$. We also give some applications for functions which have values in the space of almost periodic measures $ap(G)$. The set $ap(G)$, of all almost periodic measures on a locally compact abelian group $G$, is a locally convex space with respect to a topology which is called *the product topology.* If $\nu$ is a bounded measure and $\mu$ is an almost periodic measure, their convolution, $\nu * \mu$, is also an almost periodic measure. These considerations allow us to discuss the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = \nu * u(t), \quad t \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Here $u \in \mathcal{C}^1(\mathbb{R}, ap(G))$, $\nu$ is a bounded measure on $G$ and $u_0 \in ap(G)$.

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35690-7_44

V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

## 2. Preliminaries

Consider a Hausdorff locally compact abelian group $G$ and let $\lambda$ be the Haar measure on $G$. Let us denote by $\mathcal{C}(G)$ the set of all bounded continuous complex-valued functions on $G$, and by $\mathcal{C}_U(G)$ the subset of $\mathcal{C}(G)$ containing the uniformly continuous functions. The sets $\mathcal{C}(G)$ and $\mathcal{C}_U(G)$ are Banach algebras endowed with the supremum norm. Throughout this paper, $\| \cdot \|_u$ denotes the supremum norm on $\mathcal{C}(G)$. For $f \in \mathcal{C}(G)$ and $a \in G$, the translate of $f$ by a is the function $f_a(x) = f(xa)$ for all $x \in G$. Denote by $K(G)$ the linear space of all continuous complex-valued functions on $G$, having a compact support. We denote by $m(G)$ the space of complex Radon measures on $G$. That is, the space of all complex linear functionals $\mu$ on $K(G)$ satisfying the following: for each compact subset $A$ of $G$ there exists a positive number $m_{\mu,A}$ such that $| \mu(f) | \leq m_{\mu,A}\|f\|_u$ whenever $f \in K(G)$ and the support of $f$ is contained in $A$. We use $m_F(G)$ to denote the subspace of $m(G)$ consisting of all bounded measures, i.e. all linear functionals which are continuous with respect to the supremum norm on $K(G)$. The action of a measure $\mu \in m(G)$ on a function $f \in K(G)$ will be denoted either $\mu(f)$ or $\int_G f(x)d\mu(x)$. Corresponding to a measure $\mu \in m(G)$, one defines the variation measure $| \mu | \in m(G)$ by $|\mu|(f) = \sup\{|\mu(g)| : g \in K(G),\ |g| \leq f\}$ for all $f \in K(G), f \geq 0$. For the Borel functions $f, g$ and the measures $\mu, \nu \in m(G)$, we can define their convolutions $f * g$, $f * \mu$, $\nu * \mu$, when that is possible. A translation-bounded measure is a measure $\mu \in m(G)$ with the property that for every compact set $A \subseteq G$, $m_\mu(A) = \sup_{x \in G} |\mu| (xA) < \infty$ (L. N. Argabright and J. G. Lamadrid [1974]). The linear space of the translation-bounded measures will be denoted by $m_B(G)$. We identify an arbitrary measure $\mu \in m_B(G)$ with an element of the space $[\mathcal{C}_U(G)]^{K(G)}$ in the following way: $\mu \equiv \{f * \mu\}_{f \in K(G)}$. From this identification we have the inclusion $m_B(G) \subset [\mathcal{C}_U(G)]^{K(G)}$. The space $[\mathcal{C}_U(G)]^{K(G)}$ has the *product topology* defined by the Banach space structure on $\mathcal{C}_U(G)$, hence, $m_B(G)$ is a locally convex space of measures with the relative topology. A system of seminorms for the product topology on $m_B(G)$ is given by the family $\{\| \cdot \|_f\}_{f \in K(G)}$, where, for a function $f \in K(G)$, $\|\mu\|_f = \|f * \mu\|_u$, for all $\mu \in m_B(G)$. Next we give the definition of an almost periodic function (see E. Hewitt and K. A. Ross [1963]).

**Definition 2.1** A function $g \in \mathcal{C}(G)$ is called an almost periodic function on $G$, if the family of translates of $g$, $\{g_a : a \in G\}$ is relatively compact in the sense of uniform convergence on $G$.

The set $AP(G)$ of all almost periodic functions on $G$ is a Banach algebra with respect to the supremum norm, closed to conjugation. Denote by $\hat{G}$ the dual of $G$ and by $[\hat{G}]$ the linear space generated by $\hat{G}$ in $\mathcal{C}(G)$. It is easy to see that $[\hat{G}] \subset AP(G)$. The almost periodic measures are introduced and studied by L. N. Argabright and J. G. Lamadrid (J. G. Lamadrid [1973], L. N. Argabright and J. G. Lamadrid [1990]).

**Definition 2.2** The measure $\mu \in m_B(G)$ is said to be an almost periodic measure, if for every $f \in K(G)$, $f * \mu \in AP(G)$.

The set $ap(G)$ of all almost periodic measures is a locally convex space with respect to the product topology. If $\nu \in m_F(G)$, $\mu \in ap(G)$ then $\nu * \mu \in ap(G)$. Also, if $f \in AP(G)$ and $\mu \in ap(G)$, then the measure $f\mu$, defined by $f\mu(g) = \mu(gf)$, $g \in K(G)$ is an almost periodic measure (see J. G. Lamadrid [1973]). It is also proved that there exists a unique linear functional $M : ap(G) \to C$ such that $M$ is continuous on $ap(G)$, $M(\mu) = M(\delta_x * \mu)$, for all $x \in G$, $\mu \in ap(G)$ and $M(\lambda) = 1$ (see J. G. Lamadrid [1973]). We used the notation $\delta_x$ for the the Dirac measure at $x \in G$. If $\mu \in ap(G)$ we define the mean value of $\mu$ as being the above complex number $M(\mu)$.

## 3. A Cauchy problem

The framework of this section has been suggested by some concrete cases of the Cauchy problem, which are presented in the next section. Let $X$ be a locally convex space with a sufficient family of seminorms $\mathcal{P}$, $(Y, \|\cdot\|)$ a commutative Banach algebra with the unit element denoted by $\theta$. We also consider the Banach algebra $\mathcal{B}(Y)$ of bounded linear operators on $Y$ and a bilinear mapping $\bullet : Y \times X \to X$ such that

$$\begin{cases} (C_1) & \theta \bullet x = x, \quad \forall x \in X, \\ (C_2) & y \bullet (z \bullet x) = yz \bullet x, \quad \forall y, z \in Y, \quad \forall x \in X, \\ (C_3) & \forall p \in \mathcal{P}, \quad p(y \bullet x) \le \|y\|\, p(x), \quad \forall y \in Y, \quad \forall x \in X. \end{cases}$$

The following proposition contains simple consequences of the properties regarding the series in commutative Banach algebras.

**Proposition 3.1** *Consider $y \in Y$. Then for every $t \in \mathbb{R}$, the sequence $\theta + t\frac{y}{1!} + \cdots + t^n \frac{y^n}{n!}$ is a Cauchy sequence in the Banach space $Y$ and the limit, which is denoted by $e^{ty}$, has the following properties:*

(a) $e^{ty} = \theta \quad for \quad t = 0;$

(b) $e^{(t+s)y} = e^{ty} e^{sy} \quad for \quad t, \ s \in \mathbb{R};$

(c) $\lim\limits_{t \to 0} \|e^{ty} - \theta\| = 0;$

(d) $\lim\limits_{t \to 0} \|\frac{e^{ty} - \theta}{t} - y\| = 0.$

Next we consider $y \in Y$ and we associate to $y$ the bounded linear operator $A \in \mathcal{B}(Y)$, $Az = yz$, $z \in Y$ and the family of bounded linear operators depending on parameter $t \in \mathbb{R}$, $T(t)$, where for all $t \in \mathbb{R}$, $T(t)z = e^{ty}z$, $z \in Y$. Then $T(t)$, $t \in \mathbb{R}$ is a group of bounded linear operators on $Y$ and $A$ is its generator.

**Proposition 3.2** *Consider the Cauchy problem*

$$\begin{cases} \dfrac{du(t)}{dt} = yu(t), & t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{3.3}$$

*where $y, u_0 \in Y$ and $u \in \mathcal{C}^1(\mathbb{R}, Y)$ is the unknown function. Then there exists a unique solution and that is $u(t) = e^{ty}u_0$, $t \in \mathbb{R}$.*

**Proof.** We observe that if we consider the function $A : Y \to Y$, $A(z) = yz$ we have that $A \in \mathcal{B}(Y)$, therefore, according to the general theory (see J. A. Goldstein [1985]), the solution of (3.3) is $u(t) = T(t)u_0$, $t \in \mathbb{R}$, where $T(t)$, $t \in \mathbb{R}$ is the group associated to $A$. In fact we can represent the solution in the form $u(t) = e^{ty}u_0$, $t \in \mathbb{R}$. ■

A function $u \in \mathcal{C}^1(\mathbb{R}, X)$ is called solution of the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = y \bullet u(t), & t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{3.4}$$

if $u$ satisfies the differential equation and the initial condition, which form this problem. The derivative $\dfrac{du}{dt}(t)$ of the function $u : \mathbb{R} \to X$ is defined taking the limit of differences quotient in the topology of the locally convex space $X$. We say that $u \in \mathcal{C}^1(\mathbb{R}, X)$ if there exists the derivative $\dfrac{du}{dt} : \mathbb{R} \to X$ and this is a continuous function.

**Lemma 3.1** *Consider the functions $u \in \mathcal{C}^1(\mathbb{R}, Y)$, $v \in \mathcal{C}^1(\mathbb{R}, X)$. Then we have the formula*

$$\frac{d}{dt}[u(t) \bullet v(t)] = u'(t) \bullet v(t) + u(t) \bullet v'(t), \quad t \in \mathbb{R}. \tag{3.5}$$

**Proof.** Let $p \in \mathcal{P}$ and $t \in \mathbb{R}$. Using $(C_3)$ it follows that for all $h \in \mathbb{R}$, $h \neq 0$ we have

$$p\left[\frac{u(t+h)\bullet v(t+h)-u(t)\bullet v(t)}{h}-u'(t)\bullet v(t)-u(t)\bullet v'(t)\right]$$
$$\leq \|\frac{u(t+h)-u(t)}{h}-u'(t)\|\; p[v(t+h)]$$
$$+\|u(t)\|\; p\left[\frac{v(t+h)-v(t)}{h}-v'(t)\right]+\|u'(t)\|\; p[v(t+h)-v(t)].$$

Therefore we obtain

$$\lim_{h\to 0} p\left[\frac{u(t+h)\bullet v(t+h)-u(t)\bullet v(t)}{h}-u'(t)\bullet v(t)-u(t)\bullet v'(t)\right]=0.$$

Consequently, formula (3.5) has been proved. ∎

**Theorem 3.1** *Let $y\in Y$ and consider the family of linear operators on $X$, $T(t)$, $t\in\mathbb{R}$, where for every $t\in\mathbb{R}$, $T(t)x=e^{ty}\bullet x$, $x\in X$. Then for all $x\in X$ we have that*

$$\frac{d}{dt}T(t)x=y\bullet T(t)x, \quad t\in\mathbb{R}. \tag{3.6}$$

**Proof.** Let $p$ be in $\mathcal{P}$, $t\in\mathbb{R}$ and $x\in X$. For all $h\in\mathbb{R}$, $h\neq 0$ we have

$$p\left(\frac{T(t+h)x-T(t)x}{h}-y\bullet T(t)x\right)=p\left(\frac{e^{(t+h)y}\bullet x-e^{ty}\bullet x}{h}-ye^{ty}\bullet x\right)$$
$$\leq \|\frac{e^{(t+h)y}-e^{ty}}{h}-ye^{ty}\|\; p(x)\;\leq \|e^{ty}\|\;\|\frac{e^{hy}-\theta}{h}-y\|\; p(x).$$

Therefore

$$\lim_{h\to 0} p\left(\frac{T(t+h)x-T(t)x}{h}-y\bullet T(t)x\right)=0, \text{ for all } p\in\mathcal{P}$$

and this shows (3.6). ∎

**Corollary 3.1** *Consider $y\in Y, u_0\in X$ and the Cauchy problem*

$$\begin{cases}\dfrac{du(t)}{dt}=y\bullet u(t), & t\in\mathbb{R},\\ u(0)=u_0,\end{cases} \tag{3.7}$$

*where $u\in\mathcal{C}^1(\mathbb{R},X)$ is the unknown function. Then* (3.7) *has a unique solution $u\in\mathcal{C}^1(\mathbb{R},X)$ and $u(t)=e^{ty}\bullet u_0$, $t\in\mathbb{R}$.*

**Proof.** From Theorem 3.1 it follows that the function $u(t)=e^{yt}\bullet u_0$, $t\in\mathbb{R}$ verifies the differential equation contained in (3.7) and from Proposition 3.1 it results that this function satisfies the initial condition. Consider $t>0$. We now prove that if $u\in\mathcal{C}^1(\mathbb{R},X)$ is a solution of (3.7) then the function $s \in [0,t] \to e^{(t-s)y} \bullet u(s) \in X$ has the property that

$$\frac{d}{ds}[e^{(t-s)y} \bullet u(s)] = 0_X, \quad s \in [0,t]. \tag{3.8}$$

From Lemma 3.1 and Proposition 3.1 it results that

$$\frac{d}{ds}[e^{(t-s)y} \bullet u(s)] = -ye^{(t-s)y} \bullet u(s) + e^{(t-s)y} \bullet u'(s), \quad s \in [0,t]. \tag{3.9}$$

Combining (3.7) and (3.9) we obtain (3.8). Taking into account that $\mathcal{P}$ is a sufficient family of seminorms on $X$ it results that function $s \in [0,t] \to e^{(t-s)y} \bullet u(s) \in X$ is constant. The equality $u(0) = u(t)$ gives us that $u(t) = e^{ty} \bullet u_0$, and from the fact that $t > 0$ is arbitrary we obtain $u(t) = e^{ty} \bullet u_0$, $t \in [0,\infty)$. Finally, similar arguments for $t < 0$ lead us to the conclusion that $u(t) = e^{ty} \bullet u_0$, $t \in \mathbb{R}$. ∎

## 4. Particular cases of Cauchy problem

In this section we present some particular cases of the Cauchy problem (3.7). In the first case $Y$ is the Banach algebra $m_F(G)$ of bounded measures on a locally compact abelian group $G$. We remind that $m_F(G)$ is a commutative Banach algebra with respect the usual norm of bounded measures which is denoted by $\|\cdot\|$. The third operation of $m_F(G)$ is the convolution of bounded measures and the unit element is the Dirac measure $\delta_e \in m_F(G)$, $e$ being the unit element of the group $G$. On the other hand $X$ is the locally convex space $\left(ap(G), \{\|\cdot\|_f\}_{f\in K(G)}\right)$ of all almost periodic measures on $G$. The bilinear mapping $\bullet : Y \times X \to X$ is given by the convolution between a bounded measure and an almost periodic measure which is an almost periodic measure. We start by giving the following lemma which will play an imporant role in proving Corollary 4.1.

**Lemma 4.1** *If $\nu \in m_F(G)$, $\mu \in ap(G)$ and $f \in K(G)$ we have that*

$$\|\nu * \mu\|_f \leq \|\mu\|_f \|\nu\|.$$

**Proof.** We notice that

$$\|\nu * \mu\|_f = \|f * \nu * \mu\|_u = \sup_{x\in G} \left| \int_G f * \mu(xy^{-1}) d\nu(y) \right| .$$

Consider $x \in G$. Well-known properties of the integral yield

$$\left| \int_G f*\mu(xy^{-1})d\nu(y)\right| \leq \int_G |f*\mu|(xy^{-1})d|\nu|(y) \leq \|f*\mu\|_u\|\nu\| = \|\mu\|_f\|\nu\|.$$

Hence $||\nu * \mu||_f \leq ||\mu||_f ||\nu||$. ■

**Corollary 4.1** *Consider $\nu \in m_F(G)$, $u_0 \in ap(G)$ and the Cauchy problem*

$$\begin{cases} \dfrac{du(t)}{dt} = \nu * u(t), & t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{4.10}$$

*where $u \in \mathcal{C}^1(\mathbb{R}, ap(G))$ is an unknown function. Then* (4.10) *has a unique solution and this is $u \in \mathcal{C}^1(\mathbb{R}, ap(G))$, $u(t) = e^{t\nu} * u_0$, $t \in \mathbb{R}$.*

**Proof.** The properties of the convolution between a bounded measure and an almost periodic measure and Lemma 4.1 enable us to see that the bilinear mapping $* : m_F(G) \times ap(G) \to ap(G)$, satisfies conditions $(C_1)$, $(C_2)$, $(C_3)$ from the preceding section. So, the Cauchy problem (4.10) is a particular case of (3.7). In what follows we can apply Corollary 3.1 and we find the conclusion. ■

**Remark 4.1** In the previous corollary we can replace the locally convex space $ap(G)$ by the Banach space $AP(G)$ and we obtain, in the same manner, a similar result. So, if $\nu \in m_F(G)$, $u_0 \in AP(G)$, then the Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = \nu * u(t), & t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{4.11}$$

has a unique solution and this is $u \in \mathcal{C}^1(\mathbb{R}, AP(G))$, $u(t) = e^{t\nu} * u_0$, $t \in \mathbb{R}$.

In the second case $Y$ is the commutative Banach algebra $AP(G)$ of all almost periodic functions on $G$ and $X$ is the locally convex space of all almost periodic measures on $G$, $\left(ap(G), \{||\cdot||\}_{f \in K(G)}\right)$. The unit element of the $AP(G)$ is the constant function denoted by $\mathbf{1}$, which takes the value 1 for all $x \in G$. This time, the bilinear mapping associates to an almost periodic function $g \in AP(G)$ and an almost periodic measure $\mu \in ap(G)$ the almost periodic measure having the density $g$ and the base $\mu$.

**Lemma 4.2** *If $g \in AP(G), \mu \in ap(G)$ and $f \in K(G)$ we have that*

$$||g\mu||_f \leq ||g||_u \, ||\,|f| * |\mu|\,||_u.$$

**Proof.** We have that $||g\mu||_f = ||f * g\mu||_u = \sup_{x \in G} \left| \int_G f(xy^{-1}) g(y) d\mu(y) \right|$.

Consider $x \in G$. Clearly, the following inequalities hold

$$\left| \int_G f(xy^{-1}) g(y) d\mu(y) \right| \leq \int_G |f(xy^{-1})| \, |g(y)| \, d|\mu|(y)$$
$$\leq ||g||_u \int_G |f(xy^{-1})| \, d|\mu|(y) \leq ||g||_u ||\,|f| * |\mu|\,||_u.$$

Therefore $\|g\mu\|_f \leq \|g\|_u \, \| \, |f| * |\mu| \, \|_u$. ■

**Corollary 4.2** *Consider $g \in AP(G)$, $u_0 \in ap(G)$ and the Cauchy problem*

$$\begin{cases} \dfrac{du(t)}{dt} = gu(t), \quad t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \tag{4.12}$$

*where $u \in C^1(\mathbb{R}, ap(G))$ is an unknown function. Then* (4.12) *has a unique solution $u \in C^1(\mathbb{R}, ap(G))$ and $u(t) = e^{tg}u_0$, $t \in \mathbb{R}$.*

**Proof.** We consider the bilinear mapping which associates to an almost periodic function $g \in AP(G)$ and an almost periodic measure $\mu \in ap(G)$ the almost periodic measure having the density $g$ and the base $\mu$. It is easy to see that this bilinear mapping verifies $(C_1)$, $(C_2)$ but does not satisfy $(C_3)$. However, there exists a unique solution and this is $u \in C^1(\mathbb{R}, ap(G))$, $u(t)=e^{tg}u_0$, $t \in \mathbb{R}$. First let us see that, by Proposition 3.1 and Lemma 4.2 it results that for all $f \in K(G)$, $t \in \mathbb{R}$, $h \neq 0$ we have

$$\left\| \frac{e^{(t+h)g}u_0 - e^{tg}u_0}{h} - ge^{tg}u_0 \right\|_f \leq \|e^{tg}\|_u \left\| \frac{e^{hg} - 1}{h} - g \right\|_u \| \, |f| * |u_0| \, \|_u.$$

Hence, for all $t \in \mathbb{R}$ and $f \in K(G)$ we obtain that

$$\lim_{h \to 0} \left\| \frac{e^{(t+h)g}u_0 - e^{tg}u_0}{h} - ge^{tg}u_0 \right\|_f = 0$$

and this means that $u(t) = e^{tg}u_0$, $t \in \mathbb{R}$ verifies the differential equation contained in (4.12). By virtue of the properties of the variation measure and taking into consideration Lemma 4.2 we can establish that for a fixed $t > 0$ we have

$$\frac{d}{ds}[e^{(t-s)g}u(s)] = 0_{ap(G)}, \quad s \in [0, t].$$

Here, we have denoted by $0_{ap(G)}$, the null measure on $G$. Thus, similar arguments as in the proof of Corollary 3.1 lead us to the conclusion $u(t) = e^{tg}u_0$, $t \in \mathbb{R}$ is the unique solution of (4.12). ■

## 5. Examples and applications

To illustrate the technique of finding the solution for the Cauchy problems (4.10) and (4.11), we consider some examples. The solution of our first example is a function $u \in C^1(\mathbb{R}, AP(\mathbb{R}))$.

**Example 5.1** Consider $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x\in\mathbb{R}$, $u_0(x)=\cos x$, $x\in\mathbb{R}$ and $\nu=f\lambda$ where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. It is obvious that $\nu\in m_F(\mathbb{R})$ and $u_0\in AP(\mathbb{R})$. With these notations (4.11) becomes

$$\begin{cases} \frac{du(t)}{dt}(x)=\int_{\mathbb{R}} u(t)(x-y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy, & x,\ t\in\mathbb{R}, \\ u(0)(x)=\cos x, & x\in\mathbb{R}, \end{cases} \tag{5.13}$$

where $u\in\mathcal{C}^1(\mathbb{R},AP(\mathbb{R}))$. For all $t\in\mathbb{R}$ we have that

$$e^{t\nu}=\delta_0+t\frac{f\lambda}{1!}+\cdots+t^n\frac{f^{[n]}\lambda}{n!}+\cdots,$$

where $f^{[n]}=f*f*\cdots*f$. Some calculations give us that

$$f^{[n]}(x)=\frac{1}{\sqrt{2n\pi}}e^{-\frac{x^2}{2n}},\quad x\in\mathbb{R},\quad n\in\mathbb{N}^*,$$

$$f^{[n]}\lambda*u_0(x)=\frac{1}{\sqrt{2n\pi}}\int_{\mathbb{R}}\cos(x-y)e^{-\frac{y^2}{2n}}dy=e^{-\frac{n}{2}}\cos x,\ x\in\mathbb{R},\ n\in\mathbb{N}^*.$$

Finally we obtain

$$u(t)(x)=\cos x+\sum_{n=1}^{\infty}\frac{e^{-\frac{n}{2}}t^n}{n!}\cos x=e^{\frac{t}{\sqrt{e}}}\cos x,\quad x,\ t\in\mathbb{R}.$$

In the second Cauchy problem, the solution is a function $u\in\mathcal{C}^1(\mathbb{R},ap(\mathbb{R}))$.

**Example 5.2** Consider $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x\in\mathbb{R}$, $u_0=\cos(\cdot)\lambda$ and $\nu=f\lambda$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$. It is obvious that $\nu\in m_F(\mathbb{R})$ and $u_0\in ap(\mathbb{R})$. With these notations (4.10) becomes

$$\begin{cases} \frac{du(t)}{dt}(\varphi)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\varphi(x+y)d[u(t)](x)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy, & t\in\mathbb{R}, \\ u(0)=\cos(\cdot)\lambda, \end{cases}$$

where $\varphi\in K(\mathbb{R})$ and $u\in\mathcal{C}^1(\mathbb{R},ap(\mathbb{R}))$. In a similar manner by that used in Example 5.1 we obtain that

$$u(t)=\left[e^{\frac{t}{\sqrt{e}}}\cos(\cdot)\right]\lambda,\quad t\in\mathbb{R}.$$

Finally, we establish equalities for the mean of some classes of almost periodic measures depending on the parameter $t\in\mathbb{R}$. Let $u\in\mathcal{C}^1(\mathbb{R},ap(G))$. It is easy to see that the function $t\in\mathbb{R}\to M[u(t)]\in\mathbb{C}$ satisfies the equality

$$\frac{d}{dt}M[u(t)]=M\left[\frac{du}{dt}(t)\right],\quad t\in\mathbb{R}. \tag{5.14}$$

Next, we apply (5.14) to the solutions of the Cauchy problems (4.10) and (4.12). Consider $\nu\in m_F(G)$, $\mu\in ap(G)$ such that $M(\mu)\neq 0$ and $g\in AP(G)$. First, from (5.14) we deduce the equality

$$\frac{d}{dt}M[e^{t\nu}*\mu]=[\nu(G)]M[e^{t\nu}*\mu],\quad t\in\mathbb{R}. \tag{5.15}$$

We use a property of the mean, respectively, the equality

$$M(e^{t\nu} * \mu) = e^{t\nu}(G)M(\mu), \quad t \in \mathbb{R},$$

and we obtain that the following equality holds true:

$$\frac{d}{dt}[e^{t\nu}(G)] = \nu(G)e^{t\nu}(G), \quad t \in \mathbb{R}. \tag{5.16}$$

On the other hand, from (5.14), it also follows that

$$\frac{d}{dt}M[e^{tg}\mu] = M[ge^{tg}\mu], \quad t \in \mathbb{R}.$$

In the same manner we obtain that for all $n \in \mathbb{N}^*$ we have,

$$\frac{d^n}{dt^n}M[e^{t\nu} * \mu] = [\nu(G)]^n M[e^{t\nu} * \mu], \quad t \in \mathbb{R},$$
$$\frac{d^n}{dt^n}[e^{t\nu}(G)] = [\nu(G)]^n[e^{t\nu}(G)], \quad t \in \mathbb{R}.$$

## References

[1] L. N. Argabright and J. G. Lamadrid, *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups*, Mem. Amer. Math. Soc. No. 145, 1974.

[2] L. N. Argabright and J. G. Lamadrid, *Almost Periodic Measures*, Mem. Amer. Math. Soc. No. 428, 1990.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei, Bucureşti, 1976.

[4] N. Dinculeanu, *Vector Measures*, Veb Deutscher Verlag Der Wissenschaften, Berlin, 1966.

[5] N. Dinculeanu, *Integrarea pe Spaţii Local Compacte*, (in romanian), Editura Academiei R.P.R., Bucureşti, 1965.

[6] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, New York, 1985.

[7] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis.* Vol. I. Springer - Verlag, Berlin, Göttingen, Heidelberg, 1963.

[8] E. Hille and R. S. Phillips, *Functional Analysis and Semi - Groups*, American Mathematical Society, Providence, 1957.

[9] J. G. Lamadrid, *Sur les Mesures Presque Périodiques*, Séminaire KGB sur les marches aléatoires, Astérisque 4, 1973.

[10] W. Rudin, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics, Number 12, Interscience Publishers – John Wiley and Sons, New York, London, 1962.

[11] T. Vladislav and I. Raşa, *Analiză Numerică. Aproximare, Problema lui Cauchy Abstractă, Proiectori Altomare*, (in romanian), Editura Tehnică, Bucureşti, 1999.