AN OBSERVABILITY ESTIMATE IN $L_2(\Omega) \times H^{-1}(\Omega)$ FOR SECOND-ORDER HYPERBOLIC EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract: For second-order hyperbolic equations with variable coefficient principal part, we establish a global observability-type estimate, whereby the energy of the solutions in $L_2(\Omega) \times H^{-1}(\Omega)$ is bounded by appropriate boundary traces, modulo lower-order terms. This extends the proof and the results of [L-T.2] from the constant coefficient to the variable coefficient case. The pseudo-differential analysis of [L-T.2] is combined with Riemann geometric multipliers [Y.1], which replace the Euclidean multipliers of [L-T.2] in the constant coefficient case.

1 INTRODUCTION. STATEMENT OF MAIN RESULT

Let Ω be an open bounded domain in $\mathbb{R}^n$, with smooth boundary $\Gamma = \partial\Omega$. Let Γ_0 be a (possibly empty) portion of Γ. We shall consider regular solutions of

$$\begin{cases} w_{tt} + \mathcal{A}w \equiv f & \text{in } Q = (0,\infty) \times \Omega, & (1.1a) \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0 = (0,\infty) \times \Gamma_0, & (1.1b) \end{cases}$$

*Research partially supported by the National Science Foundation under grant DMS-9504822 and by the Army Research Office under grand DAAH04-96-1-0059.
†Research partially supported by the National Science Foundation and the Pangdeng Project of China.

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35359-3_40

S. Chen et al. (eds.), *Control of Distributed Parameter and Stochastic Systems*

and establish an *a-priori* energy estimate one unit *below* the usual energy level of $H^1(\Omega) \times L_2(\Omega)$. In (1.1a) we have set

$$\mathcal{A}w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x))\frac{\partial w}{\partial x_j}, \quad x = [x_1, x_2, \ldots, x_n], \tag{1.2}$$

$a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^n)$, to be a second-order differential operator, which is assumed throughout to satisfy the ellipticity condition
(H.1)

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0, \quad \forall\, x \in \mathbb{R}^n,\ 0 \neq \xi = [\xi_1, \xi_2, \ldots, \xi_n] \in \mathbb{R}^n. \tag{1.3}$$

Let $G(x) = (g_{ij}(x)) = [A(x)]^{-1}$ be the inverse of the symmetric positive definite coefficient matrix $A(x) = (a_{ij}(x))$. For each $x \in \mathbb{R}^n$, the matrix $G(x)$ defines in a natural way a Riemann metric $g(X,Y) = \langle X, Y\rangle_g$ on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n \ni X, Y$ [L-T-Y.1], [Y.1], so that $(\mathbb{R}^n, g)$ is a Riemann manifold. Let D be the Levi-Civita connection in the Riemann metric g, $D_X H$ be the covariant derivative of the vector field $H \in \mathbb{R}^n_x$ with respect to the vector field $X \in \mathbb{R}^n_x$; and let $DH(\,\cdot\,,\,\cdot\,)$ be the resulting covariant differential $DH(X,X) = \langle D_X H, X\rangle_g$. We may now state the main assumption, as in [Y.1], imposed throughout this paper upon the differential operator $\mathcal{A}$ in (1.2). Several nontrivial examples where this assumption is verified to hold true, by using Riemann geometry, are given in [L-T-Y.1], [Y.1].

Main Assumption (H.2). There exists a vector field H on the Riemann manifold $(\mathbb{R}^n, g)$ such that

$$DH(X,X) = \langle D_X H, X\rangle_g \geq a|X|^2_g, \quad \forall\, X \in \mathbb{R}^n_x,\ x \in \Omega, \tag{1.4}$$

where $a > 0$ is a constant. □

Main result. To state our main result, we first introduce the positive, self-adjoint operator on $L_2(\Omega)$:

$$\begin{cases} \mathcal{A}_0 w = \mathcal{A}w, \quad \mathcal{D}(\mathcal{A}_0) = H^2(\Omega) \cap H^1_0(\Omega); & (1.5a) \\ \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) = H^1_0(\Omega),\ [\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})]' = H^{-1}(\Omega) \text{ (equivalent norms)} & (1.5b) \end{cases}$$

Next, we introduce the desired 'energy' of the solution w of Eqn. (1.1):

$$\begin{cases} \mathcal{E}_w(t) \equiv \|w(t)\|^2_{L_2(\Omega)} + \|\mathcal{A}_0^{-\frac{1}{2}} w_t(t)\|^2_{L_2(\Omega)} \equiv \|\{w(t), w_t(t)\}\|^2_Z & (1.6a) \\ \text{equivalent to } \|\{w(t), w_t(t)\}\|^2_{L_2(\Omega)\times H^{-1}(\Omega)}; & (1.6b) \end{cases}$$

$$Z \equiv L_2(\Omega) \times [\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})]' \equiv L_2(\Omega) \times H^{-1}(\Omega) \text{ (equivalent norms)}. \tag{1.7}$$

Our main result below refers to solutions of problem (1.1) within the following class:

$$\begin{cases} \{w, w_t\} \in C([0,T]; L_2(\Omega) \times H^{-1}(\Omega)); & (1.8a) \\ w|_{\Sigma_1} \in L_2(\Sigma_1); \ \left.\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} \in H^{-1}(\Sigma_1), & (1.8b) \end{cases}$$

where $\Gamma_1 \equiv \Gamma \setminus \Gamma_0$, $\Sigma_1 = (0,T] \times \Gamma_1$, $L_2(\Sigma_1) \equiv L_2(0,T; L_2(\Gamma_1))$, and where $\frac{\partial}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} \nu_i$ is the co-normal derivative, $\nu = [\nu_1, \nu_2, \dots, \nu_n]$ being the unit outward normal at Γ.

Theorem 1 *Assume (H.1) = (1.3), (H.2) = (1.4). Let $f \in L_2(0,T; H^{-1}(\Omega)) \cap L_2(R^1_x; H^{-1}(\Sigma_T))$ where here x denotes the space variable in the normal direction. Let w be a solution of problem (1.1) within the class (1.8a–b). Assume, moreover, that the vector field H of hypothesis (1.4) satisfies the condition:*
(H.3)

$$H(x) \cdot \nu(x) \leq 0, \ x \in \Gamma_0, \ \nu = [\nu_1, \nu_2, \dots, \nu_n]. \tag{1.9}$$

Then, the following inequality holds true: If T is large enough, then

$$\int_0^T \mathcal{E}_w(t)dt \leq C_T \Bigg\{ \int_0^T \int_{\Gamma_1} w^2 d\Sigma_1 + \left\| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \right\|^2_{H^{-1}(\Sigma_{1T})} + \|w\|^2_{H^{-1}(Q)}$$
$$+ \left| \int_0^T \left(w|_{\Gamma_1}, \frac{\partial \mathcal{A}_0^{-1} w_t}{\partial \nu_{\mathcal{A}}} \right)_{L_2(\Gamma_1)} dt \right| + \int_0^T |f|^2_{H^{-1}(\Omega)} dt + \|\Lambda f\|^2_{L_2(Q_T)} \Bigg\}, \tag{1.10}$$

where Λ is the operator in (2.2) below lifting by one unit in time and in the tangential space variables. □

Remark 1 Inequality (1.10) is a global observability-type estimate, whereby the $L_2(\Omega) \times H^{-1}(\Omega)$-norm of the solution in the interior is controlled (bounded) by the indicated boundary traces, modulo lower-order terms. When $f \equiv 0$ and $\mathcal{A} = -\Delta$ this inequality (1.10) can be extracted from the 1989 results of [L-T.2; e.g., Eqn. (4.34)]. See Remark 2 below. A micro-local (as opposed to global) version of this inequality (1.10) was later given in the variable coefficient case in [B-H-L-R-Z] expanding upon [B-L-R], subject to geometric optics assumptions. Still, in the variable coefficient case, a *related* version of (1.10) was later given in [T.1], subject to the assumption of existence of a pseudo-convex function. Our Riemann geometric assumption (H.2) is more general than the existence of a pseudo-convex function [Y.1]. □

Consequence: Uniform stabilization of Eqn. (1.1a–b) with boundary feedback in the Dirichlet B.C. We next consider the uniform stabilization problem of the following (closed-loop) wave equation with an explicit dissipative feedback control in the Dirichlet B.C. of Γ_1:

$$\begin{cases} w_{tt} + \mathcal{A}w = 0 & \text{in } Q \equiv (0,\infty) \times \Omega; & \text{(1.11a)} \\ w(0,\ \cdot\ ) = w_0;\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; & \text{(1.11b)} \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0 \equiv (0,\infty) \times \Gamma_0; & \text{(1.11c)} \\ w|_{\Sigma_1} = \dfrac{\partial(\mathcal{A}_0^{-1} w_t)}{\partial \nu_A} & \text{in } \Sigma_1 \equiv (0,\infty) \times \Gamma_1, & \text{(1.11d)} \end{cases}$$

where $\frac{\partial}{\partial \nu_A}$ is the co-normal derivative, defined below (1.8b). Moreover, $\mathcal{A}_0$ is the operator defined in (1.5a). In the case of the pure wave equation (constant coefficients $a_{ij}(x)$ = Kronecker δ_{ij}), the explicit, dissipative feedback control in (1.11d) was introduced in [L-T.1]: this reference then provided, for the first time (the semigroup well-posedness result in Theorem 2 below and) a uniform stabilization result of the wave equation on the space $L_2(\Omega) \times H^{-1}(\Omega)$ (which had been discovered a few years earlier by I. Lasiecka-R. Triggiani and J. L. Lions to be the space of optimal regularity for the solution of second-order hyperbolic equations under $L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma)$-Dirichlet control), by means of an $L_2(0,\infty;L_2(\Gamma_1))$-feedback control as in (1.11d).

Theorem 2 (Well-posedness [L-T.1, Thm. 1.1, p. 345]) *Assume (H.1) = (1.3). Problem (1.11) defines a s.c. contraction semigroup $T_D(t)$ on Z, see (1.7).* □

Theorem 3 (Uniform stabilization) *Assume (H.1), (H.2), (H.3). Then, the s.c. contraction semigruop $T_D(t)$ guaranteed by Theorem 1 is uniformly stable on Z: there exist constants $M \geq 1$, $\delta > 0$ such that*

$$\|T_D(t)\|_{\mathcal{L}(Z)} \leq M e^{-\delta t},\ t \geq 0;\ \textit{or}\ \mathcal{E}_w(t) \leq M_1 e^{-\delta t} \mathcal{E}_w(0). \quad \square \qquad \textit{(1.12)}$$

Theorem 3 may be obtained as a corollary of Theorem 2.

2 PROOF OF THEOREM 1 FOR $F \equiv 0$: ENERGY ESTIMATE

Orientation. The proof follows closely the strategy of [L-T.2], which consists of two main steps, as critically generalized in its second step to the variable coefficient case, by a Riemann geometric approach [Y.1]. These two main steps are:

(a) a preliminary pseudo-differential change of variable (for the corresponding half-space problem) from the original variable $w(t,x)$ of problem (1.1) to a new variable $v(t,x)$ of problem (2.4) below. This change of variable has the purpose of lifting by one unit upward the regularity of w in t and in the tangential space variables and thus, via the governing equation, of w in the normal space variable as well. Thus, the required topology $L_2(\Omega) \times H^{-1}(\Omega)$ for $\{w, w_t\}$ is lifted to the $H^1(\Omega) \times L_2(\Omega)$-topology for $\{v, v_t\}$, a level where energy methods apply.

(b) An energy (multiplier) analysis at the $H^1(\Omega) \times L_2(\Omega)$-level of the resulting v-problem (2.4), which in [L-T.2] was carried out via (by now) "classical" multipliers suitable for the constant coefficient case ($\mathcal{A} = -\Delta$) considered there (i.e., the multipliers $h \cdot \nabla v$, $v \operatorname{div}_0 h$, v_t in the Euclidean metric, where h is a smooth coercive vector field on Ω).

The present proof then uses step (a) verbatim and replaces the classical multiplier analysis for step (b) in the Euclidean metric with the Riemann geometry multipliers version recently introduced by [Y.1] to handle the variable coefficient case of an operator such as $\mathcal{A}$: i.e., $\langle H, \nabla_g v\rangle_g$, $v \operatorname{div}_0 H$, v_t in the Riemann metric g.

2.1 *From the original w-problem to the new v-problem*

For simplicity of exposition, we take here $f \equiv 0$ in order to fall into [L-T.2]. Keeping track of $f \neq 0$ is readily done. Let w be a solution of (1.1a–b), in the class (1.8a–b), subject to assumption (1.9). Starting with the original variable w of the problem in (1.1), we define the truncation of w by

$$\bar{w} = w, \quad 0 < t < T; \ \bar{w} = 0 \text{ elsewhere.} \tag{2.1}$$

From [L-T.2, p. 211–212] there is a bounded operator

$$\Lambda : H^s(R^1_t \times R^{n-1}_y) \to H^{s+1}(R^1_t \times R^{n-1}_y) \tag{2.2}$$

for $s \geq 0$, given explicitly by going over to the half-space problem in [L-T.2, Sec. 5], where y is the tangential variable, such that the change of variable

$$v = \Lambda \bar{w} \tag{2.3}$$

transforms the w-problem (1.1a–b) into the problem

$$\begin{cases} v_{tt} = \mathcal{A}v + K\bar{w}, & \text{in } (0,T) \times \Omega, \\ v|_{\Gamma_0} = 0, & \text{in } (-\infty,\infty) \times \Gamma_0, \end{cases} \tag{2.4}$$

K being a commutator operator, possessing the crucial property of Proposition 5 below. Denote $\Sigma_{iT} = (0,T) \times \Gamma_i$, $\Sigma_{i\infty} = (-\infty,\infty) \times \Gamma_i$, $\Sigma_T = (0,T) \times \Gamma$, $Q_T = (0,T) \times \Omega$, and $Q_\infty = (-\infty,\infty) \times \Omega$.

Proposition 4 (L-T.2, Lemma 3.1, p. 199) *Let w be a solution of problem (1.1) with $f \equiv 0$ in the class (1.8a-b), and let v be given by (2.3). Then*

$$\|v|_{\Sigma_1}\|^2_{H^1(\Sigma_{1,\infty})} + \left\|\frac{\partial v}{\partial \nu_{\mathcal{A}}}\right\|^2_{L_2(\Sigma_{1,\infty})} \leq C\left\{\|w|_{\Sigma_1}\|^2_{L_2(\Sigma_{1T})} + \left\|\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right\|^2_{H^{-1}(\Sigma_{1T})}\right\}, \tag{2.5}$$

where the constant $C > 0$ does not depend on T.

Proposition 5 (L-T.2, Corollary 3.3, p. 200) *Let w be a solution of (1.1a–b) with $f \equiv 0$ in the class (1.8a–b), and let v be given by (2.3). Then,* *with reference to $\mathcal{E}_w(t)$ defined in (1.6a), the following inequality holds true:*

$$\int_0^T \|Kw\|^2_{L^2(\Omega)} dt \leq C\left[\int_0^T \|w|_{\Gamma_1}\|^2_{L^2(\Gamma_1)}\, dt + \left\|\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right\|^2_{H^{-1}(\Sigma_{1T})} + \int_{-\infty}^{\infty} \|v\|^2_{L^2(\Omega)} dt + \mathcal{E}_w(T) + \mathcal{E}_w(0)\right], \qquad (2.6)$$

where the constant C does not depend on T.

Remark 2 In effect, the proof in [L-T.2] refers explicitly to the feedback problem (1.11) rather than to problem (1.1a–b) in the class (1.8a–b). But problem (1.11) does satisfy the property (1.8a) (see Thm. 2) as well as (1.8b): see [L-T.1] for $w|_{\Sigma_1}$ and see [L-T.2, Lemma 2.1] stating that

$$\left\|\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right\|_{H^{-1}(\Sigma_\infty)} \leq C\|w|_{\Sigma}\|_{L_2(\Sigma_{1T})}. \qquad (2.7)$$

The proof of [L-T.2] takes advantage of estimate (2.7), see [L-T.2, below (5.27)], as well as of the dissipativity relation for (1.11): in the general case of Eqn. (1.1) (with $f \equiv 0$) with no boundary conditions imposed, the counterpart of the dissipativity relation for problem (1.11) is now

$$\mathcal{E}_w(t) + 2\int_0^t \left(w|_{\Gamma_1}, \frac{\partial \mathcal{A}_0^{-1} w_t}{\partial \nu_{\mathcal{A}}}\right)_{L_2(\Gamma_1)} d\tau = \mathcal{E}_w(0). \qquad (2.8)$$

Thus, in treating the more general case (1.1a–b) within the class (1.8a–b), we have that the estimates of Lemma 3.1 and of Corollary 3.3 in [L-T.2] must now be augmented with the term $\left\|\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right\|^2_{H^{-1}(\Sigma_{1T})}$ to obtain (2.5) of Proposition 4, and (2.6) of Proposition 5, while this term was absorbed by (2.7) in [L-T.2] following Eqn. (5.27) there, p. 212. Moreover, in the present proof, (2.8) replaces the dissipative version specialized for (1.11) in [L-T.2], which was used following (4.24) there, p. 203.

2.2 Analysis of the v-problem (2.4) by Riemann geometric energy methods

The following result is the counterpart of [L-T.2, Theorem 4.1, p. 200], where it was obtained for $\mathcal{A} = -\Delta$ by using "classical" multipliers in the Euclidean metric. The present proof treats instead the variable coefficient operator $\mathcal{A}$ in (1.2) and uses energy methods in the Riemann metric $(\mathbb{R}^n, g)$.

Theorem 6 *Let w be a solution of (1.1a–b) with $f \equiv 0$ in the class (1.8a–b) subject to hypothesis (1.9). Assume (H.2) = (1.4). Then, for T large enough,* *the following inequality holds true*

$$\int_0^T E_v(t)dt \leq C_T\left[\|w|_{\Sigma_1}\|^2_{L^2(\Sigma_{1T})} + \left\|\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right\|^2_{H^{-1}(\Sigma_{1T})}\right] + \|v\|^2_{L_2(Q_T)} + \|w\|^2_{H^{-1}((0,T)\times\Omega)} + \left|\int_0^T \left(w|_{\Gamma_1}, \frac{\partial \mathcal{A}_0^{-1} w_t}{\partial \nu_{\mathcal{A}}}\right)_{L_2(\Gamma_1)} dt\right|\Bigg], \tag{2.9}$$

where $\mathcal{E}_w(t)$ is defined in (1.6a), and where we have set

$$E_v(t) = \int_\Omega [v_t(t) + |\nabla_g v(t)|_g^2]dx. \quad \square \tag{2.10}$$

Proof. We will use the Riemann geometry energy method given in [Y.1], combined with the strategy of [L-T.2, Theorem 4.1, p. 200]. To do so, we need several multiplier identities.

(i) Multiplying (2.4) by $H(v) = \langle \nabla_g v, H\rangle_g$ and integrating by parts, we obtain e.g., [Y.1, Proposition 2.1(1)],

$$\int_{Q_T} DH(\nabla_g v, \nabla_g v)dQ = \int_{\Sigma_T} \frac{\partial v}{\partial \nu_{\mathcal{A}}} H(v)d\Sigma + \frac{1}{2}\int_{\Sigma_T} \left(v_t^2 - |\nabla_g v|_g^2\right) H\cdot\nu\, d\Sigma - \frac{1}{2}\int_{Q_T}\left(v_t^2 - |\nabla_g v|_g^2\right)\operatorname{div}_0 H\, dQ + \int_{Q_T}(Kw)H(v)dQ - [(v_t, H(v))]_0^T \tag{2.11}$$

(counterpart of [L-T.2, Eqn. (4.3)] in the constant coefficient case $\mathcal{A} = -\Delta$) where $DH(\,\cdot\,,\,\cdot\,)$ is the covariant differential of the vector field H.

(ii) Multiplying (2.4) by $v\ \operatorname{div}_0 H$ and integrating by parts, we obtain, by the boundary condition $v|_{\Sigma_0}$ in (2.4b), e.g., [Y.1, Proposition 2.1(2)],

$$\int_{Q_T}(v_t^2 - |\nabla_g v|_g^2)P\, dQ = [(v_t, vP)_{L_2(\Omega)}]_0^T + \int_{Q_T} v\nabla_g P(v)dQ - \int_{Q_T}(Kw)vP\, dQ - \int_{\Sigma_{1T}} \frac{\partial v}{\partial \nu_{\mathcal{A}}} vP\, d\Sigma, \tag{2.12}$$

where $P = \operatorname{div}_0 H$, $\nabla_g P(v) = \langle \nabla_g P, \nabla_g v\rangle_g = (A\nabla_0 P, \nabla_g v\rangle_g = \nabla_0 P\cdot\nabla_g v$. (Counterpart of [L-T.2, Eqn. (4.4)]) in the constant coefficient case.

(iii) Multiplying (2.4) by v_t and integrating by parts, we obtain, by the boundary condition in (2.4b),

$$E_v(t) - E_v(0) = 2\int_{Q_T}(Kw)v_t dQ + \int_{\Sigma_{1T}} \frac{\partial v}{\partial \nu_{\mathcal{A}}} v_t\, d\Sigma \tag{2.13}$$

as in [L-T.2, Eqn. (4.5)]. With the identities (2.11)–(2.13) at hand and using the inequality in (1.4) to take the place of that in [L-T.2, Eqn. (1.10)], we may get the inequality in (2.9) by following the proof of [L-T.2, Theorem 4.1, p. 200] using (2.8) in place of its dissipative version for (1.11) valid there. $\square$

2.3 Return from variable v to original variable w: Proof of Theorem 1

We shall establish the desired estimate (1.10) of Theorem 1 for w starting from inequality (2.9) of Theorem 6 for v. Our proof here follows closely [L-T.2, Theorem 4.2, p. 204–205]. Moreover, the norm of $v = \Lambda\bar{w}$ in $L_2(Q_T)$ is dominated by the norm of w in $H^{-1}(Q_T)$: Λ lifts in t and in the tangential direction; then, Eqn. (1.1a) lifts by one in the normal direction as well.

References

[B-L-R] C. Bardos, J. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, *SIAM J. Control & Optimiz.* 30, N.S. (1992), 1024–1065.

[B-H-L-R-Z] C. Bardos, L. Halperin, G. Lebeau, J. Rauch, E. Zuazua, Stabilization de l'equation des ondes au moyen d'un feedback portant sur la condition aux limites de Dirichlet, *Asymptotic Analysis.*

[L-T.1] I. Lasiecka and R. Triggiani, Uniform exponential energy decay of the wave equation in a bounded region with $L_2(0, \infty; L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions, *J. Diff. Eqns.* 66 (1987), 340-390.

[L-T.2] I. Lasiecka and R. Triggiani, Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions, *Appl. Math. & Optim.* 25 (1992), 189–224. Preliminary version in LNCIS vol. 147, Springer-Verlag, pp. 62–108, Proceedings of the Third Working Conference, Montpellier, France, January 1989.

[L-T-Y.1] I. Lasiecka, R. Triggiani, P. F. Yao, Exact controllability for second-order hyperbolic equations with variable coefficients–principal part and and first order terms, *Non-Linear Analysis Theory, Methods & Applications* 30 (1997), 111–122.

[T.1] D. Tataru, A-priori estimates of Carleman's type in domains with boundary, *J. Math. Pure & Appl.* 73 (1994), 355–387.

[Y.1] P. F. Yao, Exact controllability of wave equations with variable coefficients in a bounded region, to appear.