MULTIVARIATE CONSTRAINED PORTFOLIO RULES: DERIVATION OF MONGE-AMPÈRE EQUATIONS

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1 STATEMENT OF THE PROBLEM

Consider following classical problem in mathematical finance (see [16, 8, 4, 17, 3]).

$N+1$ assets are traded continuously. The $N+1$st of these assets, called *bank account*, has the value $B(t)$ at time t, which evolves according to the equation

$$dB(t) = r(t)B(t)dt, \quad B(0) = b \in [0, \infty). \tag{1.1}$$

N assets, called *stocks*, are subject to the systematic risk, and have the price-per-share $P_{(i)}(t)$ at time t, which evolve according to the equations

$$dP_{(i)}(t) = b_{(i)}(t)P_{(i)}(t)dt + \sum_{j=1}^{N} \sigma_{(ij)}(t)P_{(i)}(t)dW_j(t)$$
$$P_{(i)}(0) = p_{(i)} \in [0, \infty). \tag{1.2}$$

for $i = 1, ..., N$. In (1.1,1.2) $r(t) \geq 0$ (the *interest rate*), $b_{(i)}(t)$ (the *appreciation rates*), $\sigma_{(ij)}(t)$ (the *volatility coefficients*), are given deterministic functions of time $t \in [0, T]$, and $W_j(t)$ are independent 1-dimensional Brownian motions.

We shall assume, throughout this paper, that matrix

$$\sigma(t) \quad \text{is invertible, for every} \quad t \in [0, T). \tag{1.3}$$

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Further, we consider an economic agent who can decide, at each instant $t \in [0, T)$, knowing the amount of his current wealth $X(t) \in [0, \infty)$, and knowing functions $r, b_{(i)}$ and $\sigma_{(ij)}$, the following:

how much money $\pi_{(i)}(t, X(t)) \in (-\infty, +\infty)$ to invest in the ith stock. Vector function

$$\pi = \left(\pi_{(1)}, ..., \pi_{(N)}\right)^T : (t, x) \mapsto R^N \tag{1.4}$$

is called a *portfolio rule.*

Then (see [8]), the agent's wealth $X(t) = X^\pi(t)$ evolves according to the equation

$$\begin{aligned} dX(t) = \left\{r(t)X(t) + \pi^T(t, X(t))[b(t) - r(t)1_N]\right\} dt \\ +\pi^T(t, X(t))\sigma(t)dW(t) \\ X(0) = x \in [0, \infty). \end{aligned} \tag{1.5}$$

In (1.5) π^T denotes the transpose of π, while

$$1_N = (1, ..., 1)^T \in R^N. \tag{1.6}$$

It is assumed that if the wealth of the agent becomes zero at some time, the evolution stops there.

For a given portfolio rule $\pi(\cdot, \cdot)$, we define the expected *payoff*

$$E_{(0,x)} \left\{e^{-\int_0^T \beta(\tau)d\tau} \varphi\left(X^\pi(T)\right)\right\}. \tag{1.7}$$

In (1.7), $E_{(t,x)}$ denotes the expectation given the condition that at time t the wealth of the agent was equal to x. Also, $\beta(t) \geq 0$ is the *discount factor* (or the *inflation rate*), while φ is the *utility function* for the terminal wealth $X^\pi(T)$.

So, the payoff is the expectation of the present value of the utility of the final amount of the wealth of the agent.

Different stochastic control problems can be considered for the dynamics (1.5) and payoff (1.7), depending on the set of admissible strategies $\mathcal{AS}$. We shall consider following problems.

Problem 0. *Find the optimal (feedback) strategy*

$$\tilde{\pi}_0(\cdot, \cdot) \in \mathcal{AS}_0 = \left\{\pi(\cdot, \cdot) \left|\pi_{(i)}(t, x) \in (-\infty, +\infty) \right. \ \forall (t, x) \ \ 1 \leq i \leq N\right\} \tag{1.8}$$

such that the expected payoff is maximized on $\mathcal{AS}_0$.

This is a classical problem (see [8]).

Let function $a(t, x)$ be given.

Problem 1. *Find the optimal (feedback) strategy*

$$\tilde{\pi}_1(\cdot, \cdot) \in \mathcal{AS}_1 = \left\{\pi(\cdot, \cdot) \left|\sum_{i=1}^N \pi_{(i)}(t, x)\mu_{(i)}(t, x) = a(t, x) \ \ \forall (t, x)\right.\right\} \tag{1.9}$$

such that the expected payoff is maximized on $\mathcal{AS}_1$.

Let functions $a_1(t,x) \le a_2(t,x)$ be given.

Problem 2. *Find the optimal (feedback) strategy*

$$\tilde{\pi}_2(\cdot,\cdot) \in \mathcal{AS}_2$$

$$= \left\{ \pi(\cdot,\cdot) \,\middle|\, a_1(t,x) \le \sum_{i=1}^{N} \pi_{(i)}(t,x)\mu_{(i)}(t,x) \le a_2(t,x) \quad \forall (t,x) \right\} \tag{1.10}$$

such that the expected payoff is maximized on $\mathcal{AS}_2$.

We notice that constrained portfolio selection has been studied (see [19, 2, 10], with applicable results mainly in the case when portfolio is one-dimensional.

2 MONGE-AMPÈRE-TYPE EQUATION OF THE PORTFOLIO SELECTION PROBLEM UNDER THE CONSTRAINT $\sum_{I=1}^{N} \pi_{(I)}(T,X)\mu_{(I)}(T,X) = A(T,X)$

As it is usual in stochastic control, one considers the *value function*

$$V_i(t,x) = \sup_{\pi \in \mathcal{AS}_i} E_{(t,x)} \left\{ e^{-\int_t^T \beta(\tau)d\tau} \varphi\left(X^{\pi}(T)\right) \right\} \tag{1.11}$$

$i = 0,1,2$. Then, by the stochastic calculus,[1] and dynamic programming arguments, the value function $V_0(t,x)$ is the solution[2] of the following Dirichlet problem for the Hamilton-Jacobi-Bellman partial differential equation:

$$\left\{ \begin{array}{l} V_{0t}(t,x) + \sup\limits_{\pi \in R^N} \left[\dfrac{1}{2} ||\sigma(t)^T \pi||^2 V_{0xx}(t,x) \right. \\ \left. +(r(t)x + \pi^T(b(t) - r(t)1_N))V_{0x}(t,x) \right] - \beta(t)V_0(t,x) = 0 \end{array} \right. \tag{1.12}$$

$$t \in [0,T),\ x \in (0,\infty)$$

$$V_0(T,x) = \varphi(x),\ \ x \in (0,\infty) \tag{1.13}$$

$$V_0(t,0) = 0,\ \ t \in [0,T), \tag{1.14}$$

while, V_1 and V_2 are solutions of

$$\left\{ \begin{array}{l} V_{1t}(t,x) \\ \quad + \sup\limits_{\left\{\pi \in R^N : \sum_{i=1}^{N} \pi_{(i)}\mu_{(i)}(t,x) = a(t,x)\right\}} \left[\dfrac{1}{2} ||\sigma(t)^T \pi||^2 V_{1xx}(t,x) \right. \\ \left. + \left(r(t)x + \pi^T(b(t) - r(t)1_N)\right) V_{1x}(t,x) \right] - \beta(t)V_1(t,x) = 0 \end{array} \right. \tag{1.15}$$

and

$$\begin{cases} V_{2t}(t,x) \\ + \sup_{\{\pi \in R^N : a_1(t,x) \leq \sum_{i=1}^N \pi_{(i)}\mu_{(i)}(t,x) \leq a_2(t,x)\}} \left[\frac{1}{2}\|\sigma(t)^T\pi\|^2 V_{1xx}(t,x)\right. \\ \left. + \left(r(t)x + \pi^T(b(t) - r(t)1_N)\right) V_{1x}(t,x)\right] - \beta(t)V_1(t,x) = 0 \end{cases} \tag{1.16}$$

respectively, with the same boundary value as V_0. We do not address here the issue of the "boundary condition" at infinity, i.e., the behavior of $V(t,x)$ as $x \to \infty$. This is very important issue also in numerical calculations, when one has to truncate the problem to the finite domain, and then to impose a boundary condition.

We shall need the following notation. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^N, then

$$\text{orth}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}. \tag{1.17}$$

where $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

Theorem 1. *Let V be a classical solution of (1.15,1.13,1.14). Then, V solves the following* Monge-Ampère-type equation

$$\begin{aligned} V_t(t,x)V_{xx}(t,x) - \frac{1}{2}\left\|\text{orth}_{[\sigma(t)^{-1}\mu(t,x)]}\left[\sigma(t)^{-1}(b(t) - r(t)1_N)\right]\right\|^2 V_x(t,x)^2 \\ + \frac{1}{2}\frac{a(t,x)^2}{\|\sigma(t)^{-1}\mu(t,x)\|^2}V_{xx}(t,x)^2 \\ + \left[a(t,x)\frac{(\sigma(t)^{-1}\mu(t,x))^T\sigma(t)^{-1}(b(t) - r(t)1_N)}{\|\sigma(t)^{-1}\mu(t,x)\|^2} + r(t)x\right] V_x(t,x)V_{xx}(t,x) \\ -\beta(t)V(t,x)V_{xx}(t,x) = 0 \end{aligned} \tag{1.18}$$

Moreover, the optimal portfolio rule *is given by*

$$\begin{aligned} \tilde{\pi}(t,x) = -(\sigma(t)^{-1})^T \left[\frac{V_x(t,x)}{V_{xx}(t,x)}\text{orth}_{[\sigma(t)^{-1}\mu(t,x)]}\left[\sigma(t)^{-1}(b(t) - r(t)1_N)\right]\right. \\ \left. -a(t,x)\frac{\sigma(t)^{-1}\mu(t,x)}{\|\sigma(t)^{-1}\mu(t,x)\|^2}\right]. \end{aligned} \tag{1.19}$$

Remark 1. *So, the problem of constrained portfolio selection is completely solved provided one has an efficient, stable,*[3] *and reliable numerical method for finding* proper *solution of (1.18,1.13,1.14). Proper in a sense that it coincides with the solution of the Hamilton-Jacobi-Bellman equation, i.e., coincides with the value function, since one should notice that uniqueness, in general, does* not *hold either for (1.18,1.13,1.14), nor for (1.25,1.13,1.14), and (1.31,1.13,1.14). So, the crucial property of any numerical algorithm attempting to solve the* *above equation is the ability of a selection of the proper solution. Such an algorithm was developed by the author (see [18]).*

Remark 2. *The equation (1.18), and therefore also equations (1.12), (1.15), and (1.16), become* degenerate, *when* $\varphi_{xx} \not< 0$. *An application would be the probability maximization of reaching certain level (say* $x = 1$*) of wealth (see [9]), in which case* $\varphi(x) = 0, x < 1$, *and* $\varphi(x) = 1, x \geq 1$. *Problems like that can be solved numerically nevertheless. The terminal condition (1.13) is* not *taken in a continuous way if* $\varphi_{xx} > 0$ *at some* x *(in the sense of distributions), rather concave envelope of* φ *is taken.*

Remark 3. *The superiority of the equation (1.18) over (1.15), in the view of the author, is manifold. For example:*

1) while equations (1.12), (1.15) , and (1.16) are vastly different, that is barely observable from their formulations, as opposed to equations (1.25), (1.18), and (1.31). The difficulty is, therefore, hidden, which does not, as a rule, simplify the matters;

2) the richness of the "calculus structure" is fully implemented in formulations (1.25), (1.18), and (1.31), as opposed to the Hamilton-Jacobi-Bellman formulation, which contributes to the accuracy and efficency of the numerical algorithm, to say the least.

Consider now the special case when $N = 2$. We use the following notation:

$$\sigma(t) = \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \quad b(t) = b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.20}$$

$$a(t,x) = a, \quad r(t) = r, \quad \beta(t) = \beta, \quad V(t,x) = V, \quad \tilde{\pi}(t,x) = \begin{pmatrix} \tilde{\pi}_1 \\ \tilde{\pi}_2 \end{pmatrix}. \tag{1.21}$$

Corollary 1. *Let* $N = 2$, $\mu = 1_2$. *Let* V *be a classical solution of (1.15,1.13,1.14). Then,* V *solves the following* Monge-Ampère-type equation

$$V_t V_{xx} - \frac{1}{2}\frac{(b_1 - b_2)^2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{22} - \sigma_{12})^2}(V_x)^2$$

$$+\frac{1}{2}\frac{a^2(\sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12})^2}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{22} - \sigma_{12})^2}(V_{xx})^2$$

$$+\frac{b_1(\sigma_{21}(\sigma_{21} - \sigma_{11}) + \sigma_{22}(\sigma_{22} - \sigma_{12})) + b_2(\sigma_{12}(\sigma_{12} - \sigma_{22}) + \sigma_{11}(\sigma_{11} - \sigma_{21}))}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{22} - \sigma_{12})^2}$$

$$\cdot a V_x V_{xx}$$

$$+r(x - a)V_x V_{xx} - \beta V V_{xx} = 0 \tag{1.22}$$

Moreover, the optimal portfolio rule *is equal to*

$$\tilde{\pi}_1 = \frac{1}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{22} - \sigma_{12})^2}$$

$$\cdot\left((b_2-b_1)\frac{V_x}{V_{xx}}+a((\sigma_{21}(\sigma_{21}-\sigma_{11})+\sigma_{22}(\sigma_{22}-\sigma_{12}))\right) \tag{1.23}$$

$$\tilde{\pi}_2=\frac{1}{(\sigma_{11}-\sigma_{21})^2+(\sigma_{22}-\sigma_{12})^2}$$
$$\cdot\left((b_1-b_2)\frac{V_x}{V_{xx}}+a((\sigma_{12}(\sigma_{12}-\sigma_{22})+\sigma_{11}(\sigma_{11}-\sigma_{21}))\right) \tag{1.24}$$

3 CONSTRAINT OF THE FORM $A_1(T,X) \leq \sum_{I=1}^{N} \pi_{(I)}(T,X)\mu_{(I)}(T,X) \leq A_2(T,X)$

Proposition 1. *Let V_0 be a classical solution of (1.12,1.13,1.14). Then, V_0 solves the following* Monge-Ampère-type equation

$$V_{0t}(t,x)V_{0xx}(t,x)-\frac{1}{2}||\sigma(t)^{-1}(b(t)-r(t)1_N)||^2V_{0x}(t,x)^2$$
$$+r(t)xV_0(t,x)_xV_{0xx}(t,x)-\beta(t)V_0(t,x)V_{0xx}(t,x)=0 \tag{1.25}$$

Moreover, the optimal portfolio rule is given by

$$\tilde{\pi}_0=-(\sigma(t)^{-1})^T\sigma(t)^{-1}(b(t)-r(t)1_N)\frac{V_{0x}(t,x)}{V_{0xx}(t,x)}. \tag{1.26}$$

It is not difficult now to solve the optimal multivariate constrained portfolio selection problem, when the constraint has the form

$$a_1(t,x)\leq\sum_{i=1}^{N}\pi_{(i)}(t,x)\mu_{(i)}(t,x)\leq a_2(t,x) \tag{1.27}$$

for some given functions

$$a_1(t,x)\leq a_2(t,x) \tag{1.28}$$

and $\mu(t,x)=(\mu_{(1)}(t,x),...,\mu_{(N)}(t,x))^T$ as before.

For each $t\in[0,T)$ define (nonlinear) differential operator $A=A(t)$

$$\begin{gathered} A:u\mapsto Au \\ \text{domain}(A)=\{u\in C^2|u_{xx}<0\} \end{gathered} \tag{1.29}$$

by the following elaborate formula

$$Au = \begin{cases} -\frac{1}{2}\|\sigma^{-1}(b - r1_N)\|^2 (u_x)^2 & \text{if} \quad a_1 \leq -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{u_x}{u_{xx}} \leq a_2 \\ -\frac{1}{2}\|\mathrm{orth}_{[\sigma^{-1}\mu]}[\sigma^{-1}(b - r1_N)]\|^2 \cdot (u_x)^2 + \frac{1}{2}\frac{(a_1)^2}{\|\sigma^{-1}\mu\|^2}(u_{xx})^2 + \frac{a_1(\sigma^{-1}\mu)^T\sigma^{-1}(b-r1_N)}{\|\sigma^{-1}\mu\|^2}u_x u_{xx} & \text{if} \quad a_1 > -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{u_x}{u_{xx}} \\ -\frac{1}{2}\|\mathrm{orth}_{[\sigma^{-1}\mu]}[\sigma^{-1}(b - r1_N)]\|^2 \cdot (u_x)^2 + \frac{1}{2}\frac{(a_2)^2}{\|\sigma^{-1}\mu\|^2}(u_{xx})^2 + \frac{a_2(\sigma^{-1}\mu)^T\sigma^{-1}(b-r1_N)}{\|\sigma^{-1}\mu\|^2}u_x u_{xx} & \text{if} \quad a_2 < -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{u_x}{u_{xx}} \end{cases} . \tag{1.30}$$

Theorem 2. *Let V_2 be a classical solution of (1.16,1.13,1.14). Then, V_2 solves the following* Monge-Ampère-type equation

$$V_{2t}(t,x)V_{2xx}(t,x) + A(t)V_2(t,x) + r(t)xV_{2x}(t,x)V_{2xx}(t,x) - \beta(t)V_2(t,x)V_{2xx}(t,x) = 0 \tag{1.31}$$

Moreover, the optimal portfolio rule is given by

$$\tilde{\pi}_2 = \begin{cases} -{\sigma^{-1}}^T \sigma^{-1}(b - r1_N)\frac{V_{2x}}{V_{2xx}} & \text{if} \quad a_1 \leq -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{V_{2x}}{V_{2xx}} \leq a_2 \\ -{\sigma^{-1}}^T \left[\mathrm{orth}_{[\sigma^{-1}\mu]}[\sigma^{-1}(b - r1_N)] \cdot \frac{V_{2x}}{V_{2xx}} - \frac{a_1}{\|\sigma^{-1}\mu\|^2}\sigma^{-1}\mu\right] & \text{if} \quad a_1 > -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{V_{2x}}{V_{2xx}} \\ -{\sigma^{-1}}^T \left[\mathrm{orth}_{[\sigma^{-1}\mu]}[\sigma^{-1}(b - r1_N)] \cdot \frac{V_{2x}}{V_{2xx}} - \frac{a_2}{\|\sigma^{-1}\mu\|^2}\sigma^{-1}\mu\right] & \text{if} \quad a_2 < -\mu^T {\sigma^{-1}}^T \sigma^{-1} \cdot (b - r1_N)\frac{V_{2x}}{V_{2xx}} \end{cases} .$$

Notes

1. cf. [6, 11]
2. cf. [8, 5]
3. Stability is very important since the optimal portfolio rule depends on $\frac{V_x}{V_{xx}}$.

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