A ONE-DIMENSIONAL RATIO ERGODIC CONTROL PROBLEM

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Abstract: In this paper we consider a one-dimensional ratio ergodic control problem. We obtain both the optimal control and the minimum value by solving the corresponding Bellman equation rigidly in C^2- class.

1. INTRODUCTION

Let $\kappa \leq a(\cdot) \in C_b(\mathbb{R})$, $b(\cdot,\cdot) \in C_b(\mathbb{R} \times \Gamma)$, $f(\cdot), g(\cdot) \in C(\mathbb{R})$ be given functions, where $\kappa > 0$ is a positive constant and Γ is a compact set in a separable metric space. We consider the ergodic control problem to minimize the cost :

$$K(v) = \liminf_{T \to \infty} \frac{\displaystyle\int_0^T f(x_t)\, dt}{\displaystyle\int_0^T g(x_t)\, dt} \quad \text{a.s.} \tag{1.1}$$

subject to one–dimensional stochastic differential equations

$$dx_t = b(x_t, v(x_t))\, dt + \sqrt{a(x_t)}\, dw_t\ , \qquad x_0 = \xi\ , \tag{1.2}$$

over a class of $\Gamma-$ valued Borel measurable functions $v = v(x)$ on $\mathbb{R}$, where ξ is a constant in $\mathbb{R}$ and $w = (w_t)$ is a one–dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$. Such a function $v(\cdot)$ is called a Markov control with control set Γ.

When g is a constant function, the problem above is called a classical ergodic control problem. Classical ergodic control problems have been investigated by

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many authors [1; 2; 3; 5] etc. We are interested in the case for non–constant function g, which we call a ratio ergodic control problem. The Bellman equation corresponding to the ratio ergodic control problem above is given by

$$\lambda g(x) = \frac{1}{2}\, a(x)\, \phi''(x) \;+\; \inf_{\gamma\in\Gamma}\left[\phi'(x)\, b(x,\gamma)\right] + f(x)\,, \qquad x \in \mathbb{R}\,. \quad (1.3)$$

If the function g is bounded below and bounded above by some positive constants, then the ratio ergodic control problem above is reduced to a classical ergodic control problem. Indeed, in this case, dividing the both sides of (3) by $g(x)$, we can consider the control problem above as the classical ergodic control problem to minimize the cost

$$L(v) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T f(y_t)/g(y_t)\, dt \quad \text{a.s.}$$

subject to one–dimensional stochastic differential equations

$$dy_t = \frac{b(y_t, v(y_t))}{g(y_t)}\, dt + \sqrt{\frac{a(y_t)}{g(y_t)}}\, d\tilde{w}_t\,, \qquad y_0 = \xi\,,$$

over the same class of Markov controls as above. Here $(\tilde{w}_t)$ is a one–dimensional standard Brownian motion. change method.

Our aim of the present paper is to obtain the optimal control and the minimum value of the ratio ergodic control problem (1.1), (1.2) for functions g which may vanish at some points or may be unbounded. For example, we can treat the case that $g(x) = |x|^\gamma$ for some constant $\gamma > 0$. For this aim, we have two keys. The first key is the pathwise asymptotic behavior of (x_t) as $t \to \infty$. This is necessary to treat the cost $K(v)$ defined almost surely. The second key is to solve corresponding Bellman equation (1.3).

The present paper is organized as follows : In § 2, we solve Bellman equation (1.3) rigidly in C^2 - class. In §3 we consider ratio ergodic control problem (1.1), (1.2).

2. BELLMAN EQUATION

Throughout this paper we assume :

(A1) $0 \le f\,, g \in C(\mathbb{R})$, $0 < g(x)$ $(x \neq 0)$, $\lim_{R\to\infty} \inf_{|x|>R} g(x) > 0$. There exist constants $\beta \ge 0$ and $C > 0$ such that

$$f(x) + g(x) \le C\,(1 + |x|^\beta)\,, \qquad x \in \mathbb{R}\,.$$

(A2) $f/g \in C^1(\mathbb{R}\backslash\{0\})$. f/g is strictly decreasing on $(-\infty, 0)$ and is strictly increasing on $(0,\infty)$. In addition

$$\lim_{x\downarrow 0} \frac{f(x)}{g(x)} = \lim_{x\uparrow 0} \frac{f(x)}{g(x)}\,,$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to-\infty}\frac{f(x)}{g(x)} .$$

(A3) Γ is a compact set in a separable metric space.

(A4) $a \in C_b(\mathbb{R})$ satisfies $\kappa \le a(\cdot) \le \kappa^{-1}$ on $\mathbb{R}$ for some constant $0<\kappa<1$ and

$$\left| \sqrt{a(x)}-\sqrt{a(y)} \right| \le K|x-y| , \qquad x,y\in\mathbb{R}$$

for some constant $K>0$. $b\in C_b(\mathbb{R}\times\Gamma)$ satisfies

$$\sup_{\gamma\in\Gamma} \mid b(x,\gamma)-b(y,\gamma) \mid \to 0 \qquad (|x-y|\to 0) .$$

(A5)

$$\limsup_{x\to\infty}\frac{x}{a(x)}\inf_{\gamma\in\Gamma} b(x,\gamma) < -\frac{\beta+1}{2} , \tag{2.1}$$

$$\liminf_{x\to-\infty}\frac{x}{a(x)}\inf_{\gamma\in\Gamma} b(x,\gamma) > 0 , \tag{2.2}$$

$$\liminf_{x\to\infty}\frac{x}{a(x)}\sup_{\gamma\in\Gamma} b(x,\gamma) > 0 , \tag{2.3}$$

$$\limsup_{x\to-\infty}\frac{x}{a(x)}\sup_{\gamma\in\Gamma} b(x,\gamma) < -\frac{\beta+1}{2} . \tag{2.4}$$

Example 1 below shows that conditions (2.1) and (2.4) can not be weakened in general.

Example 1 *Let* $\Gamma=[-1,1]$, $f(x)=|x|^\beta$ ($\beta>0$), $g(x)\equiv 1$, $a(x)\equiv 1$ *and* $b(x,\gamma)=\dfrac{\gamma(\beta+1)}{2(1+|x|)}$. *Then, except (2.1) and (2.4), (A1) ~ (A5) hold and we have*

$$\limsup_{x\to\infty}\frac{x}{a(x)}\inf_{\gamma\in\Gamma} b(x,\gamma)=\limsup_{x\to-\infty}\frac{x}{a(x)}\sup_{\gamma\in\Gamma} b(x,\gamma)=-\frac{\beta+1}{2} .$$

In this case we can consider that $K(v)=\infty$ *a.s. for all* Γ *- valued Borel measurable functions* v *on* $\mathbb{R}$.

Indeed, fix an arbitrary Γ - valued Borel measurable function v on $\mathbb{R}$, and let (x_t) be the response to v. Furthermore, let (x_t^*) be the response to $v^*(x)=-\mathrm{sgn}(x)$. By Theorem 2.1 in p.357 of [6], we can consider that $|x_t|\ge|x_t^*|$ $(t\ge 0)$ a.s. Thus $K(v)\ge K(v^*)$ a.s. On the other hand, by Theorem 16 in p.46 of [7], the invariant measure $p^*(x)\,dx$ of (x_t^*) is given by $p^*(x)=(1+|x|)^{-(\beta+1)}$ for

$x \in \mathbb{R}$. Thus p^* belongs to $L^1(\mathbb{R}, dx)$, because $\beta > 0$. Let $N > 0$ be arbitrary. By equality (80) in p.52 of [7], we have with probability 1

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |x_t^*|^\beta \chi_{\{|x_t^*|\le N\}} \, dt = \frac{\int_{\mathbb{R}} |x|^\beta \chi_{\{|x|\le N\}} \, p^*(x) \, dx}{\int_{\mathbb{R}} p^*(x) dx}$$

$$= \frac{\beta}{2} \int_{-N}^{N} |x|^\beta (1+|x|)^{-(\beta+1)} \, dx \, .$$

Since $|x|^\beta (1+|x|)^{-(\beta+1)}$ does not belong to $L^1(\mathbb{R}, dx)$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |x_t^*|^\beta \, dt \ge \lim_{T\to\infty} \frac{1}{T} \int_0^T |x_t^*|^\beta \chi_{\{|x_t^*|\le N\}} \, dt \to \infty \qquad (N \to \infty) \, .$$

Thus $K(v^*) = \infty$ a.s., and we can consider that $K(v) = \infty$ a.s. for all Γ -valued Borel measurable functions v on $\mathbb{R}$. Thus we have shown the assertion of Example 1.

Let

$$Q(x) = \exp\left\{ 2 \int_0^x \frac{1}{a(t)} \sup_{\gamma\in\Gamma} b(t,\gamma) \, dt \right\}, \qquad x \in \mathbb{R} \, ,$$

$$R(x) = \exp\left\{ 2 \int_0^x \frac{1}{a(t)} \inf_{\gamma\in\Gamma} b(t,\gamma) \, dt \right\}, \qquad x \in \mathbb{R} \, .$$

For $x \in \mathbb{R}$, let

$$F(x) = \frac{\int_{-\infty}^x \frac{f(t)}{a(t)} Q(t) \, dt}{\int_{-\infty}^x \frac{g(t)}{a(t)} Q(t) \, dt} \, , \quad G(x) = \frac{\int_x^\infty \frac{f(t)}{a(t)} R(t) \, dt}{\int_x^\infty \frac{g(t)}{a(t)} R(t) \, dt} \, .$$

The functions F and G are well–defined by (A1) $\sim$ (A5).

Lemma 1 *We assume (A1) $\sim$ (A5). Then there exists a constant $\alpha \in \mathbb{R}$ such that $F(\alpha) = G(\alpha)$ and*

$$F(x) < G(x) \, , \qquad \alpha < x \, ,$$
$$G(x) < F(x) \, , \qquad x < \alpha \, .$$

Furthermore, $F(x)$ is strictly decreasing on $(-\infty, \alpha]$ and $G(x)$ is strictly increasing on $[\alpha, \infty)$.

Using the differential calculus and describing the graphs of F and G, we can show Lemma 1.

We introduce the class in which a unique solution of (1.3) is constructed.

Definition We define the class $\mathcal{H}$ by all pairs (μ, ϕ) such that
(1) $\mu \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R})$,
(2) there exists a constant $C > 0$ such that

$$|\phi'(x)| \leq C(1 + |x|^{\beta+1}) , \qquad x \in \mathbb{R} .$$

(3) there exists a piecewise continuous function $L(x) > 0$ on $\mathbb{R}$ such that $\frac{\phi'(x)}{L(x)}$ is strictly increasing function on $\mathbb{R}$ and

$$\lim_{x \to -\infty} \frac{\phi'(x)}{L(x)} < 0 < \lim_{x \to \infty} \frac{\phi'(x)}{L(x)} .$$

Theorem 1 *Under (A1) $\sim$ (A5), Bellman equation (1.3) has a unique solution (λ, ϕ) in $\mathcal{H}$, and it is given by*

$$\lambda = F(\alpha) = G(\alpha) , \tag{2.5}$$

$$\phi'(x) = \begin{cases} 2 \displaystyle\int_x^\infty \left[\frac{f(t)}{a(t)} - \lambda \frac{g(t)}{a(t)} \right] \frac{R(t)}{R(x)} \, dt , & \alpha \leq x , \\ -2 \displaystyle\int_{-\infty}^x \left[\frac{f(t)}{a(t)} - \lambda \frac{g(t)}{a(t)} \right] \frac{Q(t)}{Q(x)} \, dt , & x < \alpha . \end{cases} \tag{2.6}$$

In addition, $\operatorname{sgn}(\phi'(x)) = \operatorname{sgn}(x - \alpha)$.

The proof of this theorem is similar to that of the case for the classical ergodic control (cf. [5]). So we omit it.

Example 2 . *Let $\Gamma = [-1, 1]$, $f(x) = |x|^2$, $g(x) = |x|$, $a(x) \equiv 2$ and $b(x, \gamma) = \gamma$. In this case, Bellman equation (1.3) is reduced to*

$$\lambda |x| = \phi''(x) - |\phi'(x)| + |x|^2 , \qquad x \in \mathbb{R} .$$

Its solution (λ, ϕ) is given by $\lambda = 2$, $\phi'(x) = x|x|$.

3. RATIO ERGODIC CONTROL

In this section we consider the ratio ergodic control problem (1.1), (1.2). Define $\mathcal{V}$ by the class of all $\Gamma-$ valued Borel measurable functions v on $\mathbb{R}$ such that

$$\lim_{R\to\infty} \sup_{|x|>R} \frac{x}{a(x)} b(x, v(x)) < -\frac{\beta+1}{2} . \tag{3.1}$$

For each $v \in \mathcal{V}$, (1.2) has a unique strong solution (x_t) by [8]. We investigate several properties of the response (x_t) to $v \in \mathcal{V}$. For $v \in \mathcal{V}$, set

$$p_v(x) = \frac{1}{a(x)} \exp\left\{ 2 \int_0^x \frac{1}{a(t)} b(t, v(t))\, dt \right\} , \qquad x \in \mathbb{R} , \tag{3.2}$$

and choose $\delta = \delta(v) > 0$ so that

$$\lim_{R\to\infty} \sup_{|x|>R} \frac{x}{a(x)} b(x, v(x)) < -\delta < -\frac{\beta+1}{2} \tag{3.3}$$

Since there exists a constant $r_1 > 0$ such that

$$p_v(x) \le \kappa^{-2} \max\{ p_v(r_1) , p_v(-r_1) \} \, |x/r_1|^{-2\delta} , \quad r_1 < |x|$$

and since $\beta - 2\delta < -1$, we have

Lemma 2 *We assume (A1) $\sim$ (A5). Then, for every $v \in \mathcal{V}$,*

$$p_v^{-1} \notin L^1(0,\infty) , \quad p_v^{-1} \notin L^1(-\infty, 0) , \quad |x|^\beta p_v(x) \in L^1(\mathbb{R}) . \tag{3.4}$$

In particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(x_t)\, dt = \frac{\int_{\mathbb{R}} g(x)\, p_v(x)\, dx}{\int_{\mathbb{R}} p_v(x) dx} > 0 \quad \text{a.s.} \tag{3.5}$$

Equality (3.5) follows (80) in p.52 of [7]. Throughout Lemmas 3 $\sim$ 5 below, we denote by $C_1, C_2, \cdots$ constants which are independent of the time variable $T > 0$.

Lemma 3 *We assume (A1) $\sim$ (A5). Then*

$$\mathbb{E} \sup_{0\le t\le T} |x_t|^{2\delta+1} \le C_1 \left(|\xi|^{2\delta+1} + T \right) , \qquad T \ge 0 .$$

Lemma 4 *We assume (A1) ~ (A5). Then, there exists a constant* $T_0 > 0$ *such that*

$$\mathbb{E} \sup_{0 \le t \le T} |x_t|^{\beta+2} \le C_2 T^{\frac{\beta+2}{2\delta+1}} , \qquad T \ge T_0 ,$$

$$\mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t \sqrt{a(x_s)}\, \phi'(x_s)\, dw_s \right| \le C_3 T^{\frac{1}{2} + \frac{\beta+2}{2(2\delta+1)}} , \qquad T \ge T_0 .$$

Lemma 5 *Let* (X_t) *be a stochastic process satisfying*

$$\mathbb{E} \sup_{0 \le t \le T} |X_t| \le C_4 T^{\theta} , \qquad T \ge T_1$$

for some constants $0 < \theta < 1$ *and* $T_1 > 0$. *Then, with probability 1,*

$$\frac{1}{T} \sup_{0 \le t \le T} |X_t| \to 0 , \qquad T \to \infty .$$

Lemmas 3 and 4 follow from the stochastic calculus. Lemma 5 follows from Borel– Cantelli lemma. Combining Lemmas 4, 5 and remarking that $\dfrac{\beta+2}{2\delta+1} < 1$, we have

Lemma 6 *We assume (A1) ~ (A5). Then, for every* $v \in \mathcal{V}$, *we have with probability 1*

$$\frac{1}{T} \sup_{0 \le t \le T} |x_t|^{\beta+2} \to 0 , \qquad (T \to 0) ,$$

$$\frac{1}{T} \sup_{0 \le t \le T} \left| \int_0^t \sqrt{a(x_s)}\, \phi'(x_s)\, dw_s \right| \to 0 , \qquad (T \to 0) .$$

By the measurable selection theorem (cf. Lemma 16.30 of [4]), we have

Lemma 7 *We assume (A1) ~ (A5). Then there exists* $\rho \in \mathcal{V}$ *satisfying*

$$b(x, \rho(x)) = \begin{cases} \inf_{\gamma \in \Gamma} b(x, \gamma) , & \alpha \le x , \\ \sup_{\gamma \in \Gamma} b(x, \gamma) , & x < \alpha . \end{cases}$$

Now we state the main results of this section.

Theorem 2 *We assume (A1) $\sim$ (A5). Then,*

(1) for each $v \in \mathcal{V}$, with probability 1, the cost $K(v)$ of (1.1) is equal to a constant which is independent of $\xi \in \mathbb{R}$ of (1.2).

(2) $\lambda = \min\{ K(v) \ : \ v \in \mathcal{V} \} = K(\rho)$.

Sketch of the proof. We obtain (1) by Lemma 2 and equality (80) in p.52 of [7]. Next, applying Ito formula to $\phi(x_t^*)$, where (x_t^*) is the response to ρ, and using the fact that (λ, ϕ) is a solution of (1.3), we obtain $\lambda = K(\rho)$ by (3.5) and Lemmas 6, 7. For $v \in \mathcal{V}$, we obtain $\lambda \leq K(\rho)$ similarly.

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