LINEAR QUADRATIC OPTIMAL CONTROL: FROM DETERMINISTIC TO STOCHASTIC CASES

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1 DETERMINISTIC LQ PROBLEM: A BRIEF HISTORICAL OVERVIEW

Given the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \tag{1.1}$$

and the cost functional

$$J(x_0, u(\cdot)) = \langle Gx(T), x(T)\rangle + \int_0^T [\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle]\, dt \tag{1.2}$$

where $x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for all $t \geq 0$, A, B, G, Q, R are matrices of correspondent dimension, and G, Q, R are symmetric. Our problem is:

Problem (DLQ). Minimize (1.2), subject to (1.1).

In the optimal control theory, the above problem is referred to as deterministic linear quadratic optimal control problem (DLQ problem, for short). Following are two definitions associated with it. Problem (DLQ) is said to be *finite* if

$$\forall x_0 \in \mathbb{R}^n, \;\; V(x_0) \triangleq \inf_{u(\cdot)\in L^2(0,T;\mathbb{R}^m)} J(x_0, u(\cdot)) > -\infty.$$

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Problem (DLQ) is said to be *(uniquely) solvable* if

$$\forall x_0 \in \mathbb{R}^n,\ \exists \bar{u}(\cdot) \in L^2(0,T;\mathbb{R}^m),\ \text{ s.t. } V(x_0) = J(x_0,\ \bar{u}(\cdot)).$$

LQ problem has roots going back to the very beginning of optimal control theory, the fundamental work was due to R. E. Kalman [5] in 1960. Since then, a lot of works have been done on DLQ problem, among them, we mention A. M. Letov [7], J. C. Willems [11], B. P. Molinari [10] and V. A. Yakubovic [13] etc. for finite dimensional cases, and I. Lasiecka and R. Triggiani [6], X. Li and J. Yong [8], S. Chen [2] and J. L. Lions [9] for infinite dimensional cases.

By now, it is very well known that

- If Problem (DLQ) is finite, by which, we mean that the infimum of the cost functional is finite, then $R \geq 0$.
- If $R \geq 0$, then Problem (DLQ) is (uniquely) solvable if and only if

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ \dot{p}(t) = -Qx(t) - A^T p(t), & p(T) = Gx(T), \\ Ru(t) + B^T p(t) = 0, \end{cases} \tag{1.3}$$

admits a (unique) solution (x, u, p).

- If $R > 0$ and Riccati equation

$$\dot{P} = -(PA + A^T P + Q) + PB^T R^{-1} BP, \quad P(T) = G. \tag{1.4}$$

admits a solution $P(\cdot)$, then Problem (DLQ) is uniquely solvable with the optimal control

$$\bar{u}(t) = -R^{-1} B^T P \bar{x}(t). \tag{1.5}$$

- If $Q \geq 0$, $G \geq 0$, $R > 0$, then Problem (DLQ) is uniquely solvable.

2 STOCHASTIC LQ PROBLEM: A NEW PHENOMENON

Let $(\Omega,\ \mathcal{F},\ \mathcal{P},\ \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space on which defined a one dimensional standard Brownian motion $w(\cdot)$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filteration generated by $w(\cdot)$, augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. Consider the following linear controlled stochastic diffenrential equation:

$$dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dw(t), \quad x(\tau) = \xi \tag{2.1}$$

and the cost functional

$$J(\tau, \xi; u(\cdot)) = E\Big\{\langle Gx(T), x(T)\rangle + \int_\tau^T \{\langle Qx(t), x(t)\rangle + \langle Ru(t), u(t)\rangle\}dt\Big\}, \tag{2.2}$$

where $\tau \in \mathcal{T}[0,T]$, the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times taking values in $[0,T]$, $\xi \in \chi_\tau \stackrel{\Delta}{=} L^2_{\mathcal{F}_\tau}(\Omega,\ \mathbb{R}^n)$, the set of all $\mathbb{R}^n$-valued $\mathcal{F}_\tau$-measurable square-integrable random variables; A, B, C, D are matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes, Q, R are symmetric matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes, G is symmetric matrix-valued $\mathcal{F}_T$-measurable bounded random variable, $u(\cdot)$ is a control process and $x(\cdot)$ is the corresponding state process. Our problem is:

Problem (SLQ). Minimize (2.2) subject to (2.1).
Here, our control process $u(\cdot)$ is taken from $\mathcal{U}[\tau, T] = L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m)$, the set of all $\mathbb{R}^m$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted square-integrable processes defined on the random interval $[\tau, T]$ with $\tau \in \mathcal{T}[0,T]$. Let $\Delta[0,T] = \cup_{r\in\mathcal{T}[0,T]}[\{r\} \times \chi_r]$, $V(\tau,\xi) = \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau,\xi,u(\cdot))$. Then one has

$$V : \Delta[0,t] \to \mathbb{R}.$$

Problem (SLQ) has recently been extensively studied in [3, 4], which is referred to as the stochastic linear quadratic optimal control problem (SLQ problem, for short) with random coefficient. Problem (SLQ) is said to be *finite* at $(\tau, \xi) \in \Delta[0,T]$, if

$$V(\tau,\xi) > -\infty$$

Problem (SLQ) is said to be *solvable* at $(\tau,\xi) \in \Delta[0,T]$, if there exists a $\bar{u}(\cdot)$, such that

$$J(\tau,\xi;\bar{u}(\cdot)) = V(\tau,\xi).$$

Comparing with its deterministic counterpart, Problem (SLQ) have some new interesting features. Firstly, in the deterministic case, $R \geq 0$ is a necessary condition for the finiteness; but in the stochastic case, $R \geq 0$ is no longer necessary for the finiteness. Example 2.1 shows that SLQ problem with $R < 0$ may uniquely solvable. Secondly, $R > 0$ can assure the unique solvability of Problem (DLQ), but can't even assure the finiteness of Problem (SLQ) sometimes. Example 2.2 shows this fact.

Example 2.1 Consider:

$$\dot{x}(t) = u(t), \quad x(0) = x_0 \in \mathbb{R},$$

with the cost functional

$$J(u(\cdot)) = x(T)^2 - \int_0^T u(t)^2 dt.$$

Thus, $R = -1 < 0$.

$$\inf_{u(\cdot)} J(u(\cdot)) = -\infty.$$

Consider the stochastic version:

$$dx(t) = u(t)dt + \delta u(t)dw(t), \quad x(0) = x_0,$$

with the cost functional

$$J(u(\cdot)) = E\Big\{x(T)^2 - \int_0^T u(t)^2 dt\Big\}.$$

It can be proved that, If $|\delta| > 0$ and

$$\delta^2(2\ln|\delta| - 1) > T - 1,$$

then Problem (SLQ) is solvable on $[0, T]$.

Example 2.2 Consider: $(0 < T < 1)$

$$\dot{x}(t) = u(t), \quad x(0) = x_0 \in \mathbb{R}.$$

with the cost functional

$$J(u(\cdot)) = -x(T)^2 + \int_0^T u(t)^2 dt.$$

Thus, $R = 1 > 0$. We can show that Problem (DLQ) is uniquely solvable with optimal control:

$$\bar{u}(t) = \frac{x(t)}{t + 1 - T}.$$

Now, consider the stochastic version:

$$dx(t) = u(t)dt + \delta u(t)dw(t), \quad x(0) = x_0.$$

with the cost functional

$$J(u(\cdot)) = E\Big\{ - x(T)^2 + \int_0^T u(t)^2 dt \Big\}.$$

It can be proved: If $|\delta| > 1$, then Problem (SLQ) is not finite.

Then one will ask: *what's the necessary condition for the finiteness of Problem (SLQ)*? The following theorem answer this.

Theorem 2.1 ([4]) *If Problem (SLQ) is finite at some* $(\tau, \xi) \in \Delta[0, T]$, *then*

$$R(T) + D^T(T)GD(T) \geq 0, \quad \text{a.s. } \omega \in \Omega. \tag{2.3}$$

From Theorem 2.1 and the arguments above, we can easy see that D plays a very interesting role in the finiteness and solvability of problem (SLQ). When A, B, C, D, G, Q, R are matrices, (2.3) becomes

$$R + D^T GD \geq 0. \tag{2.4}$$

In this case, if $D = 0$, i.e., control does not appear in the diffusion, (2.4) becomes $R \geq 0$, the same as the DLQ problem. In Example 2.2,

$$R + D^T GD = 1 - \delta^2.$$

Thus, $\delta > 1$ violates the necessary condition.

3 STOCHASTIC MAXIMUM PRINCIPLE AND STOCHASTIC RICCATI EQUATION

The following forward-backward stochastic differential equation (FBSDE, for short)

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dw(t), \\ dp(t) = -[A^T p(t) + C^T q(t) + Qx(t)]dt + q(t)dw(t), \\ x(\tau) = \xi, \qquad p(T) = Gx(T). \end{cases} \tag{3.1}$$

will play a very important role in the *stochastic maximun principle*:

Theorem 3.1 ([4]) *Problem (SLQ) is solvable at* $(\tau, \xi) \in \Delta[0, T]$, *with an optimal pair* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *if and only if the following forward-backward stochastic differential equation (FBSDE)*

$$\begin{cases} d\bar{x}(t) = [A\bar{x}(t) + B\bar{u}(t)]dt + [C\bar{x}(t) + D\bar{u}(t)]dw(t), \\ d\bar{p}(t) = -[A^T \bar{p}(t) + C^T \bar{q}(t) + Q\bar{x}(t)]dt + \bar{q}(t)dw(t), \\ \bar{x}(\tau) = \xi, \qquad \bar{p}(T) = G\bar{x}(T). \end{cases} \tag{3.2}$$

admits an adapted solution $(\bar{x}(\cdot), \bar{p}(\cdot), \bar{q}(\cdot))$ *such that*

$$I_{\Omega(\tau,\xi)}\Big[R\bar{u}(t) + B^T\bar{p}(t) + D^T\bar{q}(t)\Big] = 0, \quad in\ L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m), \tag{3.3}$$

and for any $u(\cdot) \in \mathcal{U}[\tau, T]$, *the unique adapted solution* $(x(\cdot), p(\cdot), q(\cdot))$ *of (3.1) with* $\xi = 0$ *satisfies*

$$E\int_\tau^T \langle Ru(t) + B^T p(t) + D^T q(t),\, u(t)\rangle\, dt \geq 0. \tag{3.4}$$

Another important thing associated with the solvability of Problem (SLQ) is the following *stochastic backward Riccati equation* for symmetric matrix-valued process $(P(\cdot), \Lambda(\cdot))$ in a random time interval:

$$\begin{cases} dP = -\{PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q \\ \qquad -(PB^T + C^T PD + \Lambda D)(R + D^T PD)^{-1} \\ \qquad\qquad \cdot(B^T P + D^T PC + D^T \Lambda)\}dt + \Lambda dw(t), \quad t \in [\tau, T], \\ P(T) = G. \\ \det[R(t) + D(t)^T P(t) D(t)] \neq 0, \quad t \in [\tau, T], \text{ a.s. } \omega \in \Omega, \end{cases} \tag{3.5}$$

where $\tau \in \mathcal{T}[0.T]$.

Theorem 3.2 ([3, 4]) *If (3.5) admits an adapted solution* (P, Λ), *then Problem (SLQ) is uniquely solvable at* (τ, ξ), *with the optimal control*

$$u(t) = -(R + D^T PD)^{-1}(B^T P + D^T PC + D^T \Lambda)x(t).$$

4 SOLVABILITY OF RICCATI EQUATION WITH DETERMINISTIC COEFFICIENT

Now, we consider Problem (SLQ) with deterministic coefficient, i.e., $\tau \in [0,T]$, $\xi \in \mathbb{R}^n$, $A, C, Q, G \in \mathbb{R}^{n\times n}$, $B, D \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{m\times m}$. In this case, the stochastic maximum principle is as following

Theorem 4.1 ([4]) *Let Problem (SLQ) be uniquely solvable. Then (3.1) admits a solution satisfies*

$$Ru(t) + B^T p(t) + D^T q(t) = 0. \tag{4.1}$$

Conversely, if (3.1)-(3.2) admits a (unique) solution $(\bar{x}, \bar{u}, \bar{p}, \bar{q})$, and any adapted solution (x, u, p, q) of (3.1) with $\xi = 0$ satisfies

$$E\int_0^T \langle Ru(t) + B^T p(t) + D^T q(t), u(t)\rangle dt \geq 0, \tag{4.2}$$

then Problem (SDLQ) is (uniquely) solvable; in addition, if R^{-1} exists, then

$$\bar{u}(t) = -R^{-1}[B^T \bar{p}(t) + D^T \bar{q}(t)], \tag{4.3}$$

is the optimal control.

The stochastic backward Riccati equation (3.5) becomes:

$$\begin{cases} P = -(PA + A^T P + C^T PC + Q) \\ \qquad +(PB^T + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC), \quad t \in [\tau, T], \\ P(T) = G. \end{cases} \tag{4.4}$$

For Riccati equation (4.4), we have (see [3, 4])

(1) If $R > 0, Q \geq 0, G \geq 0$, then (4.4) is globally solvable.

(2) If $C = 0$. Then Riccati equation (4.4) is globally solvable if and only if there exists $K^{\pm}$, such that

$$K^+ \geq R + D^T \Psi(K^+) \geq R + D^T \Psi(K^-) D \geq K^-, \tag{4.5}$$

where $\Psi(K)$ is the solution of

$$\begin{cases} \dot{P} = -(PA + A^T P + C^T PC + Q) \\ \quad +(PB^T + C^T PD)K^{-1}(B^T P + D^T PC), \\ P(T) = G. \end{cases} \tag{4.6}$$

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