OPTIMAL CONTROL PROBLEMS GOVERNED BY AN ELLIPTIC DIFFERENTIAL EQUATION WITH CRITICAL EXPONENT

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Abstract: This work is concerned with analysis and optimal control of a semi-linear elliptic partial differential equation involving critical exponent. Some necessary conditions for optimality are given.

Key words and phrases: Minimal positive solution, state constraint, finite codimensionality, Penalty functional.

1 INTRODUCTION

We discuss the optimal control problem for which the state is governed by a semilinear elliptic partial differential equation with a distributed control.

The system reads

$$\begin{cases} -\Delta y = y^p + u, & \text{in} \quad \Omega \\ y|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Where $\Omega \subset R^N$ (with $N \geq 3$) is a bounded region with $\partial\Omega$ smooth. $p = \frac{N+2}{N-2}$ is the critical sobolev exponent.

The cost functional is given by

$$L(y,u) = G(y) + H(u) \tag{1.2}$$

We assume that

$(\mathbf{H_L})$ G and H: $L^2(\Omega) \longrightarrow \bar{R} = (-\infty, +\infty]$ are proper, convex and lower semicontinuous.

In section 2 , we will see that there is a minimal positive solution $y(x;u) \in H_0^1(\Omega)$ for each $u \in B_r^+(0) \subset L^\infty(\Omega) \subset L^2(\Omega)$, where $B_r^+(0)$ is given by

$$B_r^+(0) = \{u \in L^\infty(\Omega) | \, ||u||_\infty \leq r \quad \text{and}\, u(x) \geq 0, \quad \text{a.e.} \quad x \in \Omega\} \tag{1.3}$$

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and $r > 0$ is a constant given in §2.

Note that here we define $y(x;0) \equiv 0$. Thus we may consider $u \in B_r^+(0)$ as the control and $y(x;u)$, the minimal positive solution of (1.1) corresponding to the state.

We assume

$(\mathbf{H_F})$ Let Y be a Banach space with strict convex dual Y^* , $F : H_0^1(\Omega) \to Y$ be continuously Frechet differentiable and $Q \subset Y$ be a closed and convex subset.

Set

$$A = \{(y(x;u), u) \in L^2(\Omega) \times L^2(\Omega) \,|\, u \in B_r^+(0),\ y(x;u) \text{ is the minimal positive solution of (1.1) corresponding to } u \text{ and } y(x;0) \equiv 0\}$$

A pair $(y,u) \in A$ is called a feasible pair.

$$A_{ad} = \{(y(x;u), u) \in A | F(y(x;u)) \subset Q\}$$

A pair$(y,u) \in A_{ad}$ is called an admissible pair.

Note that $F(y) \subset Q$ is a kind of state constraint which was given by X.Li and J.Yong(cf.[3]). For its applications, we refer readers to [2] and [3].

We formulate the optimal control problem as follows

$$(\mathbf{P}) \qquad \text{Inf } L(y,u) \qquad \text{over all} \quad (y,u) \in A_{ad}$$

We shall study the necessary conditions for the problem (P) in this paper.

2 THE MINIMAL POSITIVE SOLUTION

We first quote a result of [5] as follows.

Theorem A: For any $u \in H^{-1}(\Omega)$ with $||u||_{H^{-1}} \leq C_N S^{\frac{N}{2}}$, problem (2.1) possesses at least one positive solution y with $y \not\equiv 0$ in Ω . Where $C_N = \frac{4}{N-2}(\frac{N-2}{N+2})^{\frac{N+2}{4}}$ and S is the best sobolev constant for the embedding $H_0^1(\Omega) \to L^p(\Omega)$.

From Theorem A and the methods of monotone interation we can prove the existence of minimal positive solution for problem (2.1).

Theorem 2.1 *Under the assumption of Theorem A, Problem (2.1) possesses a unique minimal positive solution $y \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ if $u \in L^\infty(\Omega)$.*

In the following we discuss some properties of the minimal positive solution of (1.1).

Lemma 2.1 *Let $y(x;u)$ be the minimal positive solution of (1.1), then the corresponding eigenvalue problem*

$$\begin{cases} -\Delta\varphi = \lambda p[y(x;u)]^{p-1}\varphi, & in \quad \Omega \\ \varphi \in H_0^1(\Omega) \end{cases} \tag{2.1}$$

has the first eigenvalue $\lambda_1(u) > 1$ *for all* $u \in B_R^+(0)$ *and the corresponding eigenfunction* $\varphi_1 > 0$ *in* Ω.

Where

$$B_R^+ = \left\{ u \in H_0^{-1}(\Omega) \mid ||u||_{H_0^1(\Omega)} \leq R, u \geq 0 \quad in \quad \Omega \right\}$$

and $R < C_N S^{\frac{N}{2}}$.

Proof: By the standard argument we can prove that the minimum

$$\lambda_1 = \operatorname{Inf} \left\{ \int_\Omega |\nabla v|^2 \, dx \mid v \in H_0^1(\Omega), \int_\Omega p y^{p-1}(x;u) v^2 \, dx = 1 \right\} \tag{2.2}$$

can be achieved by some function $\varphi_1 > 0$. Thus eigenvalue problem (2.2) has a solution (λ_1, φ_1). Now we prove that $\lambda_1 > 1$.

Indeed, for any $u \in B_R^+(0)$, we can find a function $w \in H_0^{-1}(\Omega)$ with $||w||_{H_0^{-1}} \leq C_N S^{\frac{N}{2}}$, $w \geq u$, $w \not\equiv u$ a.e. in Ω such that problem (2.1) (corresponding to w) possesses a minimal positive solution $y(x;w)$. Let $y(x;u)$ be the solution of (2.1), we have

$$\lambda_1 \int_\Omega p y^{p-1}(x;u)\varphi_1[y(x;w) - y(x:u)]dx \int_\Omega p y^{p-1}(x;u)\varphi_1(y(x;w) - y(x;u)) \, dx \tag{2.3}$$

Which gives $\lambda_1 > 1$ for all $u \in B_R^+(0)$. This completes the proof.

Theorem 2.2 *Assume* $u \in B_R^+(0)$ *and* $y(x;u)$ *be a minimal positive solution of (1.1) corresponding to* u. *Then for any* $g(x) \in H_0^{-1}(\Omega)$, *The problem*

$$\begin{cases} -\Delta\omega = p y^{p-1}(x;u)\omega + g(x) \\ \omega \in H_0^1(\Omega) \end{cases} \tag{2.4}$$

has a unique solution ω *satisfying*

$$||\omega||_{H_0^1(\Omega)} \leq C|g||_{H_0^{-1}(\Omega)} \tag{2.5}$$

for some constant $C > 0$.

Proof: By a standard argument and Lemma 2.1, one can get the existence of the solution of the equation (2.4).

Now we are on the position to prove(2.5).

Let ω be the solution of (2.5). Multiplying (2.5) by ω and integrating by parts we have

$$\int_\Omega |\nabla\omega|^2 \, dx = \int_\Omega p y^{p-1}(x;u)\omega^2 \, dx + \int_\Omega g\omega \, dx.$$

Now Lemma 2.1 implies

$$(1-\frac{1}{\lambda_1})||\omega||^2_{H^1_0(\Omega)} \le C||g||_{H_0^{-1}(\Omega)}||\omega||_{H^1_0(\Omega)}$$
$$\le \epsilon||\omega||^2_{H^1_0(\Omega)} + C_\epsilon||g||^2_{H_0^{-1}(\Omega)}.$$

From Lemma2.1 we can choose ϵ small enough so that $(1-\frac{1}{\lambda_1}-\epsilon) \ge \lambda_2 > 0$ for some constant $\lambda_2 > 0$.

Thus

$$||\omega||_{H^1_0(\Omega)} \le \frac{C_\epsilon}{\lambda_2}||g||_{H_0^{-1}(\Omega)}$$

This gives (2.5) by taking $C = \frac{C_\epsilon}{\lambda_2}$.

The uniqueness of the solution for (2.4) comes from (2.5).

Corollary 2.1 *Let $u \in B_R^+(0)$ and $y(x;u)$ be the minimal solution of (1.1). Then $y(x,u)$ is continuous in $H_0^{-1}(\Omega)$ with respect to control function u .*

Proof: Define

$$F : H_0^{-1}(\Omega) \times H_0^1(\Omega) \to H_0^{-1}(\Omega) \quad \text{by}$$

$$F(u,y) = \Delta y + y^p + u, \quad \text{for} \quad (u,y) \in H_0^{-1}(\Omega) \times H_0^1(\Omega) \tag{2.6}$$

From Lemma 2.1 and Theorem 2.2, we know that

$$F_y(u,y)\omega = \Delta\omega + py^{p-1} + py^{p-1}(x;u)\omega$$

is an isomorphism of $H_0^1(\Omega)$ onto $H_0^{-1}(\Omega)$.

It follows from Implicit Function Theorem that the solution of $F(u,y) = 0$ near $(u, y(x;u))$ is given by a continuous curve.

Theorem 2.3 *Let $u, v \in B_R^+(0)$ and $y(x,u), y(x,v)$ be the minimal positive solution of (1.1) corresponding to u, v respectively. If $u \to v$ in $H_0^{-1}(\Omega)$ and $u - v$ dosen't change the sign. Then*

$$||y(x;u) - y(x;v)||_{H^1_0(\Omega)} \le C||u-v||_{H_0^{-1}(\Omega)},$$

for $||u-v||_{H^1_0(\Omega)}$ small enough.

Where C is a constant independent of u.

Proof: Without loss of generality, we may assume that $u \ge v$, a,e. in Ω.
By Remark 2.1 and (1.1) we have

$$\int_\Omega |\nabla(y(x;u) - y(x;v))|^2\,dx$$
$$\le p\int_\Omega y^{p-1}(x;u)(y(x;u)-y(x;v))^2\,dx + \int_\Omega (u-v)(y(x;u)-y(x;v))\,dx$$

By lemma 2.1, Holder's inequality and Young's inequality, we have

$$(1-\frac{1}{\lambda_1(u)})\int_\Omega |\nabla(y(x;u)-y(x;v))|^2\,dx$$
$$\le \epsilon||y(x;u)-y(x;v)||^2_{H^1_0(\Omega)} + C_\epsilon||u-v||^2_{H_0^{-1}(\Omega)}$$

for any $\epsilon > 0$. Where C_ϵ is a positive constant depending on ϵ .

Note that $\lambda_1(u)$ is the first eigenvalue for the problem (2.2) corresponding to $y(x;u)$. By corollary 2.1, as $u \to v$ in $H_0^{-1}(\Omega)$, $y(x;u) \to y(x;v)$ in $H_0^1(\Omega)$. Then by (2.3), $\lambda_1(u) \to \lambda_1(v)$.

Thus, as $||u-v||_{H_0^{-1}(\Omega)}$ small enough, we have

$$(1 - \frac{1}{\lambda_1(v)} - \epsilon_1 - \epsilon)||y(x;u) - y(x;v)||_{H_0^1(\Omega)} \leq C_\epsilon ||u-v||_{H_0^{-1}(\Omega)}$$

for some $\epsilon_1 > 0$ with $1 - \frac{1}{\lambda_1(v)} - \epsilon_1 > 0$.

This completes the proof.

3 FINITE CODIMENSIONALITY

In this section, we will give some results relative to finite codimensionality of a set. For the detail, we refer readers to [3] and [6].

Lemma 3.1 *Let Q_1 and Q_2 be subsets of some Banach space X . Let Q_1 be finite codimensional in X . Then for any $\alpha \in R \setminus \{0\}$, $\beta \in R$,*

$$\alpha Q_1 - \beta Q_2 \equiv \{\alpha x_1 - \beta x_2 | x_1 \in Q_1, x_2 \in Q_2\}$$

is finite codimensional in X .

Lemma 3.2 *Let Q be finite codimensional in X . Let $\{f_n\}_{n \geq 1} \subset X^*$ with $|f_n| \geq \delta > 0$, $f_n \to f \in X^*$ in the weak-star topology, and*

$$\langle f_n, x \rangle \geq -\epsilon_n, \quad \forall x \in Q, \quad n \geq 1$$

when $\epsilon_n \to 0$. Then $f \neq 0$.

4 NECESSARY CONDITION FOR OPTIMALITY

In this section, we discuss the necessary conditions for (y^*, u^*) to be an optimal pair for (P).

Our basic assumptions are given by (H_L) and (H_F) in §1 .

Let (y^*, u^*) be optimal for the problem (P).

Consider the variational systems of (1.1) as follows:

$$\begin{cases} -\Delta z = py^{*(p-1)}z + (v-u^*)^+, & x \in \Omega \\ z|_{\partial\Omega} = 0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} -\Delta z = py^{*(p-1)}z + (v-u^*)^-, & x \in \Omega \\ z|_{\partial\Omega} = 0 \end{cases} \tag{4.2}$$

By Theorem 2.2, for each $v \in B_r^+(0)$, both (4.1) and (4.2) have a unique solution $z(.;v^+)$ and $z(.;v^-)$ in $H_0^1(\Omega)$.

Thus we may define

$R^+ = \{z(.;v^+)|v \in B_r^+(0), z(.;v^+)$ is the solution of (4.1) corresponding to $v\}$

$R^- = \{z(\cdot\,;v^-)\,|\,v \in B_r^+(0), z(\cdot\,;v^-)$is the solution of (4.2) corresponding to $v\}$

Our another basic assumption which plays a key role in dealing with the state constraint is as follows:

(H_R) Both $F'(y^*)R^+ - Q$ and $F'(y^*)R^- - Q$ have finite codimentionality.

The following results is important for us to introduce our penalty functionals in the proof of our main Theorem 4.1.

Lemma 4.1 *Let H be a Hilbert space, $f : H \to \bar{R}$ be proper convex and lower semicontinuous. Suppose that ∂f (the subdifferential of f) is locally bounded at y^*. Then there exists a neighborhood $O(y^*)$ of y^* such that $f_\lambda(y) \to f(y)$ uniformly in $O(y^*)$, where f_λ is the regularization of f (cf. [4], [7]).*

Proof: It is trivial from [4] and [7].

Our main results on the necessary condition for (y^*, u^*) to be an optimal pair are as follows:

Theorem 4.1 *Let (y^*, u^*) be an optimal pair for $P)$ and $(H_L), (H_F)$ and (H_R) hold. Assume that ∂G and ∂H, the subdifferentials of G and H are locally bounded at y^* and u^* (in $L^2(\Omega)$) respectively. Then there exists a triplet*

$$(\lambda, \psi, q) \in [-1, 0] \times H_0^1(\Omega) \times Y^*, \quad \textit{such that} \quad (\lambda, q) \neq 0,$$

$$\langle q, \eta - F(y^*)\rangle \leq 0, \quad \forall \eta \in Q. \tag{4.3}$$

$$\begin{cases} -\Delta\Psi = py^{*(p-1)}\Psi + \lambda\alpha - [F'(y^*)]^*q & \textit{in} \quad \Omega \\ \Psi|_{\partial\Omega} = 0, \end{cases} \tag{4.4}$$

where $\alpha \in \partial G(y^*)$
and

$$\langle \Psi + \lambda\beta, v - u^*\rangle \leq 0 \quad \textit{for any} \quad v \in B_r^+(0) \tag{4.5}$$

Where $\beta \in \partial H(u^*)$.
In the case $N[F'(y^*)]^* = 0$ (i.e. $[F'(y^*)]^*$ is injective), $(\lambda, \psi) \neq 0$.

Proof: Without lose of generality, assume that

$$L(y^*, u^*) = G(y^*) + H(u^*) = 0.$$

Since ∂G and ∂H are locally bounded at y^* and u^* respectively, by lemma 4.1, we obtain that there exist neighborhoods $O(y^*)$ and $O(u^*)$ of y^* and u^* in $L^2(\Omega)$ respectively, such that

$$G(y) + H(u) \geq G_\lambda(y) + H_\lambda(u) \to G(y) + H(u)$$
$$\text{as} \quad \lambda \to 0 \quad \text{uniformly in} \quad y \in O(y^*) \quad \text{and} \quad u \in O(y^*).$$

Thus for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ ($\delta(\epsilon) \to 0$ as $\epsilon \to 0$), such that

$$G(y) + H(u) + \epsilon \geq G_{\delta(\epsilon)}(y) + H_{\delta(\epsilon)}(u) + \epsilon > G(y) + H(u) \tag{4.6}$$

for all $(y, u) \in O(y^*) \times O(u^*)$.

Let $U = (B_r^+(0), d)$ with $d(u, v) = ||u - v||$, where the norm is taken in $L^2(\Omega)$. Then U is a metric space. Note that $B_r^+(0) \subset L^\infty(\Omega) \subset L^2(\Omega)$ and one can check easily that $B_r^+(0)$ is closed in $L^2(\Omega)$. Thus U is a complete metric space.

Now we define $L_\epsilon : U \to R$ by

$$L_\epsilon(u) = \{d_Q^2(F(y(u))) + [G_{\delta(\epsilon)}(y(u)) + H_{\delta(\epsilon)}(u) + \epsilon]^2\}^{\frac{1}{2}}.$$

Where $d_Q(w) = \mathrm{Inf}||w - z||, z \in Q$ and $y \equiv y(x; u)$ is the unique minimal positive solution of (1.1) corresponding to u.

By Ekeland's variational principle , there exists a $u^\epsilon \in U$ for each $\epsilon > 0$ such that

$$d(u^*, u^\epsilon) \leq \sqrt{\epsilon}, \quad \text{i.e.}: \quad ||u^* - u^\epsilon|| \leq \sqrt{\epsilon} \tag{4.7}$$

and

$$L_\epsilon(u) - L_\epsilon(u^\epsilon) \geq -\sqrt{\epsilon} d(u, u^\epsilon), \quad \forall u \in U. \tag{4.8}$$

Let $v \in U$, we define

$$u_\rho^\epsilon = u^\epsilon + \rho(v - u^\epsilon)^+. \tag{4.9}$$

It's clear that $u_\rho^\epsilon \in U$ and

$$u_\rho^\epsilon - u^\epsilon = \rho(v - u^\epsilon)^+ \to 0 \quad \text{in} \quad \mathrm{L}^\infty(\Omega) \quad \text{as} \quad \rho \to 0^+.$$

Let $y_\rho^\epsilon \equiv y_\rho^\epsilon(\cdot\,; u_\rho^\epsilon)$ be the minimal positive solution of (1.1) corresponding to u_ρ^ϵ and $z_\rho^\epsilon \equiv \frac{y_\rho^\epsilon - y^\epsilon}{\rho}$.

Consider

$$\begin{cases} -\Delta z^\epsilon = p(y^\epsilon)^{p-1} z^\epsilon + (v - u^\epsilon)^+, & x \in \Omega \\ z^\epsilon|_{\partial\Omega} = 0. \end{cases} \tag{4.10}$$

We have

$$-\Delta(z^\epsilon - z_\rho^\epsilon) - p(y_\epsilon)^{p-1}(z^\epsilon - z_\rho^\epsilon) = (p(y^\epsilon)^{p-1} - a_\rho^\epsilon) z_\rho^\epsilon, \tag{4.11}$$

where $a_\rho^\epsilon = \int_0^1 p[y^\epsilon + t(y_\rho^\epsilon - y^\epsilon)]^{p-1}\, dt$.

Multiplying (4.11) by $(z^\epsilon - z_\rho^\epsilon)$ and interating on Ω, we obtain (note that both y^ϵ and y_ρ^ϵ are positive)

$$\begin{aligned} &\int_\Omega |\nabla(z^\epsilon - z_\rho^\epsilon)|^2\, dx - \int_\Omega p(y^\epsilon)^{p-1} \cdot (z^\epsilon - z_\rho^\epsilon)^2\, dx \\ &= \int_\Omega [p(y^\epsilon)^{p-1} - a_\rho^\epsilon] z_\rho^\epsilon \cdot (z_\rho^\epsilon - z^\epsilon)\, dx \\ &\leq \frac{C_\epsilon}{2} [\int_\Omega |y_\rho^\epsilon - y^\epsilon|^{\frac{N(p-1)}{2}}\, dx]^{\frac{4}{N}} \cdot ||z_\rho^\epsilon||_{H_0^1}^2 + \frac{\epsilon}{2} ||z^\epsilon - z_\rho^\epsilon||_{H_0^1}^2 \end{aligned}$$

for any $\epsilon > 0$.

By Lemma 2.1,

$$(1-\frac{1}{\lambda_1}-\epsilon)||z^\epsilon - z_\rho^\epsilon||_{H_0^1}^2 \le \frac{C_\epsilon}{2}[\int_\Omega |y_\rho^\epsilon - y^\epsilon|^{\frac{N(p-1)}{2}}\,dx]^{\frac{4}{N}} \cdot ||z_\rho^\epsilon||_{H_0^1}^2.$$

Taking ϵ small enough s.t. $1-\frac{1}{\lambda_1}-\epsilon \le r > 0$, we have

$$||z^\epsilon - z_\rho^\epsilon||_{H_0^1}^2 \le \frac{1}{r}\cdot\frac{C}{2}[\int_\Omega |y_\rho^\epsilon - y^\epsilon|^{\frac{N(p-1)}{2}}\,dx]^{\frac{4}{N}} \cdot ||z_\rho^\epsilon||_{H_0^1}^2 \tag{4.12}$$

Since $p=\frac{N+2}{N-2}$ and $y_\rho^\epsilon \to y^\epsilon$ in $H_0^1(\Omega)$, we have

$$\left[\int_\Omega |y_\rho^\epsilon - y^\epsilon|^{\frac{N(p-1)}{2}}\right]^{\frac{4}{N}} \le C_1 \cdot ||y_\rho^\epsilon - y^\epsilon||_{H_0^1}^{\frac{4}{N-2}}$$

On the other hand , by Theorem 2.3,

$$||z_\rho^\epsilon||_{H_0^1} = ||\frac{y_\rho^\epsilon - y^\epsilon}{\rho}||_{H_0^1} \le \frac{C_2\cdot\rho||(v-u^\epsilon)^+||_{L^2(\Omega)}}{\rho} \le C_3$$

Thus (4.12) gives us

$$||z^\epsilon - z_\rho^\epsilon||_{H_0^1} \le C_4 \cdot ||y_\rho^\epsilon - y^\epsilon||_{H_0^1}^{\frac{2}{N-2}} \to 0 \quad \text{as} \quad \rho \to 0.$$

Thus we obtain

$$y_\rho^\epsilon = y^\epsilon + \rho z^\epsilon + \rho\, o(1) \quad \text{in} \quad H_0^1(\Omega) \tag{4.13}$$

Next we estimate $G_{\delta(\epsilon)}(y_\rho^\epsilon) - G_{\delta(\epsilon)}(y^\epsilon)$ and $H_{\delta(\epsilon)}(u_\rho^\epsilon) - H_{\delta(\epsilon)}(u^\epsilon)$.
Clearly, we have

$$\begin{aligned}&\tfrac{G_{\delta(\epsilon)}(y_\rho^\epsilon)-G_{\delta(\epsilon)}(y^\epsilon)}{\rho} = \langle \dot{G}_{\delta(\epsilon)}(y^\epsilon), z_\rho^\epsilon\rangle + \tfrac{1}{\rho}o(||y_\rho^\epsilon - y^\epsilon||_{L^2})\\ &= \langle \dot{G}_{\delta(\epsilon)}(y^\epsilon), z_\rho^\epsilon\rangle + o(1), \quad \text{as} \quad \rho\to 0\end{aligned} \tag{4.14}$$

and

$$\tfrac{H_{\delta(\epsilon)}(u_\rho^\epsilon)-H_{\delta(\epsilon)}(u^\epsilon)}{\rho} = \langle \dot{H}_{\delta(\epsilon)}(u^\epsilon), (v-u^\epsilon)^+\rangle + o(1), \quad \text{as} \quad \rho\to 0 \tag{4.15}$$

We have

$$\frac{L_\epsilon(u_\rho^\epsilon) - L_\epsilon(u^\epsilon)}{\rho} \ge -\sqrt{\epsilon}||(v-u^\epsilon)^+||_{L^2} \ge -\sqrt{\epsilon}M. \tag{4.16}$$

It's clear that

$$L_\epsilon(u_\rho^\epsilon) = L_\epsilon(u^\epsilon) + o(1) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \rho\to 0^+. \tag{4.17}$$

One can check that $L_\epsilon(u^\epsilon) \ne 0$ for ϵ small enough.

So we have

$$
\begin{aligned}
&-\sqrt{\epsilon}M \leq \tfrac{1}{L_\epsilon(u^\epsilon_p)+L_\epsilon(u^\epsilon)}\{\tfrac{[G_{\delta(\epsilon)}(y^\epsilon_\rho)+H_{\delta(\epsilon)}(u^\epsilon_\rho)+\epsilon]^2}{\rho} \\
&-\tfrac{[G_{\delta(\epsilon)}(y^\epsilon)+H_{\delta(\epsilon)}(u^\epsilon)+\epsilon]^2}{\rho} + \tfrac{d^2_Q(F(y^\epsilon_\rho))-d^2_Q(F(y^\epsilon))}{\rho}\} \\
&\to \tfrac{G_{\delta(\epsilon)}(y^\epsilon)+H_{\delta(\epsilon)}(u^\epsilon)+\epsilon}{L_\epsilon(u^\epsilon)}[\langle \dot{G}_{\delta(\epsilon)}(y^\epsilon), z^\epsilon\rangle + \langle \dot{H}_{\delta(\epsilon)}(u^\epsilon), (v-u^\epsilon)^+]\\
&+\langle \tfrac{d_Q(F(y^\epsilon))\xi^\epsilon}{L_\epsilon(u^\epsilon)}, F'(y^\epsilon)z^\epsilon\rangle,
\end{aligned}
\tag{4.18}
$$

Define

$$
\lambda^\epsilon_0 = \frac{G_{\delta(\epsilon)}(y^\epsilon)+H_{\delta(\epsilon)}(u^\epsilon)+\epsilon}{L_\epsilon(u^\epsilon)} \quad \in [0,1], \qquad q^\epsilon = \frac{d_Q(F(y^\epsilon))\xi^\epsilon}{L_\epsilon(u^\epsilon)} \tag{4.19}
$$

Then

$$
-\sqrt{\epsilon}M \leq \lambda^\epsilon_0[\langle \dot{G}_{\delta(\epsilon)}(y^\epsilon), z^\epsilon\rangle + \langle \dot{H}_{\delta(\epsilon)}(u^\epsilon), (v-u^\epsilon)^+\rangle] + \langle q^\epsilon, F'(y^\epsilon)z^\epsilon\rangle. \tag{4.20}
$$

Since Y^* is strictly convex, we have

$$
|\lambda^\epsilon_0|^2 + ||q^\epsilon||^2_{Y^*} = 1 \tag{4.21}
$$

and

$$
\langle q^\epsilon, \eta - F(y^\epsilon)\rangle \leq 0 \quad \forall \eta \in Q \tag{4.22}
$$

In order to pass to the limits for $\epsilon \to 0^+$,
We first consider equation as follows:

$$
\begin{cases} -\Delta z = p(y^*)^{p-1}z + (v-u^*)^+, & x \in \Omega \\ z|_{\partial\Omega} = 0 \end{cases} \tag{4.23}
$$

which has a unique solution z in $H^1_0(\Omega)$, by Theorem 2.2.
By the similar arguments in (4.12) and (4.13), We have

$$
z^\epsilon \to z \text{ in} H^1_0(\Omega). \tag{4.24}
$$

Consider $\{\dot{G}_{\delta(\epsilon)}(y^\epsilon)\}_{\epsilon>0}$ and $\{\dot{H}_{\delta(\epsilon)}(u^\epsilon)\}_{\epsilon>0}$.
By a standard argument in [4], one can get easily that

$$
\begin{aligned}
\dot{G}_{\delta(\epsilon)}(y^\epsilon) \to \alpha \in \partial G(y^*) \quad \text{weakly in} \quad L^2(\Omega) \quad \text{and} \\
\dot{H}_{\delta(\epsilon)}(u^\epsilon) \to \beta \in \partial H(u^*) \quad \text{weakly in} \quad L^2(\Omega).
\end{aligned}
\tag{4.25}
$$

Next, by the hypothese (H_F) , we have $F'(y^\epsilon) \to F'(y^*)$ in $L(H^1_0(\Omega);Y)$.
Thus (4.24) implies

$$
\begin{aligned}
\lambda^\epsilon_0[\langle \dot{G}_{\delta(\epsilon)}(y^\epsilon), z^\epsilon\rangle + \langle \dot{H}_{\delta(\epsilon)}(u^\epsilon), (v-u^\epsilon)^+\rangle] + \\
\langle q^\epsilon, F'(y^*)z - \eta + F(y^*)\rangle \geq -\theta_\epsilon,
\end{aligned}
\tag{4.26}
$$

$\forall v \in U$ and $\eta \in Q$, and $\theta_\epsilon \to 0$ uniformly on $v \in U$ as $\epsilon \to 0^+$.

Now since $F'(y^*)R^+ - Q$ is finite codimensional in Y , by Lemma 3.2 and (4.21), we can assume (relabelling if necessary) that $(\lambda_0^\epsilon, q^\epsilon) \to (\lambda_0, q) \neq 0$ weakly in $R \times Y^*$.

Thus, by taking the limits for $\epsilon \to 0$, we obtain

$$\begin{aligned} &\lambda_0[\langle \alpha, z\rangle + \langle \beta, (v-u^*)^+\rangle] + \langle q, F'(y^*)z - \eta + F(y^*)\rangle \geq 0 \\ &\quad \text{for all} \quad v \in U \quad \text{and} \quad \eta \in Q \end{aligned} \tag{4.27}$$

Note that z depends on v .

After some simple calculations, we obtain

$$0 \geq \langle \Psi + \lambda\beta, (v-u^*)^+\rangle, \quad \forall v \in U$$

Similarly, by taking consideration of $u_\rho^\epsilon = u^\epsilon + \rho(v-u^\epsilon)^-$, we obtain

$$0 \geq \langle \Psi + \lambda\beta, (v-u^*)^-\rangle, \quad \forall v \in U$$

Thus

$$\langle \Psi + \lambda\beta, v - u^*\rangle \leq 0, \quad v \in U$$

If $(\lambda, \Psi) = 0$, then by (4.4), we have $[F'(y^*)]^* q = 0$. Thus in the case where $N[F'(y^*)]^* = \{0\}$, we must have $(\lambda, \Psi) \neq 0$

This completes the proof.

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