IDENTIFICATION PROBLEM FOR A WAVE EQUATION VIA OPTIMAL CONTROL

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Abstract We approximate an identification problem by applying optimal control techniques to a Tikhonov's regularization. We seek to identify the dispersive coefficient in a wave equation and allow for the case of error or uncertainty in the observations used for the identification.

1 INTRODUCTION

We apply optimal control techniques to find approximate solutions to an identification problem for a wave equation. Consider the wave equation in $Q = \Omega \times (0,T)$:

$$\begin{aligned}
w_{tt} &= \Delta w + hw + f && \text{in } Q \\
w &= 0 && \text{on } \partial\Omega \times (0,T) \\
w &= w_0(x) && t = 0, \quad x \in \Omega \\
w_t &= w_1(x) && t = 0, \quad x \in \Omega
\end{aligned} \tag{1.1}$$

where $f \in L^2(Q)$, $w_0 \in H_0^1(\Omega)$, $w_1 \in L^2(\Omega)$ and $\Omega \subset R^n$ with C^2 boundary. We seek to identify the dispersive coefficient h from observations of the solution on a set $Q' \subset Q$.

The identification problem is to find bounded h such that the corresponding solution $w = w(h)$ to (1.1) is close to the observations z on Q'. Thus, we want

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to minimize the functional

$$J(h) = \frac{1}{2}\int_{Q'} (w(h) - z)^2 dxdt \tag{1.2}$$

over the class

$$U = \{h \in L^\infty(Q) | -M \le h(x,t) \le M\}.$$

To approximate this problem, we introduce the following control problem for $\beta > 0$:

$$\min_{h \in U} J_\beta(h) \tag{1.3}$$

with

$$J_\beta(h) = \frac{1}{2}\left(\int_{Q'} (w(h) - z)^2 dxdt + \beta \int_Q h^2 dxdt\right).$$

The coefficient to be identified is viewed as a control, which is adjusted to get the corresponding solution, $w(h)$, close to the observations z. This type of approximation is called Tikhonov's regularization [1].

See Liang [6] for some further results on this type of control problem (without the connection to identification). See Lenhart, Protopopescu and Yong [4, 5] for similar approximation techniques for other wave equation problems. See [1, 2] for other approaches to using Tikhonov's regularization for identification problems.

In section 2, we prove the existence of an optimal control to the approximate problem (1.3) and characterize it through an optimality system. In section 3, we connect these approximations to the identification problem as $\beta \to 0$. Specifically, we show that as $\beta \to 0$, the optimal control h_β for the functional J_β, converges to a feasible coefficient. We allow for the possibility that there is some error or uncertainty in the observations z. Thus the solution, which is the limit of $w_\beta = w(h_\beta)$ as $\beta \to 0$, may not match the observation z.

2 OPTIMAL CONTROL PROBLEM

We assume $f,\ f_t \in L^2(Q)$ and define the bilinear form:

$$B[u,v;t] = \int_\Omega \nabla u \nabla v dx - \int_\Omega huvdx.$$

Definition: Given $h \in U$, a function $w \in L^2(0,T;H_0^1(\Omega))$, with $w_t \in L^2(0,T;L^2(\Omega))$ and $w_{tt} \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the problem (1.1) if

$$\int_0^T < w_{tt}, \phi > dt + \int_0^T B[w,\phi;t]dt = \int_Q f\phi dxdt$$

for any $\phi \in H_0^1(\Omega)$, and $w(0) = w_0$; $w_t(0) = w_1$, Here $< \, , >$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Lemma 2.1 For $h \in U$, the problem (1.1) has a unique weak solution w and

$$\sup_{0\le t\le T}\left(||w(t)||_{H_0^1(\Omega)} + ||w_t(t)||_{L^2(\Omega)}\right) + ||w_{tt}||_{L^2(0,T;H^{-1}(\Omega))}$$
$$\le C(||f||_{L^2(Q)} + ||w_0||_{H_0^1(\Omega)} + ||w_1||_{L^2(\Omega)}).$$

The proof uses standard estimates on Galerkin approximations starting from smooth initial data [3, 6].

Theorem 2.1: There exists an optimal control $h_\beta \in U$, which minimizes the objectve functional $J_\beta(h)$ over $h \in U$.
Proof: Let $\{h^n\} \in U$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(h^n) = \inf_{h\in U} J(h).$$

Lemma 2.1 gives a priori estimates on $w^n = w(h^n)$ with bounds independent of n. On a subsequence, by weak compactness, there exists w_β in $L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{aligned} w^n &\rightharpoonup w_\beta \text{ weakly in } L^2(0,T;H_0^1(\Omega)),\\ w_t^n &\rightharpoonup (w_\beta)_t \text{ weakly in } L^2(Q),\\ h^n &\rightharpoonup h_\beta \text{ weakly in } L^2(Q),\\ w_{tt}^n &\rightharpoonup (w_\beta)_{tt} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned}$$

By a compactness result [7], we have $w^n \to w^\beta$ strongly in $L^2(Q)$ and hence $w_\beta = w(h_\beta)$. Using the lower-semicontinuity of L^2 norm with respect to weak convergence, we conclude that h_β is an optimal control. □

To characterize the optimal control, we need to differentiate the maps

$$h \to J_\beta(h) \text{ and } h \to w(h).$$

Lemma 2.2: The mapping

$$h \in U \longrightarrow w(h)$$

is differentiable in the following sense;

$$\frac{w(h+\varepsilon\ell) - w(h)}{\varepsilon} \rightharpoonup \Psi \text{ weakly in } L^2(0,T;H_0^1(\Omega))$$

as $\varepsilon \to 0$, for any $h, h+\varepsilon\ell \in U$. Moreover Ψ is a weak solution of

$$\begin{aligned} \Psi_{tt} &= \Delta\Psi + h\Psi + \ell w, \quad \text{in } Q\\ \Psi &= 0 \quad \text{on } \partial\Omega \times (0,T)\\ \Psi &= 0 \quad t=0,\ x\in\Omega\\ \Psi_t &= 0 \quad t=0,\ x\in\Omega \end{aligned} \tag{2.1}$$

where $w = w(h)$.

Proof: Denoting $w^\varepsilon = w(h+\varepsilon\ell)$ and $w = w(h)$, we have

$$\begin{aligned} &\sup_{0\le t\le T} \left\| \frac{(w^\varepsilon - w)(t)}{\varepsilon} \right\|_{H_0^1(\Omega)} + \left\| \frac{w_{tt}^\varepsilon - w_{tt}}{\varepsilon} \right\|_{L^2([0,T];H^{-1}(\Omega))} \\ \le\ & C\|\ell\|_{L^\infty}. \end{aligned}$$

On a subsequence, we have the existence of Ψ in $L^2(0,T;H_0^1(\Omega))$ and Ψ solves (2.1). □

We derive the necessary conditions to characterize an optimal control.

Theorem 2.2: Given an optimal control, h_β, and corresponding state, $w_\beta = w(h_\beta)$, there exists a weak solution p in $L^2(0,T;H_0^1(\Omega))$ to the adjoint problem:

$$\begin{aligned} p_{tt} &= \Delta p + hp + (w_\beta - z)\chi_{Q'}, \quad \text{in } Q \\ p &= 0, \qquad \text{on } \partial\Omega \times (0,T) \\ p &= p_t = 0, \qquad t = T,\ x \in \Omega \end{aligned} \tag{2.2}$$

where $\chi_{Q'}$ is the characteristic function of the set Q'. Furthermore, h_β satisfies

$$h_\beta = \max(-M,\ \min(-\frac{wp}{\beta}, M)). \tag{2.3}$$

Proof: Let $h_\beta + \varepsilon\ell \in U$ for $\varepsilon > 0$ and $w^\varepsilon = w(h_\beta + \varepsilon\ell)$. Since the minimum of J_β is achieved at h_β, we have

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0^+} \frac{J_\beta(h_\beta+\varepsilon\ell) - J_\beta(h_\beta)}{\varepsilon} \\ &= \int_{Q'} \Psi(w_\beta - z)dxdt + \beta\int_Q h_\beta \ell dxdt. \end{aligned} \tag{2.4}$$

Let p be the weak solution of the adjoint problem (2.2). Using (2.1) and (2.2) in (2.4), we obtain

$$0 \le \int_Q \ell(pw_\beta + \beta h_\beta)dxdt,$$

which gives the desired characterization (2.3). □

Thus, for a fixed β, the optimal control can be expressed from (2.3) in terms of the solution of the optimality system

$$\begin{aligned} w_{tt} &= \Delta w + \max(-M,\ \min(-\frac{wp}{\beta}, M))w + f \quad \text{in } Q \\ p_{tt} &= \Delta p + \max(-M,\ \min(-\frac{wp}{\beta}, M))p + (w-z)\chi_{Q'} \quad \text{in } Q \\ w &= p = 0 \qquad \text{on } \partial\Omega \times (0,T) \\ w &= w_0, \quad w_t = w_1 \qquad t = 0,\ x \in \Omega \\ p &= p_t = 0 \qquad t = T,\ x \in \Omega. \end{aligned} \tag{2.5}$$

Note that bounded solutions of the optimality system are unique if T is sufficiently small. See [4], [6] for similar results. If one assumes h is a function of x only or t only, better uniqueness results can be obtained [6].

3 IDENTIFICATION PROBLEM

We now use a sequence of optimal controls (as $\beta \to 0$) to identify h from observations z. We do not assume that z is in the range of the map

$$h \in U \ \to \ w(h)|_{Q'}. \tag{3.1}$$

This situation could occur from inaccurate observations.

Theorem 3.1: There exist: (i) a sequence $\beta_n \to 0$, (ii) corresponding optimal controls, h_{β_n}, for the functional $J_{\beta_n}(h)$, (iii) $h^* \in U$, and (iv) $w^* = w(h^*)$ (solution of (1.1)), such that

$$\begin{aligned} h_{\beta_n} &\rightharpoonup h^* \text{ weakly in } L^2(Q), \\ w_{\beta_n} = w(h_{\beta_n}) &\rightharpoonup w^* \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \end{aligned}$$

and

$$\int_{Q'} (w^* - z)^2 dxdt = \inf_{h \in U} \int_{Q'} (w(h) - z)^2 dxdt. \tag{3.2}$$

Note: The limit w^* can be interpreted as a (not necessarily unique) projection of z onto the range of the map (3.1).

Proof: Note that the a priori estimates from Lemma 1.1 are independent of β. Thus we can find a sequence $\beta_n \to 0$ and $h^* \in U, w^* \in L^2(0,T;H_0^1(\Omega))$ such that

$$J_{\beta_n}(h_{\beta_n}) = \inf_{h \in U} J_{\beta_n}(h)$$

and

$$\begin{aligned} h_{\beta_n} &\rightharpoonup h^* \text{ weakly in } L^2(Q), \\ w_{\beta_n} &\rightharpoonup w^* \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ w_{\beta_n} &\to w^* \text{ strongly in } L^2(Q), \\ (w_{\beta_n})_{tt} &\rightharpoonup (w^*)_{tt} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned}$$

where $w^* = w(h^*)$, is the weak solution of (1.1). Now for any $\bar{h} \in U, \bar{w} = w(\bar{h})$, we have

$$J_\beta(h_\beta) \leq J_\beta(\bar{h}). \tag{3.3}$$

Letting $\beta_n \to 0$ in (3.3) gives

$$\int_{Q'} (w^* - z)^2 dxdt \leq \int_{Q'} (\bar{w} - z)^2 dxdt$$

which yields the desired result (3.3). □

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