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Dimensioning the Continuous State Leaky Bucket for Geometric Arrivals

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Abstract : In the ATM network, conformance of the connection with the negotiated traffic contract is monitored by means of the "Continuous State Leaky Bucket", or "Virtual Scheduling Algorithm" (VSA). It allows to verify both T (peak emission interval) and τ (cell delay variation) of the connection.

Usual models of the VSA make use of a ./D/1/N queue. We show that in the general case where T and τ take arbitrary (non integer) values, the exact model is given by the queue with bounded waiting time. In the simple case of Bernoulli arrivals this allows dimensioning the VSA for any value of the parameters. An asymptotic formula is given, and an independence property is stressed, easing the dimensioning.

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1 Introduction

In the ATM network, conformance of the actual cell flow to the negotiated traffic parameters has to be verified. For peak-rate allocated connections, the *Virtual Scheduling Algorithm* (VSA), or equivalently the *Continuous State Leaky Bucket* is the reference algorithm for cell conformance at the UNI or NNI. It is defined by the two parameters T (Peak Emission Interval) and τ (Cell Delay Variation Tolerance), cf. [3].

The VSA is traditionally modelled as a finite-capacity queue, with total size $1 + \tau/T$. However, this holds only in case where the ratio τ/T takes an integer value. A general model has to be given for arbitrary values of τ/T. Another problem arises due to arbitrary value of T, when measured in cell transmission time. Since arrivals occur on a synchronous basis, a discrete-time model would be appropriate. However, T may not be integer, implying that services may begin somewhere between slots.

In the following, we first show that the general form of the VSA (i.e. with arbitrary values for T and τ) can be exactly modelled as a queue with Bounded Waiting Time. This improves previous models considering the VSA as a finite-capacity queue (see e.g. [2]). In the case of Bernoulli input, we write down the equations in a form allowing an exact, numerical solution for rational T's (i.e. T of the form r/s, with r, s integers).

The numerical results show an interesting independence property, allowing to express τ/T as a function of a single parameter. This gives a practical way to dimension the system, that is to find the bound τ on the CDV Tolerance which achieves a given Cell Loss Probability.

2 The VSA as a Queue with Bounded Waiting Time

Consider a queueing system with *limited virtual offered waiting time* (or, equivalently, with *impatient customers*). Namely, each arriving customer is characterized by an amount of time it accepts to wait before beginning being served. If the unfinished work at its arrival (the *virtual waiting time*) is larger than this delay, then the customers gives up immediately (or equivalently, it enters the queue and gives up as soon as this delay is exceeded).

Such a system has already received attention (see e.g. references [8] [9]). In what follows, one is restricted to the case where all customers have the same patience time and require the same amount of service : this corresponds to the system analysed in [9]. One may, algorithmically, define the system using the following rules:

- Let $T_i = T$ be the service time of the i-th accepted customer (a customer is accepted if his patience is long enough; otherwise it is said to be rejected).
- Let τ be the (common) patience time of customers.
- Let LT denote the last time a customer has been accepted.
- Let X denote the value of the virtual offered waiting time immediately after a customer is accepted.
- Let X'_t denote the value of the unfinished work at time t (or virtual offered waiting time).
- The queue with impatient customers works according to the following rules :
Upon an arrival at time t, the customer estimates the value of the unfinished work :

$$X'_t = \max\{0, X - (t - \mathrm{LT})\}$$

  - If $X'_t > \tau$, then the customer is rejected.
  - If $X'_t \leq \tau$, the customer is accepted, and both X and LT are updated :

$$X = X'_t + T \quad \text{and} \quad LT = t$$

These rules are the same as the *Continuous State Leaky Bucket* as defined in reference [3], and which is equivalent to the *Virtual Scheduling Algorithm* ([3], [1]).

Since all customers have the same service time T, the integer number $\left\lceil \frac{X'}{T} \right\rceil$ represents the number of customers in the system at the arrival epoch ($\lceil x \rceil$ is the smallest integer number greater than or equal to x : $\lceil 5 \rceil = 5$, $\lceil 5.1 \rceil = 6$). The fractional part of the ratio accounts for the customer being served.

Now, let us assume that $\tau/T + 1 = N$ (integer). In this case, the test "$X' > \tau$?" is equivalent to "$n > N$?". In other words, for τ/T integer, that is in the configuration of the classical *Leaky Bucket* [15], the system is equivalent to the finite-capacity queue X/D/1/N [2].

3 Analysis of the VSA with Discrete-Time Arrivals

3.1 The General Case

Let δ be the slot duration. The input flow is of discrete nature, cells arriving at epochs of the form $k\delta$. The VSA is characterized by parameters (T, τ). These values are arbitrary (integer or real). Note that T may not be a multiple of δ, so that services may begin at arbitrary epochs and the system is generally not a discrete-time queue. To simplify notations, we assume that $\delta = 1$ (equivalently, all times are measured in units of δ).

Let W_n denote the virtual offered waiting time at the slot number n (unfinished work just prior to a possible arrival in the slot). It obeys the following recurence equations :

- If an arrival occurs in slot n and is accepted :

$$W_{n+1} = W_n + T - 1 \tag{3.1}$$

- If no arrival occurs in slot n, or if the arrival is rejected :

$$W_{n+1} = \max\{0, W_n - 1\} \tag{3.2}$$

The domain in which $Wsubn$ varies is bounded : $0 \leq W_n \leq T + \tau - 1$. Moreover, W_n takes only values of the general form $(kT - j)$ with $k \geq 0$ and $j \geq 0$. For instance, let us assume that $T = 1.4$. In this case, $W_n \in \{0, 0.2, 0.4, 0.6, 0.8, 1., 1.2, \ldots\}$. However, if T is irrational W_n takes an infinite number of values in the interval $[0, T + \tau - 1]$.

3.2 The Case with Bernoulli Arrivals

In the following, we assume a Bernoulli arrival process with parameter p : in each slot, a cell arrives with probability p, independently of what happened in previous slots.

Let $\mathcal{W}_n(t) = P\{W_n \leq t\}$ be the Probability Distribution Function (PDF) of the virtual waiting time. According to eqn. (3.1) and (3.2), one has :

$$\left.\begin{array}{rcllr} \mathcal{W}_n(0) & = & (1-p)\mathcal{W}_{n-1}(1) & & \\ \mathcal{W}_n(t) & = & (1-p)\mathcal{W}_{n-1}(t+1) & \text{if} & t < T-1 \\ \mathcal{W}_n(t) & = & (1-p)\mathcal{W}_{n-1}(t+1) + p\mathcal{W}_{n-1}(t-T+1) & \text{if} & T-1 \leq t < \tau - 1 \end{array}\right\} \tag{3.3}$$

As usual, one is interested in the limiting distribution (if it exists) as $n \to \infty$. The limit always exists (see [8]), and let $\mathcal{W}(t) = \lim_{n\to\infty} P\{W_n \leq t\}$. One has :

$$\left.\begin{array}{rcllr} \mathcal{W}(t) & = & (1-p)\mathcal{W}(t+1) & \text{if} & 0 \leq t < T-1 \\ & = & (1-p)\mathcal{W}(t+1) + p\mathcal{W}(t-T+1) & & T-1 \leq t < \tau - 1 \\ & = & \mathcal{W}(t+1) - p\mathcal{W}(\tau) + p\mathcal{W}(t-T+1) & & \tau - 1 \leq t < T + \tau - 1 \\ \mathcal{W}(\tau + T - 1) & = & 1 & & \end{array}\right\} \tag{3.4}$$

4 Calculation of the Rejection Probability

4.1 Asymptotic Case of Poisson Arrivals

In the case where the parameter p of the Bernoulli process decreases, while T increases so that the product $\rho = pT$ remains constant, it is known that the system goes to a limit given by the $M/D/1$ queue (with impatient customers).

This system has been studied in depth, see e.g. [8], [9]. We refer here to the results given in [9] :

Let $a = \frac{\tau}{T}$ and $n = 1 + \lfloor a \rfloor$ (that is, n is such that $n \leq 1 + \tau/T < n + 1$). Let $\rho(= pT)$ be the load offered to the $M/D/1$ system. Then, the rejection probability π is given by :

$$\begin{array}{rcl} \pi & = & 1 - \dfrac{1-Q}{\rho} \\ \text{with} \quad \dfrac{1}{Q} & = & 1 + \rho e^{\rho a} \displaystyle\sum_{j \leq n-1} \dfrac{e^{-j\rho}}{j!}(-1)^j[\rho(a-j)]^j \end{array} \tag{4.5}$$

For low rejection probabilities the above relation gives poor results, and it is worth transforming it using a combinatorial identity (see e.g. [12]) :

$$\frac{e^{a\rho}}{1-\rho} = \sum_{j=0}^{\infty} \frac{(\rho e^{-\rho})^j}{j!}(j+a)^j \tag{4.6}$$

The transformation yields an infinite series of positive terms, instead of a finite (*but of alternating signs*) series.

4.2 Solution when T is a rational number

As already mentioned, if T is an irrational number, the W_n's can take all real values in $[0, T+\tau-1]$. On the other hand, in the case where T is a rational number, say $T = r/s$, possible values of the virtual waiting time W_n are of the form $x = k/s$ with $k \in \{0, 1, \cdots, \lfloor \tau s \rfloor + r - s\}$. That is, the PDF given by eqn. (3.4) only varies by jumps at k/s. Transitions between these points are given by Table 1.

$\mathbf{sW_n}$	$\mathbf{sW_{n+1}}$	**Probability**	**Condition**
	$k+r-s$	p	$k \leq \lfloor \tau s \rfloor$
k	$\max(0, k-s)$	$1-p$	$k \leq \lfloor \tau s \rfloor$
	$\max(0, k-s)$	1	$k > \lfloor \tau s \rfloor$

Table 1: Transition Table - $W_n = k/s \longrightarrow W_{n+1}$

The transition matrix is very sparse as it can be seen in Figure 1 which gives the affectation algorithm of the non zero values.

for $k = 0 \longrightarrow \lfloor \tau s \rfloor$ **do**
  $a_{k,\max(0,k-s)} = 1-p$
  $a_{k,k+r-s} = p$
end do
for $k = \lfloor \tau s \rfloor + 1 \longrightarrow \lfloor \tau s \rfloor + r - s$ **do**
  $a_{k,\max(0,k-s)} = 1$
end do

Figure 1: Transition Matrix construction.

The resolution proceeds by computing the pdf $P(W = k/s)$ from the transition matrix, instead of using eqns. (3.4). For some special cases a direct resolution is available, see Appendix. In any case, the GASTA property[7] is used in order to derive the Loss Probability from state probabilities:

$$P_L = \sum_{k=\lfloor \tau s \rfloor+1}^{\lfloor \tau s \rfloor + r - s} \Pr(s.W = k)$$

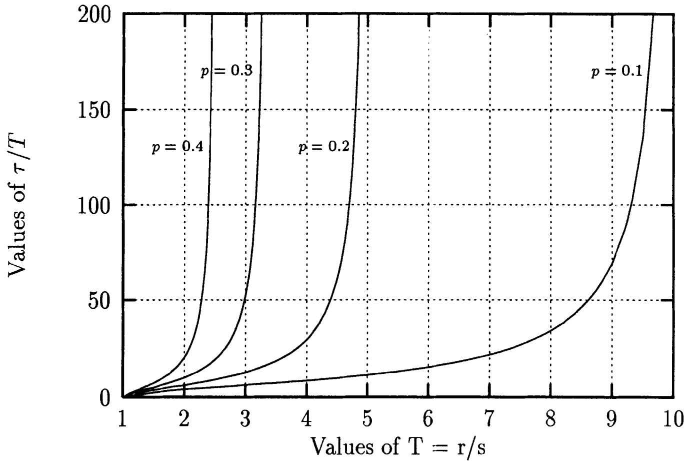

Figure 2: τ/T vs T for different values of p and for $P_L \leq 10^{-8}$.

5 Numerical results

Calculations have been made – using a stochastic matrices reduction method (see, for example [5]) – for the general case

$$T > 1 \quad \text{and} \quad \begin{cases} T \in \mathbb{N} & \text{and} \quad \tau \geq 1 \\ \text{or} \\ T \notin \mathbb{N} & \text{and} \quad \tau \geq T - \lfloor T \rfloor \end{cases}$$

Figure 2 shows the evolution of τ/T vs T for different values of the arrival probability ($p = 0.1,\ 0.2,\ 0.3$ et 0.4) in order to obtain a maximum Cell Loss Probability less or equal to 10^{-8}. There is, as expected, a vertical asymptote for $T = 1/p$ which corresponds to the load $\rho = 1$.

Note that the points cannot be chosen arbitrarily in the general case. Assume for instance $T = 1.2$, which is naturally represented as $6/5$ (that is, $r = 6, s = 5$). The state space is composed of values $k \times 0.2$, for $0 \leq k \leq \tau + 0.2$. As a consequence, the loss probability is the same for values of τ in the interval $[k \times 0.2, (k+1) \times 0.2[$.

The choice of a rational representation for T is of importance. For an arbitrary, real value of T, it is always possible to find a pair (r, s) such as $T \sim r/s$, the ratio approximating T as closely as needed. On the other hand, since the matrix size is $1 + \tau s + r - s = 1 + s(\tau + T - 1)$, s has to be chosen as low as possible.

6 An approximation formulae

This approach has been described in [10]. It is based on estimates for the tail behaviour of the virtual waiting time in the GI/G/1 queue obtained by Kingman [11] and extended by Ross [13] :

Let A be the intercell distribution for the GI arrival process and v be defined by the equation :

$$E\left[e^{-vA}\right] = e^{-vT} \tag{6.7}$$

The following upper bound applies for the tail of the virtual waiting time distribution (result given by Kingman) :

$$\Pr(W > \tau) \leq e^{-v\tau}$$

Therefore, in order to ensure a proportion of non-conforming cells smaller than 10^{-r}, it is sufficient to choose τ as

$$\tau = \frac{r \ln 10}{v}$$

The value of v is obtained by solving equation (6.7), which requires the transform for the distribution of A.

For the Bernoulli input considered in this paper, the transform is $H(z) = \dfrac{p.z}{1-(1-p).z}$ so that, finally,

$$\frac{\tau}{T} = \frac{-\gamma}{\ln\left[H(e^{-\frac{\gamma}{\tau}})\right]} \tag{6.8}$$

Figure 3 shows the accuracy of the approximation.

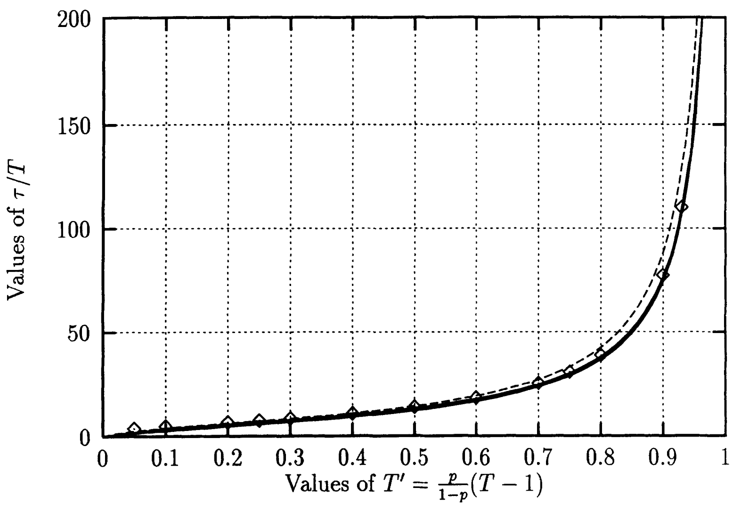

Figure 3: τ/T vs $T' = \frac{p}{1-p}(T-1)$ for different values of p and for $P_L \leq 10^{-8}$. The squares above correspond to the M/D/1 values and the dotted line to the approximation (6.8).

7 An Independence Property. Application to VSA Dimensioning

For T in the interval $]1, \frac{1}{p}]$, an interesting property appears thanks to the transformation $T \longrightarrow T' = \frac{p}{1-p}(T-1)$ (so that $T' \in]0,1]$). The result (see Figure 3) gives the curves almost perfectly superimposed.

The property is illustrated for some T' in Table 2. The rows "M/D/1" refer to the asymptotic case where $p \rightarrow \infty$, with $\rho = T'$.

T′	p	T = r/s	τ/T	Loss	τ_a/T
0.25	0.1	3.25=13/4	6.77	8.82 10^{-9}	7.17
	0.2	2.0=2/1	6.00	8.94 10^{-9}	6.64
	0.3	1.583=19/12	5.89	8.58 10^{-9}	6.23
	0.4	1.375=11/8	5.55	8.28 10^{-9}	5.89
		M/D/1	7.44	9.99 10^{-9}	
0.50	0.1	5.5=55/10	13.27	9.55 10^{-9}	14.19
	0.2	3.0=3/1	12.67	9.76 10^{-9}	13.78
	0.3	2.167=13/6	12.38	9.97 10^{-9}	13.40
	0.4	1.75=7/4	12.00	9.69 10^{-9}	13.07
		M/D/1	13.78	9.99 10^{-9}	
0.75	0.1	7.75=31/4	30.13	9.89 10^{-9}	33.12
	0.2	4.0=4/1	29.50	9.92 10^{-9}	32.77
	0.3	2.75=11/4	29.09	9.88 10^{-9}	32.43
	0.4	2.125=17/8	28.59	9.81 10^{-9}	32.11
		M/D/1	30.64	9.93 10^{-9}	
0.90	0.1	9.1=91/10	76.70	9.99 10^{-9}	88.60
	0.2	4.6=23/5	75.87	9.99 10^{-9}	88.28
	0.3	3.1=31/10	75.00	9.96 10^{-9}	87.97
	0.4	2.35=47/20	74.04	9.97 10^{-9}	87.65
		M/D/1	77.50	9.93 10^{-9}	

Table 2: A display of some of the results, illustrating the normalizing factor T'.

The results seem to show that (for a given value of T') τ/T is slightly decreasing as p increases. No satisfactory explanation has been given yet; note that Geometric burst arrival models would be appealing to this concern - since $\frac{p}{1-p}(T-\delta)$ is the remaining work at the end of a burst.

Anyway, this property can be used to dimension the VSA as follow (see Figure 4). For a given T - say $T = 4.5$ - an horizontal line crosses the staight line corresponding to a desired arrival probability - say $p = 0.2$. The abscissa obtained actually is $T' = \frac{p}{1-p}(T-\delta)$ so that the curve finally gives the requested value of $\frac{\tau}{T}$.

Table 2 also shows — as well as Figure 3 — the results obtained using approximation (6.8). That approximation gives greater values of τ/T that those computed exactly, but it remains quite good since the maximum increase is about 18.4 % for $T' = 0.9$ and $p = 0.4$. Nethertheless, its pessimistic character makes it practically usefull.

8 Conclusion

We have shown the equivalence between the general VSA (i.e. the VSA with arbitrary values of its parameters T and τ) and the discrete-time queue with impatient customers (queue with limited waiting time). This equivalence allows to write down the recurrence equation to which the waiting time distribution obeys. It must be noted that this property has been mentioned independently by [14]. In the case where the arrival process is of the Bernoulli type, the Markov chain analysis is worked out by representing the parameter T under the form r/s (r, s integers). Such a representation is always possible (with an error bounded by $1/s$). From a numerical viewpoint, the smaller s the better since the matrix size grows directly with s.

This allows to dimension the VSA, that is to calculate the bound on τ, such that for given p and T the loss probability is lower than the QoS requirement. The system exhibits a curious and interesting property : the normalized CDV Tolerance (ratio τ/T) does not depend on p and T but it depends only on the aggregated parameter $T' = \frac{p}{1-p}(T-\delta)$. This property allows an easy dimensioning procedure examplified by the abacus on Figure 4.

It remains to extend the analysis to more general input processes. Especially, it would be interesting to look for an analogous invariance property for other input processes.

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Appendix: Resolution of eqn. (4) for special values

- $\underline{T \leq 1}$:
The waiting time is always zero and the distribution is simply $\Pr(W=0)=1$. This remains true, even if T is not a rational number. Note that τ does not matter and that the loss probability is always zero.

- $\underline{p=1}$:
The terms $1-p$ vanishe in the transition matrix and

$$P_{(sW=k)} = 0 \text{ for } 0 \leq k \leq \lfloor \tau s \rfloor - s$$

The remainding equations are then

$$\begin{cases} P_{(sW=k)} = P_{(sW=k+s)} & \lfloor \tau s \rfloor - s < k \leq \lfloor \tau s \rfloor + r - 2.s \\ P_{(sW=k)} = P_{(sW=k-r+s)} & \lfloor \tau s \rfloor + r - 2.s < k \leq \lfloor \tau s \rfloor + r - s \end{cases}$$

which solution is $\quad P_{(sW=k)} = 1/r \quad \lfloor \tau s \rfloor - s < k \leq \lfloor \tau s \rfloor + r - s$

The Loss Probability is then $\quad P_L = \frac{r-s}{r} = 1 - \frac{1}{T} \quad$ which does not depend on τ so that dimensioning the VSA for an aimed Loss Probability leads to a unique value of T.

- **Case where a cell is accepted only if $W=0$** :
In that case, τ is small enougth to make all incoming cells refused, but if $W=0$. The only reachable states are then $(W=0)$ and $(W=T-k)$ for $1 \leq k \leq \lfloor T \rfloor$, linked by the relations $pP_{(W=0)} = P_{(W=T-\lfloor T \rfloor)} = \cdots = P_{(W=T-1)}$.
Two alternatives are here to be considered depending on the fact that $T - \lfloor T \rfloor$ can be nul or not. Nethertheless, the loss probability is $P_L = 1 - P_{(W=0)}$ in both cases.

  1. **$T \in \mathbb{N}$ and $\tau < 1$** :
  Here, $s=1$ and $T - \lfloor T \rfloor = 0$. The values taken by the virtual waiting time are integers and

$$\begin{cases} P_{(W=0)} = \dfrac{1}{1+(T-1)p} & \\ P_{(W=T-k)} = pP_{(W=0)} & 1 \leq k \leq T-1 \end{cases}$$

  2. **$T \notin \mathbb{N}$ and $\tau < T - \lfloor T \rfloor$** :
  Here, $s > 1$, $T - \lfloor T \rfloor > 0$ and $\tau s < r \bmod s$ so that

$$\begin{cases} P_{(W=0)} = \dfrac{1}{1+\lfloor T \rfloor p} & \\ P_{(W=T-k)} = pP_{(W=0)} & 1 \leq k \leq \lfloor T \rfloor \end{cases}$$

The cell loss probability is then a step function of T. Let n be a positive integer. If $T \in]n, n+1]$, the cell loss is constant in that interval (see Figure 5) :

$$P_L = \frac{np}{1+np} \quad \text{with} \quad n \in \mathbb{N} \quad \text{such that} \quad T \in]n, n+1]$$

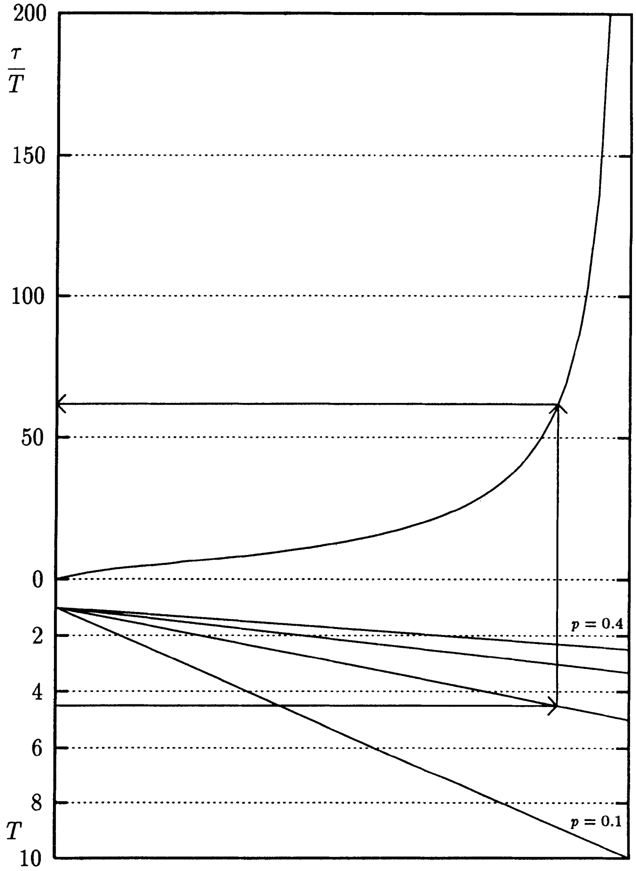

Figure 4: Abacus for finding the value of $\frac{\tau}{T}$ with given T and p to achieve a Cell Loss Probability less or equal to 10^{-8}.

Since P_L does not depend on τ, dimensioning the VSA for an aimed Loss Probability leads, as in the previous case, to a unique value of T. But here, τ has to be choosen properly, that is

$$\begin{cases} \tau < 1 & \text{if } T \in \mathbb{N} \\ \tau < T - \lfloor T \rfloor & \text{if } T \notin \mathbb{N} \end{cases}$$

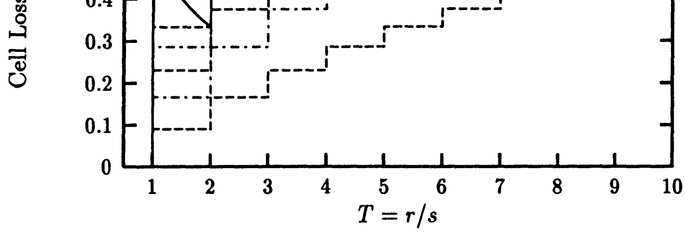

Figure 5: Cell Loss Probability vs T for $T \in \mathbb{N}$ and $\tau < 1$ or $0 < \tau < T - \lfloor T \rfloor$.