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A frequency method for H^∞ operators

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Abstract

We present numerical methods for determining the stability or instability of autonomous linear functional differential equations. These are based on some qualitative properties of analytic mappings and the ability to use standard mathematical packages to evaluate transcendental functions and approximate complex integrals.

Keywords

Functional differential equations, Poisson integral, H^∞ spaces, uniform exponential stability

1 INTRODUCTION

The use of quadratic Lyapunov functionals in the investigation of the stability of linear autonomous dynamical systems is largely passé. With the exception of finite dimensional linear differential equations their construction presents formidable computational difficulties. Consequently, when used to study infinite dimensional systems, Lyapunov functionals are often introduced in an ad hoc manner. In this paper we outline a general approach for studying the stability of a class of linear dynamical systems using what might be loosely described as a frequency domain approach. In reality it is a return to a complex analysis approach in the study of linear systems which is enhanced by the use of computational techniques such as MATLAB. The fundamental idea is quite straight-forward and is based on the following observation. The majority of linear dynamical systems one encounters in practice are uniformly exponentially stable (u.e.s. for short) if and only if their Laplace transforms are H^∞ operators in some Banach space. How does one determines whether a given analytic operator is in H^∞? Certainly not by direct methods, such as locating poles in the right half plane. However by reversing certain arguments used to study H^∞ functions and using standard numerical packages it is possible in many instances to determine whether or not a given analytic operator valued function is in H^∞.

Due to space limitations we shall state our results either without proof or with a brief sketch. However the details can easily be filled in by recourse to the associated references. All computations in this paper were carried out using the Texas Instruments TI-85 Scientific Calculator.

2 STABILITY AND H^∞ OPERATORS

Definition 2.1

(i) An operator valued function F from the complex plane, C, is in H^∞ if $F(\lambda)$ is analytic and uniformly bounded in $Re\lambda > 0$.
(ii) A function F which is in H^∞ will be said to be in $H^\infty(B)$ if it is in H^∞, $\lim_{|w|\to\infty} |F(iw)| = 0$, where $|\cdot|$ denotes a suitable norm, and if F is analytic in a neighborhood of $Re\lambda = 0$.

Property 2.2 Let $F : C \to H$, a Hilbert space, be in $H^\infty(B)$. Then if $\lambda = x + iy$, F has the following integral representation.

$$F(x+iy) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{F(it)}{(t-y)^2 + x^2}dt; \tag{1}$$

$$F(x+iy) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} F[i(x\tan\sigma + y)]d\sigma \tag{2}$$

The formula (1) is given in Hille (1973) and (2) is obtained from (1) by the change of variable

$$x\tan\sigma = t - y. \tag{3}$$

Property 2.3 If F is in $H^\infty(B)$, then

$$\int_{-\pi/2}^{\pi/2} e^{-2i\sigma} F[i(x\tan\sigma + y)]d\sigma = 0 \tag{4}$$

for all $\lambda = x + iy,\ \ x > 0$. Formula (4) is obtained from formula (2) and the requirement that $F_x = \frac{1}{i}F_y$.

Property 2.4 Let F be holomorphic in a neighborhood of $Re\lambda = 0$. Then F is in $H^\infty(B)$ if and only if equation (4) is satisfied for all $x + iy,\ \ x > 0$. If F is not analytic in $Re\lambda > 0$, but analytic and uniformly bounded in some neighborhood of $Re\lambda = 0$, then equation (2) is a harmonic function which is not analytic.

Example 1

Let $F(\lambda) = \frac{1}{\lambda - 2}$. Then, if $x > 0$,

$$\frac{x}{\pi}\int_{-\infty}^{\infty} \frac{F(it)\,dt}{x^2 + (t-y)^2} = \frac{-1}{\overline{\lambda} + 2},$$

where $\lambda = x + iy$. Also $\int_{-\pi/2}^{\pi/2} e^{-2\lambda i\sigma}F(i\tan\sigma)d\sigma = -\frac{2}{9} \neq 0$.

Theorem 2.5 *The linear dynamical system described by*

$$\frac{d}{dt}[x(t) - \sum_{j=1}^{m} D_j x(t-h_j)] = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-h_j), \tag{5}$$

where $\{A_j\}$, $0 \le j \le m$, *and* $\{D_j\}$, $1 \le j \le m$, *are real* $n \times n$ *matrices and* $\{h_j\}$, $1 \le j \le m$, *are positive constants, is u.e.s. if and only if the scalar function*

$$F(\lambda) = [\det(\lambda(I - \Sigma D_j e^{-\lambda h_j}) - A_0 - \Sigma A_j e^{-\lambda h_j})]^{-1} \tag{6}$$

is in $H^\infty(B)$. *That is, equation (6) satisfies Property 2.3 for all* $\lambda = x + iy$ *in some neighborhood,* $N(\lambda_0)$, *of* $\lambda_0 = x_0 + iy_0$, $x_0 > 0$.

Sketch of the Proof It can be shown that if (4) is satisfied at some point $\lambda_1 = x_1 + iy$, $x_1 > 0$ then $F(\lambda_1)$ satisfies the Cauchy-Riemann equations at λ_1. Thus, if (4) is satisfied in an open neighborhood of λ_0, F is analytic in that neighborhood. But F must also be analytic is a neighborhood of $Re\lambda = 0$ and uniformly bounded there (otherwise the integral (4) would not exist). This means, using an analytic continuation argument, that the Poisson integral (2) represents F in $Re\lambda > 0$. This also means F has no poles in $Re\lambda \ge 0$ and since it is uniformly bounded in some neighborhood of $Re\lambda = 0$ all the poles of F must lie in $Re\lambda \le -\alpha$, $\alpha > 0$ (see e.g. Hale and Lunel (1993) or Henry (1974)). Hence the system (5) is u.e.s. (see e.g. Henry (1974)).

Remark 2.6 The practical implementation of Theorem 2.5 would be to evaluate $F(\lambda)$ at a finite number of points and then check the Poisson integrals (2) against these points using some numerical method. If these are consistent then the system is probably u.e.s. We indicate this procedure in the following example.

Example 2

Consider the system

$$\frac{d}{dt}\left[x(t) - \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} x(t-h)\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x(t-h), \tag{7}$$

where $h > 0$. The corresponding function is

$$F(\lambda, h) = \left[\lambda^2 \left(1 + \frac{e^{-2\lambda h}}{4}\right) + 1\right]^{-1} \tag{8}$$

Let $h = 1$. Then the values of $F(1,1), F(.5,1)$ and $F(.25,1)$ and their corresponding Poisson integral approximations are given in the following table.

$F(1,1)$	$= .491682255339$,	$I(1,1)$	$= .491716045511$
$F(.5,1)$	$= .78550604137$,	$I(.5,1)$	$= .785504502663$
$F(.25,1)$	$= .932855799362$,	$I(.25,1)$	$= .932948880576$.

The integral approximations were obtained using Simpson's Rule with upper limit and lower limit respectively $\pm$ 1.57069632679 and the number of partitions $N = 100$.

We now let $h = 2$ in (8). The corresponding values of $F(1,2), F(.5,2)$ and $F(.25,2)$ and the approximate integrals are

$$\begin{array}{llll} F(1,2) & = .498857887385, & I(1,2) & = .559635925368 \\ F(.5,2) & = .794622973739, & I(.5,2) & = .815297457756 \\ F(.25,2) & = .936112111757, & I(.25,2) & = .941191927459. \end{array}$$

We conclude that for $h = 1$ (7) is u.e.s. but not for $h = 2$. Indeed when $h = 2$ the integral (4) in Property 2.3 yields $I(1,2) \doteq .659367589698$, but for $h = 1$ the integral (4) is $I(1,1) \doteq -3.48519770543 \times 10^{-5}$

3 STABILITY ESTIMATES USING LYAPUNOV FUNCTIONALS

Although we stated in the introduction that Lyapunov functionals were passé, they still have their uses in studying certain types of dynamical systems. One such class is linear autonomous functional differential equations with multiple discrete delays described by equation (5).

Assumption 3.1 The poles of the function

$$G(\lambda) = [\det(I - \sum_{j=1}^{m} D_j e^{-\lambda h_j})]^{-1} \tag{9}$$

lie in $Re\lambda \leq -\beta,\ \beta > 0$.

Assumption 3.1 is known as the D-stability condition of Cruz and Hale (1969). Henry (1974) has shown that it is a necessary condition for the system (5) to be u.e.s.

The solutions of (5) must satisfy an initial condition on the interval $[-h, 0]$, where

$$h = \max_j h_j. \tag{10}$$

If this initial condition is a continuously differentiable vector function,

$$\phi : [-h, 0] \to C^n, \tag{11}$$

then the solution has for $t \geq 0$ the representation.

$$\begin{aligned} x(t) &= [S(t) - \sum_{j=1}^{m} S(t - h_j)D_j]\phi(0) \\ &+ \sum_{j=1}^{m} \int_{-h_j}^{0} S(t - \sigma - h_j)[A_j\phi(\sigma) + D_j\dot{\phi}(\sigma)]d\sigma, \end{aligned} \tag{12}$$

where $S(t)$ is the inverse Laplace transform of the analytic matrix valued function

$$\hat{S}(\lambda) = [\lambda(I - \sum_{j=1}^{m} D_j e^{-\lambda h_j}) - A_0 - \sum A_j e^{-\lambda h_j}]^{-1}. \tag{13}$$

Definition 3.2

(i) Let $A = (a_{ij})$ be an $n \times n$ matrix. The norm of A, $|A|$, is defined by
$|A| = \sup_j \sum_{i=1}^{n} |a_{ij}|$

(ii) The norm of any n-vector $x = \{x_j\}$ is
$|x| = \sup_j |x_j|$

(iii) The inner product in C^n is denoted by $(\cdot,\cdot)$ with the conventions that for any λ in C and x, y in C^n
$(\lambda x, y) = \overline{\lambda}(x,y), \quad (x, \lambda y) = \lambda(x,y).$

(iv) The adjoint of any $n \times n$ matrix A is denoted by A^*.

Theorem 3.3 *Let the system (5) be u.e.s. with a continuously differentiable initial condition ϕ. Then there exists a continuously differentiable mapping $q : [0,\infty] \to C^n$ such that:*

(i) $$q(t) = \int_0^\infty S^*(\sigma)x(\sigma + t)d\sigma, \tag{14}$$

(ii) $$\frac{d}{dt}[q(t) - \sum_{j=1}^{m} D_j^* q(t+h_j)] = -x(t) - A_0^* q(t) - \sum_{j=1}^{m} A_j^* q(t+h_j) \tag{15}$$

and

(iii) $$(q(0) - \sum_{j=1}^{m} D_j^* q(h_j),\ \phi(0)) + \sum_{j=1}^{m} \int_{h_j}^{0} (q(\sigma + h_j),\ A_j\phi(\sigma) + D_j\dot{\phi}(\sigma))d\sigma \tag{16}$$
$$= \int_0^\infty (x(\sigma), x(\sigma))d\sigma.$$

The proof of Theorem 3.3 is an extrapolation of the results in Datko (1974). In particular (14) and (16) induce a bilinear functional on the space of continuous vector valued functions from $[-h, 0]$ into C^n. The problem is that an explicit form for this functional is impossible to obtain. However as we shall see equations (14) and (16) have their uses. These are shown in the following theorem whose proof is given in Datko (1995).

Theorem 3.4 *Let the system (5) be u.e.s. and let*

$$M_0 \geq \sup_w |\hat{S}(iw)^*| \tag{17}$$

and

$$M_1 \geq \sup_w |wS(iw)^*|. \tag{18}$$

Then all poles of the matrix operator function $\hat{S}(\lambda)$ lie in

$$Re\lambda < -\alpha, \quad \alpha > 0, \tag{19}$$

where α is the unique real solution of the equation

$$2(1 + \sum |D_j|)\, M_0\alpha + 2\sum_{j=1}^{m} [M_0|A_j| + M_1|D_j|] \sinh\alpha h_j = 1 \tag{20}$$

Remark 3.5 One consequence of Theorem 3.4 is that all solutions of the system (5) have a decay rate $e^{-\alpha t}$ (see e.g. Henry (1974)).

Example 3

Let us return to example 2. When $h = 1$ the system is u.e.s. and unstable when $h = 2$. In this case $|D| = \frac{1}{2}$, $|A_1| = 1$. When $h = 1$ the quantities M_0 and M_1 in (17) and (18) satisfy

$$M_0 < 7, \quad M_1 < 8 \tag{21}$$

(see Datko (1995)). Since the real solution of (20) decreases as M_0 and M_1 increase the solution of that equation for the values $M_0 = 7$ and $M_1 = 8$ yield an upper bound on the poles of $\hat{S}(\lambda)$. The value, $-\alpha$, in this case is

$$-\alpha = -.02354741522. \tag{22}$$

When $h = 2$ in the system of example 2 the values of M_0 and M_1 respectively satisfy

$$M_0 < 2.5, \quad M_1 < 2.3. \tag{23}$$

We substitute these values into equation (20) to obtain

$$\alpha = .045208159134. \tag{24}$$

We now estimate $\max_w |\hat{S}(-\alpha + iw)^*|$ for this value of α. We obtain
$\max_w |\hat{S}(-\alpha + iw)^*| > 2.7 > \max_w |S(iw)^*|$.

This implies that $\hat{S}(-\overline{\lambda})^*$ is not analytic in the entire left half plane $Re\lambda < 0$. Hence $\hat{S}(\lambda)$ cannot be in H^∞, which means that for $h = 2$ the system is unstable.

Another method for arriving at the same conclusion is to observe that $F(\lambda, 2)$ in (8) satisfies the inequalities $\sup_w |F(iw, 2)| < 4.3 < 8.9 < \sup_w |F(\alpha + iw, 2)|$, where α is given by (24). Thus if $F(\lambda, 2)$ is not in H^∞ neither is $\hat{S}(\lambda)$.

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