In the present paper, the wave theory of transverse impact upon Timoshenko beams is developed by the ray method, which was proposed by Achenbach and Reddy (1967) and has been extended by Rossikhin (1986) and Rossikhin and Shitikova (1992, 1993) to dynamic contact problems.

2 DETERMINING RELATIONS OF THE RAY THEORY FOR A TIMOSHENKO BEAM

The dynamic behaviour of an elastic homogeneous prismatic beam with due account for the rotary inertia and transverse shear deformations is described by the equations:

$$\partial M/\partial z - Q = -\rho I\dot{\beta}, \qquad \partial Q/\partial z = \rho F\dot{W}, \tag{1}$$

$$\dot{M} = -EI\,\partial\beta/\partial z, \qquad \dot{Q} = K\mu F(\partial W/\partial z - \beta), \tag{2}$$

where M is the bending moment, Q is the transverse force, W is the transverse displacement velocity of a beam central axis (velocity of deflection), β is the angular velocity of a cross section about the x-axis which is perpendicular to the plane of flexure $y-z$ (the axes z and y are directed along the beam axis and vertically down, respectively), E is Young's modulus, μ is the shear modulus, ρ is the density, I is the moment of inertia about the x-axis (vibrations occur in the y-direction), F is the cross-section area, K is the shear coefficient depending on the form of a cross section, and an over dot denotes the derivative with respect to time t.

Assume that as a result of the transverse impact upon the beam, a plane wave Σ of strong discontinuity propagates along the z-direction with the velocity G. Behind the wave surface Σ up to the boundary of the contact region, a certain desired function $Z(z,t)$ is represented by a series in terms of powers $t-(z-l)G^{-1} \geq 0$

$$Z(z,t) = \sum_{k=0}^{\infty} \frac{1}{k!}[Z_{,(k)}]\left(t - \frac{z-l}{G}\right)^k H\left(t - \frac{z-l}{G}\right), \tag{3}$$

where $[Z_{,(k)}] = Z^+_{,(k)} - Z^-_{,(k)} = [\partial^k Z/\partial t^k]$ are the jumps in kth derivatives of function Z with respect to time t on the wave surface Σ, i.e. at $t = z/G$, the upper indices $+$ and $-$ signify that the value is calculated ahead of and behind the wave front, respectively, $2l$ is the length of the contact area, and $H(t)$ is the Heaviside function.

To determine coefficients of the ray series (3) for the desired functions, it is necessary to differentiate the governing Eqs.(1) k times with respect to time, take their difference on the different sides of the wave surface Σ, and apply the condition of compatibility (Thomas, 1961) for discontinuities of $(k+1)$th order derivatives of a certain function Z with respect to time

$$G\left[\frac{\partial Z_{,(k)}}{\partial z}\right] = -[Z_{,(k+1)}] + \frac{d[Z_{,(k)}]}{dt}, \tag{4}$$

where d/dt is the total time derivative of a function defined on the moving surface Σ. As a result we obtain

$$\rho I\left(1 - \frac{E}{\rho G^2}\right)[\beta_{,(k+1)}] = -2EIG^{-2}\frac{d[\beta_{,(k)}]}{dt} - K\mu FG^{-1}[W_{,(k)}] + EIG^{-2}\frac{d^2[\beta_{,(k-1)}]}{dt^2}$$

K. Malanowski et al. (eds.), *Modelling and Optimization of Distributed Parameter Systems Applications to engineering*

$$+K\mu FG^{-1}\frac{d[W_{,(k-1)}]}{dt} - K\mu F[\beta_{,(k-1)}], \tag{5}$$

$$\rho F\left(1-\frac{K\mu}{\rho G^2}\right)[W_{,(k+1)}] =$$

$$-2K\mu FG^{-2}\frac{d[W_{,(k)}]}{dt} + G[\beta_{,(k)}] + \frac{d^2[W_{,(k-1)}]}{dt^2} - G\frac{d[\beta_{,(k-1)}]}{dt}. \tag{6}$$

From Eqs.(5) and (6), one can obtain the values $[\beta_{,(k)}]$ and $[W_{,(k)}]$ (k = -1,0,1,..) with an accuracy of arbitrary constants on the two waves: quasi-flexural wave and quasi-shear wave propagating with the velocities $G^{(1)} = (E/\rho)^{1/2}$ and $G^{(2)} = (K\mu/\rho)^{1/2}$, respectively. Arbitrary constants are determined from the condition of compatibility for deformations on the boundaries of the contact region.

3 IMPACT OF A THIN BAR UPON A TIMOSHENKO BEAM

Let a thin elastic bar of a rectangular cross section, whose axis coincides with the y-axis, move along this axis with the velocity V and bump by its end against the centre of a Timoshenko beam.

During impact two types of plane waves propagate along the beam, behind the wave fronts up to the contact area boundary the solution is determined by Eq.(3). At the same time, a longitudinal wave propagates along the thin bar with the velocity $G_b = (E_b/\rho_b)^{1/2}$ (E_b and ρ_b are the Young's modulus and density of the striking bar, respectively), representing itself a plane of strong discontinuity. In virtue of the fact that the jumps of derivatives of the displacement velocities of the bar's particles $[V_{,(k)}]$ = const, on this wave plane the dynamic condition of compatibility is satisfied what allows one to connect the stress σ^- and displacement velocity V behind the wave front with each other at every instant of time. Specifically, in the contact area, i.e. at $y = 0$, we have

$$\sigma' = \rho_b G_b(V - W), \tag{7}$$

where $\sigma' = \sigma|_{y=0}$ is the contact stress, $W = V|_{y=0}$ is the displacement velocity of the beam part being in contact.

In view of Eq.(7), the equation of motion of the beam part, which is in contact with the bar, may be written as

$$-2lF\rho\dot{W} + 2la\rho_b G_b(V - W) + 2Q = 0, \tag{8}$$

where a is the width of the beam.

The quantities W and Q entering into Eq.(8) are defined by Eqs.(3) where $z = l$. It is necessary to add the initial condition

$$W|_{t=o} = 0, \tag{9}$$

as well as the relation

$$\partial W/\partial z|_{z=l} = 0 \tag{10}$$

to Eq.(8). Substituting (3) into Eqs.(8) and (10) with due account of (9), and equating the coefficients associated with the same powers of t, we can find all necessary arbitrary constants. Thus the approximate solution of our problem can be obtained. As an example, we consider the transverse impact of a steel bar of the length s =120 mm and rectangular cross section 38.1 × 38.1 mm upon an aluminum beam with 38.1 × 38.1 mm cross section.

The initial velocity of impact is V=1 m/s. After calculation of ray series coefficients, the truncated ray series for the beam transverse displacement w, transverse force Q, and bending moment M can be written at $z = l$, respectively, as

$$w = 3.9\,10^5 \tfrac{t^2}{2} - 3.07\,10^{11} \tfrac{t^3}{6} + 3.62\,10^{17} \tfrac{t^4}{24},$$
$$Q = -1.16\,10^{10} t + 9.11\,10^{15} \tfrac{t^2}{2} - 8.12\,10^{21} \tfrac{t^3}{6},$$
$$M = 1.12\,10^3 - 8.84\,10^8 t + 7.05\,10^{14} \tfrac{t^2}{2} - 6.55\,10^{20} \tfrac{t^3}{6}.$$

4 IMPACT OF A SPHERE UPON A TIMOSHENKO BEAM

Let an elastic sphere of the radius R and mass m move along the y-axis with the constant velocity V_0 towards an elastic beam. The impact occurs at $t = 0$.

When $t > 0$, the sphere displacement may be represented as

$$y = w + \alpha. \tag{11}$$

Then the equation of motion of the beam part being in contact without regard for an inertia term (due to infinitesimal of the contact region), and the equation of the sphere motion have the form

$$2Q + P(t) = 0, \tag{12}$$
$$m\ddot{y} = -P(t). \tag{13}$$

Equations (12) and (13) are solved with the following initial conditions to be taken into account:

$$y|_{t=0} = 0, \quad \dot{y}|_{t=0} = V_0, \quad w|_{t=0} = 0, \quad W|_{t=0} = 0. \tag{14}$$

The relation between the contact force $P(t)$ and penetration $\alpha(t)$ has the form

$$P = k\alpha^{3/2}, \tag{15}$$

where $k = 4R^{1/2}/3\pi(k_s + k_b)$, $k_s = (1 - \nu_s^2)/E_s$, $k_b = (1 - \nu^2)/E$, ν is the Poisson's ratio, and the indices s and b concern the sphere and beam, respectively.

The value Q entering into Eq.(12) is determined by the dynamic condition of compatibility as

$$Q = -\rho F G^{(2)} W. \tag{16}$$

This condition can be obtained if we interpret the discontinuity surface as the limiting layer of the width h at $h \to 0$, wherein the value Z to be found changes monotonically and infinitely from the magnitude Z^+ to the magnitude Z^-. Considering that on the wave surface (Thomas, 1961)

$$\frac{\partial}{\partial z} = \frac{d}{dn}, \qquad \frac{\partial}{\partial t} = -G\frac{d}{dn}, \tag{17}$$

where d/dn is the derivative with respect to the normal to the wave surface, and changing the partial derivatives in the second equation of (1) by its expressions (17), after the integration of the resulting relations with respect to n from $-h/2$ to $h/2$ and passage to the limit at $h \to 0$ we are led to the formula (16).

Substituting Eqs.(11), (15), and (16) into Eqs.(12)-(13), we arrive at the differential equation about the value $\alpha(t)$:

$$\ddot{\alpha} + \frac{k}{m}\alpha^{3/2} + \frac{3}{2}b\dot{\alpha}\alpha^{1/2} = 0, \quad b = \frac{k}{2\rho F G^{(2)}}. \tag{18}$$

Introducing $A = \dot{\alpha}$ and converting from the variable t to the new independent variable α, we are led to the equation

$$A\frac{dA}{d\alpha} + \frac{3}{2}bA\alpha^{1/2} = -\frac{k}{m}\alpha^{3/2} \tag{19}$$

with the initial conditions

$$\alpha|_{t=0} = 0, \quad \dot{\alpha}|_{t=0} = V_0. \tag{20}$$

Note that Eq.(19) is the Abel equation of the second kind (Kamke, 1959).

We seek the solution to Eq.(19) in the form

$$A = V_0 + \sum_{i=1}^{7} a_i\alpha^{(2i+1)/2} + \sum_{i=1}^{7} b_i\alpha^i + O(\alpha^8). \tag{21}$$

Substituting (21) into (19) and equating coefficients at equal powers of α, we determine the coefficients to be found

$$a_1 = -b, \quad a_2 = -\frac{2}{5}\frac{k}{V_0 m}, \quad a_3 = a_4 = b_1 = b_2 = b_3 = b_6 = 0,$$

$$b_4 = -\frac{5}{8}a_1a_2/V_0, \quad b_5 = -\frac{1}{2}a_2^2/V_0, \quad b_7 = -\frac{11}{14}a_1a_5/V_0,$$

$$a_5 = -\frac{8}{11}b_4a_1/V_0, \quad a_6 = -\left(\frac{10}{13}a_1b_5 + a_2b_4\right)/V_0, \quad a_7 = -\frac{2}{3}a_2b_5/V_0. \tag{22}$$

To obtain connection between the value α and the time, it is necessary to integrate Eq.(21).

The coefficients a_1 and a_2 are defined by the two processes being caused by the shock interaction: the coefficient a_1 is responsible for the dynamic processes arising in the beam during the propagation of the surfaces of discontinuity, but the coefficient a_2 answers for the quasi-static processes occurring at local bearing of the material due to the Hertz's contact theory. If $a_1 \to 0$, what realizes at an infinitely large velocity of shear wave propagation (Bernoulli-Euler beam), then the solution (21) for small α goes over into the quasi-static solution obtained by Timoshenko (1928).

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