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Intrinsic modelling of shells

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Abstract
We present recent results on the use of the oriented boundary (signed, algebraic) distance function and the tangential differential calculus in the intrinsic modelling of thin/shallow shells. We provide the link with covariant operators and show how to express them without Christoffel symbols. Such models are mathematically more tractable than classical ones.

Keywords
Shell, distance function, intrinsic differential operators, tangential calculus

1 INTRODUCTION

In this paper we review and announce some recent results on intrinsic models of linear shells by Delfour and Zolésio (1994-4, 1995-1 to 3) and show how to reformulate the classical models of Naghdi, Koiter, and the asymptotic membrane model of Ciarlet and Sanchez-Palencia (1993) in terms of intrinsic differential operators. This link was so far missing making comparisons difficult between our intrinsic model and the classical ones. It is now our belief that models expressed in terms of intrinsic differential operators are mathematically more tractable and natural to use in associated control, optimal design, and shape sensitivity problems (cf. for instance Lions (1968), Lagnese and Lions (1988), Sokolowski and Zolésio (1992)). It can also open the way to different parametrizations of the mean surface in the numerical analysis of partial differential equations on submanifolds of $\mathbb{R}^N$. An illustration of this viewpoint is given in the companion paper of Delfour and Zolésio (1995-5).

The new linear model of Delfour and Zolésio (1995-2) only uses two assumptions: (i) the displacement vector is equal to the displacement of the mean surface plus a tangential vector times the normal coordinate z to the mean surface, and (ii) a truncation of the infinite expansion with respect to z of the corresponding strain (deformation) tensor after its second power in z. No other approximations is involved. The simplest rheological law has been used, but our development readily extends to more complex laws. This approach

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was developed in a sequence of papers (Delfour and Zolésio (1994-4, 1995-1)) starting with a truncation of the strain tensor after its first power in z . However for this model it was difficult to exactly recover the rigid displacements as the kernel of the deformation tensor. The difficulty completely disappeared by going to the second power of z. It was possible to give a completely self-contained treatment for both static and dynamical models extending to thin/shallow shells the "Natural Theory" and the Love-Kirchhoff theory of plates (cf. for instance Germain (1986)) in the general spirit of completely intrinsic methods of Valid (1981). Finally it is interesting to emphasize that the Love-Kirchhoff theory comes out of the analysis as a special case of the natural theory by looking at the same variational equation over a closed linear subspace of the Hilbert space $\mathcal{V}$ associated with the natural theory.

2 DEFINITIONS AND NOTATION

2.1 Oriented boundary distance function

Let $\mathbb{R}^N$ be the N-dimensional Euclidean space for some integer $N > 1$ (in practice $N = 3$). Let Ω be a subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$ which is a C^2 $(N-1)$-dimensional submanifold of $\mathbb{R}^N$. Associate with Ω the ***oriented boundary (resp. algebraic or signed) distance function***

$$b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\Omega^c}(x), \quad \Omega^c \stackrel{\text{def}}{=} \mathbb{R}^N \backslash \Omega = \{x \in \mathbb{R}^N : x \notin \Omega\}, \tag{1}$$

where d_A is the usual distance function to a subset A of $\mathbb{R}^N$. This function captures the geometrical properties of the boundary $\partial\Omega$. Moreover for any integer $k \geq 2$, a domain Ω has a C^k boundary $\partial\Omega$ if and only if in each point $X \in \partial\Omega$ there exists a bounded open neighbourhood $N(X)$ of X such that $b_\Omega \in C^k(N(X))$ (cf. Gilbarg and Trudinger (1983), Delfour and Zolésio (1994-1, 1994-2)). At each point X of $\partial\Omega$, its gradient $\nabla b_\Omega(X)$ coincides with the unitary exterior normal n to $\partial\Omega$ and the eigenvalues of the symmetrical matrix of second order partial derivatives $D^2 b_\Omega$ are 0 and the *principal curvatures*, κ_i, $1 \leq i \leq N-1$, of the submanifold $\partial\Omega$. The trace of $D^2 b_\Omega(X)$ is the *mean curvature*

$$H(X) \stackrel{\text{def}}{=} \operatorname{tr}(D^2 b_\Omega(X)) = \Delta b_\Omega(X), \tag{2}$$

up to the multiplying factor $(N-1)$ which is used as a normalization factor to make the mean curvature of the unit sphere equal to one in all dimensions. *We choose to modify the classical definition since it is the term Δb_Ω which will naturally occur and not* $\Delta b_\Omega/(N-1)$. If we really want to make a distinction, our definition of $H(X)$ would be the *additive curvature.* The trace of the matrix of cofactors $M(D^2 b_\Omega)$ is the *total* or *Gaussian curvature*

$$K(X) \stackrel{\text{def}}{=} \operatorname{tr} M(D^2 b_\Omega(X)). \tag{3}$$

Since the domain Ω is fixed throughout this paper, we shall now drop the subscript Ω. For each $X \in \partial\Omega$, the *projection mapping* $p : N(X) \to \partial\Omega$ and its Jacobian matrix are obtained directly from the oriented distance function b as

$$p(x) = x - b(x)\nabla b(x), \qquad Dp(x) = I - b(x)D^2b(x) - \nabla b(x)^*\nabla b(x), \tag{4}$$

where ${}^*\nabla b(x)$ is the transposed of the vector $\nabla b(x)$ and I is the identity matrix. For $x \in N(X)$, the *linear projector* onto the *tangent plane* $T_{p(x)}\partial\Omega$ at $p(x)$ of $\partial\Omega$ is given by

$$P(x) = I - \nabla b(x)^*\nabla b(x). \tag{5}$$

2.2 Definition of the "shell"

In practice *the mean surface* Γ is the mean surface of a thin piece of material called the shell, but in order to make sense of all the differential operators defined on Γ it is a posteriori assumed that it is a sufficiently smooth submanifold of $\mathbb{R}^N$. So we might as well start from a mathematical description of the mean surface and *build the shell around it.* Therefore a *(mathematical) shell* is characterized by its *mean surface* Γ and its *thickness (function)* $\tilde{h} : \Gamma \to \mathbb{R}^+$. Assume that Γ is a bounded open domain in the $(N-1)$-submanifold $\partial\Omega$ of $\mathbb{R}^N$. When $\Gamma = \partial\Omega$ (hence $\partial\Omega$ is compact), the shell has no boundary. When $\Gamma \neq \partial\Omega$, the (relative) boundary $\partial_{\partial\Omega}\Gamma$ is assumed to be uniformly Lipschitzian in $\partial\Omega$. Since Γ is bounded and $\partial\Omega$ is C^2, there exist $h > 0$ and a bounded neighbourhood

$$S_h \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : p(x) \in \Gamma,\ |b(x)| < h\}, \tag{6}$$

where b is C^2. The set S_h is a bounded open domain in $\mathbb{R}^N$ with a Lipschitzian boundary. When $\Gamma \neq \partial\Omega$, S_h has a *lateral boundary*

$$\Sigma_h = \{x \in \mathbb{R}^N : p(x) \in \partial_{\partial\Omega}\Gamma,\ |b(x)| < h\} \tag{7}$$

which is an $(N-1)$-dimensional surface normal to the mean surface Γ. It is important to keep in mind that we use the distance function $b = b_\Omega$ and not the distance function to Γ.

2.3 Flow of the gradient of b and local coordinates

Since $\nabla b \in C^1(S_h)$, consider the *flow mapping*

$$T_z(X) = x(z), \quad \frac{dx}{dz}(z) = \nabla b(x(z)), \quad |z| < h, \quad x(0) = X. \tag{8}$$

It is a homeomorphism from Γ onto $\Gamma_z = \{x \in \mathbb{R}^N : b(x) = z,\ p(x) \in \Gamma\}$. In particular

$$T_z(X) = X + z\nabla b(X), \qquad DT_z = I + zD^2b \tag{9}$$

for $|z| < h$. This induces a *curvilinear coordinate system* $(X, z) \in \Gamma\times\,]-h, h[$ in S_h. The points on the level set Γ_z are given by $\{X + z\nabla b(X) : X \in \Gamma\}$ and for each

$(X,z) \in \Gamma \times]-h,h[$, $\nabla b(T_z(X)) = \nabla b(X)$. Therefore for all $x \in S_h$, $\nabla b(x) = n(p(x))$, where n is the normal to Γ at $p(x)$. *So we shall often simply write n instead of ∇b.* Moreover $\det DT_z(X)$ is a polynomial of degree at most $N-1$

$$j(z) \stackrel{\text{def}}{=} \det DT_z(X) = \sum_{i=0}^{N-1} K_i(X) z^i, \tag{10}$$

where $K_0 = 1$, $K_1 = H$ for $N \geq 2$, and $K_{N-1} = K$ for $N \geq 3$. For $N = 3$, $j(z) = 1 + zH(X) + z^2 K(X)$.

3 INTRINSIC TANGENTIAL CALCULUS

Given a scalar function $w : \Gamma \to \mathbb{R}$, denote by $\nabla_\Gamma w$ the *tangential gradient*

$$\nabla_\Gamma w = \nabla W|_\Gamma - \frac{\partial W}{\partial n} n \tag{11}$$

defined in terms of an extension W of w to S_h. This definition is independent of the choice of the extension W. Moreover it is easy to check that

$$\nabla(w \circ p) = Dp\,(\nabla_\Gamma w) \circ p = [I - bD^2 b]\,\nabla_\Gamma w \circ p \quad \text{and} \quad \nabla(w \circ p)|_\Gamma = \nabla_\Gamma w, \tag{12}$$

where $\circ$ denotes the composition of two functions. The *tangential Jacobian matrix* of a vector $v : \Gamma \to \mathbb{R}^N$ is defined through an extension V of v or through the transposed ${}^*D_\Gamma v = (\nabla_\Gamma v_1, \ldots, \nabla_\Gamma v_N)$ in terms of the column tangential gradients of its components. In particular

$$D(v \circ p) = (D_\Gamma v) \circ p\, Dp = (D_\Gamma v) \circ p\, [I - bD^2 b] \quad \text{and} \quad D(v \circ p)|_\Gamma = D_\Gamma v. \tag{13}$$

In the same way define the *tangential divergence* in term of an extension V of v to a neighbourhood of Γ

$$\operatorname{div}_\Gamma v \stackrel{\text{def}}{=} \operatorname{div} V|_\Gamma - DVn \cdot n \quad \text{and} \quad \operatorname{div}(v \circ p)|_\Gamma = \operatorname{div}_\Gamma v = \operatorname{tr} D_\Gamma(v), \tag{14}$$

where $\cdot$ is the inner product in $\mathbb{R}^N$. Similarly the *tangential strain tensor* is defined as

$$\varepsilon_\Gamma(v) \stackrel{\text{def}}{=} \frac{1}{2}(D_\Gamma v + {}^*D_\Gamma v) \qquad \text{and} \qquad \varepsilon_\Gamma(v) = \varepsilon(v \circ p)|_\Gamma. \tag{15}$$

In view of identities (12) and (14) the composition of $\operatorname{div}_\Gamma$ and ∇_Γ yields the *Laplace-Beltrami* operator

$$\Delta_\Gamma w \stackrel{\text{def}}{=} \operatorname{div}_\Gamma(\nabla_\Gamma w) \quad \text{and} \quad \Delta(w \circ p)|_\Gamma = \Delta_\Gamma w, \tag{16}$$

but the *matrix of tangential second order derivatives* is generally not symmetrical

$$\begin{aligned} &D_\Gamma(\nabla_\Gamma w) - D^2b\,\nabla_\Gamma w\ {}^*n = D^2(w \circ p)|_\Gamma = D^*_\Gamma(\nabla_\Gamma w) - n\ {}^*(D^2b\,\nabla_\Gamma w) \\ &\varepsilon(\nabla(w \circ p))|_\Gamma = \varepsilon_\Gamma(\nabla_\Gamma(w)) - \tfrac{1}{2}[D^2b\,\nabla_\Gamma w\ {}^*n + n\ {}^*(D^2b\,\nabla_\Gamma w)]. \end{aligned} \tag{17}$$

Another tangential operator which will naturally occur is

$$\varepsilon^P_\Gamma(v) \stackrel{\text{def}}{=} P\varepsilon_\Gamma(v)P = \varepsilon^P_\Gamma(v) = \varepsilon_\Gamma(v) - [\varepsilon_\Gamma(v)n\,{}^*n + n^*(\varepsilon_\Gamma(v)n)]. \tag{18}$$

If we denote by w and u the normal and tangential components of v, $w \stackrel{\text{def}}{=} v \cdot n$ and $u \stackrel{\text{def}}{=} v - w\,n$, then

$$\varepsilon^P_\Gamma(v) = \varepsilon^P_\Gamma(u) + wD^2b \qquad \varepsilon^P_\Gamma(u) = \varepsilon_\Gamma(u) + \frac{1}{2}[D^2b\,u\ {}^*n + n\ {}^*(D^2b\,u)]. \tag{19}$$

In the sequel the following identity will be useful

$${}^*D_\Gamma v\,n = \nabla_\Gamma(v \cdot n) - D^2b\,v = \nabla_\Gamma(w) - D^2b\,u \quad \text{on } \Gamma. \tag{20}$$

4 INTRINSIC LINEAR MODEL FOR SHELLS

4.1 Preliminary results

For simplicity we work with a shell of constant thickness and make the following standard mechanical assumption on the displacement vector.

Assumption 1 *At each point x of the shell the displacement vector $U(x)$ is of the form*

$$U(x) = u(p(x)) + b(x)\,\ell(p(x)) \qquad (U = u \circ p + b\,\ell \circ p) \text{ in } S_h, \tag{21}$$

for vector-valued mappings u and ℓ from Γ to $\mathbb{R}^N$ such that ℓ is a tangent field, that is $\ell(X) \cdot n(X) = 0$, for all $X \in \Gamma$.

With the help of the tangential calculus the Jacobian matrix DU in S_h is given by

$$\begin{aligned} &DU = [D_\Gamma(u) \circ p + b\,D_\Gamma(\ell) \circ p + \ell \circ p\ {}^*\nabla b]\,[I - b\,D^2b] \\ &DU\,\nabla b = \ell \circ p \end{aligned} \tag{22}$$

and the *strain tensor* $\varepsilon(U)$ by

$$\begin{aligned} &2\varepsilon(U) = D(U) + {}^*D(U) \\ &= [D_\Gamma(u) \circ p + b\,D_\Gamma(\ell) \circ p + \ell \circ p\ {}^*\nabla b]\,[I - b\,D^2b] \\ &+[I - b\,D^2b]\,{}^*[D_\Gamma(u) \circ p + b\,D_\Gamma(\ell) \circ p + \ell \circ p\ {}^*\nabla b] \end{aligned} \tag{23}$$

$$2\varepsilon(U)\,n = [I - b\,D^2b]\,[2\varepsilon_\Gamma(u) \circ p\,n + \ell \circ p], \tag{24}$$

where $\varepsilon_\Gamma(e)$ is the *tangential strain tensor* defined in (15). In the (X, z) coordinate system the above expression becomes

$$DU \circ T_z = [D_\Gamma(u) + z\, D_\Gamma(\ell) + \ell\, {}^*\nabla b]\,[I + z\, D^2 b]^{-1} \tag{25}$$
$$DU \circ T_z\, \nabla b = \ell$$

$$2\varepsilon(U) \circ T_z = [D_\Gamma(u) + z\, D_\Gamma(\ell) + \ell\, {}^*n]\,[I + z\, D^2 b]^{-1}$$
$$+[I + z\, D^2 b]^{-1}\, {}^*[D_\Gamma(u) + z\, D_\Gamma(\ell) + \ell\, {}^*n]$$
$$2\varepsilon(U) \circ T_z\, n = [I + zD^2 b]^{-1}[2\,\varepsilon_\Gamma(u)\, n + \ell]. \tag{26}$$

Under Assumption 1 we always have the identity

$$\varepsilon(U) \circ T_z\, n \cdot n = 0.$$

This identity is often introduced as an assumption in the literature. It is a direct consequence of the choice of U and the fact that the vector ℓ is tangential. The identity

$$2\varepsilon_\Gamma(u)\, n + \ell = 0 \text{ on } \Gamma$$

characterizes the *Love-Kirchhoff* models. When $u \cdot \nabla b = u \cdot n \in H^1(\Gamma)$, this identity can be written as

$$\nabla_\Gamma(u \cdot n) - D^2 b\, u + \ell = 0 \text{ on } \Gamma$$

extending to shells the identity for plates in Germain (1986). If ℓ and u belong to $H^1(\Gamma)^N$ and Γ is C^3, then $b \in C^3$ and $u \cdot n \in H^2(\Gamma)$.

The nonlinear part of $\varepsilon(U) \circ T_z$ with respect to the variable z is contained in the matrix $[I + z\, D^2 b]^{-1}$. So for $h\, \|D^2 b\| < 1$, that is

$$h \max_{1 \leq i \leq N-1} |\kappa_i(X)| < 1, \quad \forall X \in \Gamma,$$

the inverse is given by

$$[I + zD^2 b]^{-1} = \sum_{i=0}^{\infty} (-D^2 b)^i\, z^i, \tag{27}$$

and we get

$$\varepsilon(U) \circ T_z = \varepsilon(u \circ p + b\ell \circ p) \circ T_z = \sum_{i=0}^{\infty} \varepsilon^i(u, \ell)\, z^i, \tag{28}$$

where

$$\begin{aligned}
2\,\varepsilon^0(u,\ell) &= 2\,\varepsilon_\Gamma(u) + \ell\, {}^*n + n\, {}^*\ell \\
2\,\varepsilon^1(u,\ell) &= 2\,\varepsilon_\Gamma(\ell) - D_\Gamma(u)\, D^2 b - D^2 b\, {}^*D_\Gamma(u) \\
2\,\varepsilon^2(u,\ell) &= [D_\Gamma(\ell) - D_\Gamma(u)\, D^2 b](-D^2 b) + (-D^2 b)\,[{}^*D_\Gamma(\ell) - D^2 b\, {}^*D_\Gamma(u)] \\
2\,\varepsilon^i(u,\ell) &= [D_\Gamma(\ell) - D_\Gamma(u)\, D^2 b](-D^2 b)^{i-1} + (-D^2 b)^{i-1}\,[{}^*D_\Gamma(\ell) - D^2 b\, {}^*D_\Gamma(u)]
\end{aligned} \tag{29}$$

for $i \geq 3$. The next theorem characterizes the *rigid displacements*.

Theorem 1 *Given u and ℓ in $H^1(\Gamma)^N$, the following statements are equivalent:*

(i) $\varepsilon(u \circ p + b\ell \circ p) = 0$ in S_h
(ii) $\varepsilon^0(u,\ell) = \varepsilon^1(u,\ell) = \frac{1}{2}[{}^*D_\Gamma(\ell)\, D^2 b + D^2 b\, D_\Gamma(\ell)] = 0$ on Γ
(iii) $\varepsilon^0(u,\ell) = \varepsilon^1(u,\ell) = \varepsilon^2(u,\ell) = 0$ on Γ
(iv) $\exists\, a \in \mathbb{R}^N$ *and an* $N \times N$ *matrix* B, $B + {}^*B = 0$, *such that*

$$\ell(X) = B\,n(X), \quad u(X) = a + BX, \quad \forall X \in \Gamma. \tag{30}$$

Therefore ℓ is tangential and $n \cdot (B\,n) = 0$ on Γ.

4.2 The second order model in the thickness variable

In order to preserve the *rigid displacements* the series is truncated as

$$\tilde{\varepsilon}(U) \circ T_z \stackrel{\text{def}}{=} \varepsilon^0(u,\ell) + \varepsilon^1(u,\ell)\, z + \varepsilon^2(u,\ell)\, z^2 \tag{31}$$

It is natural to associate with $\tilde{\varepsilon}$ the following Hilbert spaces

$$\begin{aligned} \mathcal{H} &= \{(u,\ell) \in L^2(\Gamma)^N \times L^2(\Gamma)^N : \ell \cdot n = 0 \text{ on } \Gamma\} \\ \mathcal{V} &= \left\{(u,\ell) \in \mathcal{H} : \varepsilon^i(u,\ell) \in L^2(\Gamma)^{N\times N},\ 0 \le i \le 2\right\} \\ \mathcal{N} &= \{(u,\ell) \in \mathcal{V} : \varepsilon^i(u,\ell) = 0 \text{ on } \Gamma,\ 0 \le i \le 2\} = \textstyle\bigcap_{i=0}^{i=2} \operatorname{Ker} \varepsilon^i \end{aligned} \tag{32}$$

with norms

$$|(e,\ell)|^2_{\mathcal{H}} = |e|^2_{L^2(\Gamma)} + |\ell|^2_{L^2(\Gamma)} \quad \text{and} \quad \|(e,\ell)\|^2_{\mathcal{V}} = |(e,\ell)|^2_{\mathcal{H}} + \sum_{i=0}^{2} \|\varepsilon^i(e,\ell)\|^2_{L^2(\Gamma)}. \tag{33}$$

From Theorem 1

$$\begin{aligned} \mathcal{N} = \{(u,\ell) \in \mathcal{V} :\ & u(X) = a + BX,\ \ell(X) = B\,n(X), \quad \forall a \in \mathbb{R}^N, \\ & \forall B \text{ an } N \times N \text{ matrix such that } B + {}^*B = 0\}. \end{aligned} \tag{34}$$

In order to complete the characterization of $\mathcal{V}$ we use the following *Korn's inequality.*

Theorem 2 *Assume that Γ is a bounded open domain in the C^2 $(N-1)$-dimensional submanifold $\partial\Omega$ of $\mathbb{R}^N$ with a Lipschitzian boundary $\partial_{\partial\Omega}\Gamma$ in $\partial\Omega$. As h goes to zero,* there exists a constant $c(h) > 0$ such that for all $(u,\ell) \in \mathcal{V}$

$$\begin{aligned} &\int_\Gamma 2\,h\,[|\ell|^2 + \|D_\Gamma(u)\|^2] + 2\,\frac{h^3}{3}\,\|D_\Gamma(\ell)\|^2\, d\Gamma \\ &\le c(h)^2 \int_\Gamma 2\,h\,|u|^2 + 2\,\frac{h^3}{3}\,|\ell|^2 + 2\,h\,\|\varepsilon^0(u,\ell)\|^2 + 2\,\frac{h^3}{3}\,\|\varepsilon^1(u,\ell)\|^2 + 2\,\frac{h^5}{5}\,\|\varepsilon^2(u,\ell)\|^2\, d\Gamma, \end{aligned} \tag{35}$$

where

$$\|A\|^2 = \sum_{i,j=1}^{N} A_{ij} A_{ji}, \quad |a|^2 = \sum_{i=1}^{N} a_i^2.$$

In particular

$$\mathcal{V} = \left\{(u,\ell) \in H^1(\Gamma)^N \times H^1(\Gamma)^N \,:\, \ell\cdot n = 0\right\}. \tag{36}$$

Remark 1 *(Spherical shells) For a spherical shell* $\{x \in \mathbb{R}^N \,:\, |x| = R\}$ *of radius* R *in* $\mathbb{R}^N$

$$b(x) = |x| - R, \quad \nabla b(x) = \frac{x}{|x|}, \quad p(x) = R\,\nabla b(x).$$

As a result

$$Dp(x) = R\,D^2b(x), \quad P(x) = |x|\,D^2b(x)$$

and since $P^2(x) = P(x)$

$$D^2b(x) = |x|\,(D^2b(x))^2.$$

Moreover

$$\begin{array}{ll}
\frac{1}{2}[D_\Gamma v\, D^2b + D^2b^*D_\Gamma v] = \frac{1}{R}\,\varepsilon_\Gamma(v), & \frac{1}{2}[D_\Gamma v\,(D^2b)^2 + (D^2b)^{2*}D_\Gamma v] = \frac{1}{R^2}\,\varepsilon_\Gamma(v) \\
\varepsilon^0(u,\ell) = \varepsilon_\Gamma(u) + \frac{1}{2}[\ell^*\nabla b + \nabla b^*\ell], & \varepsilon^1(u,\ell) = \varepsilon_\Gamma(\ell) - \frac{1}{R}\,\varepsilon_\Gamma(u) \\
\varepsilon^2(u,\ell) = -\frac{1}{R}\,\varepsilon^1(u,\ell) & \\
\multicolumn{2}{l}{\tilde\varepsilon(U)\circ T_z = [1 - \frac{z}{R} + \left(\frac{z}{R}\right)^2]\,\varepsilon_\Gamma(u) + \frac{1}{2}[\ell^*\nabla b + \nabla b^*\ell] + [1 - \frac{z}{R}]\,z\,\varepsilon_\Gamma(\ell).}
\end{array} \tag{37}$$

By introducing the simple rheological law

$$\sigma = 2\,\mu\,\tilde\varepsilon + \lambda\,\mathrm{tr}\,\tilde\varepsilon, \quad \mu > 0,\ \lambda \ge 0,$$

we obtain the bilinear form associated with the strain energy in terms of polynomials $\alpha_n(h)$ in h and bilinear forms a_n. The polynomials $\alpha_n(h)$ of odd powers of h are functions of X on Γ defined as

$$\alpha_n(h) \stackrel{\text{def}}{=} h^{n+1}\sum_{i=0}^{N-1}[1 - (-1)^{n+i+1}]\frac{h^i}{n+i+1}K_i, \quad 0 \le n \le 4. \tag{38}$$

For $N = 3$

$$\begin{array}{l}
\alpha_0 = 2\,h + 2\,\frac{h^3}{3}\,K \quad \alpha_1 = 2\,\frac{h^3}{3}\,H \quad \alpha_2 = 2\,\frac{h^3}{3} + 2\,\frac{h^5}{5}\,K \quad \alpha_3 = 2\,\frac{h^5}{5}\,H \\
\alpha_4 = 2\,\frac{h^5}{5} + 2\,\frac{h^7}{7}\,K.
\end{array} \tag{39}$$

If as in Naghdi's model the assumption of zero normal constraint, $\sigma\circ T_z\,n\cdot n = 0$, is used then since $\tilde\varepsilon\,n\cdot n = 0$

$$\mathrm{tr}\,\tilde\varepsilon = 0 \text{ and } \mathrm{tr}\,\varepsilon^i = 0, \quad 0 \le i \le 2.$$

This is equivalent to

$$\operatorname{div}_\Gamma(u) = 0, \quad \operatorname{div}_\Gamma(\ell) - \operatorname{tr}(D_\Gamma(u)D^2b) = 0 \quad \operatorname{tr}(D_\Gamma(\ell)D^2b) - \operatorname{tr}(D_\Gamma(u)(D^2b))^2 = 0.$$

The first condition is some kind of *inextensibility* of the mean surface. The spaces $\mathcal{H}$, $\mathcal{V}$, and $\mathcal{N}$, and their associated norms and seminorm have been defined in (33). Now define the bilinear operator $A : \mathcal{V} \to \mathcal{V}'$ and the linear operator $\mathcal{B} : L^2(\Gamma)^N \times L^2(\Gamma)^N \to \mathcal{H}'$: for all (u,ℓ) and $(\overline{u},\overline{\ell})$ in $\mathcal{V}$

$$\langle A(u,\ell),(\overline{u},\overline{\ell})\rangle_{\mathcal{V}} = \sum_{n=0}^{4} \int_\Gamma \alpha_n(h)\, a_n((u,\ell),(\overline{u},\overline{\ell}))\, d\Gamma, \tag{40}$$

where for all $\varepsilon^i = \varepsilon^i(u,\ell)$ and $\overline{\varepsilon}^i = \varepsilon^i(\overline{u},\overline{\ell})$

$$\begin{aligned}
a_0((u,\ell),(\overline{u},\overline{\ell})) &= 2\mu\, \varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^0 + \lambda \operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^0 \\
a_1((u,\ell),(\overline{u},\overline{\ell})) &= 2\mu\,[\varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^1 + \overline{\varepsilon}^0 \cdot\cdot\, \varepsilon^1] + \lambda\,[\operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^1 + \operatorname{tr}\overline{\varepsilon}^0 \operatorname{tr}\varepsilon^1] \\
a_2((u,\ell),(\overline{\varepsilon},\overline{\ell})) &= 2\mu\,[\varepsilon^1 \cdot\cdot\, \overline{\varepsilon}^1 + \varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^2 + \overline{\varepsilon}^0 \cdot\cdot\, \varepsilon^2] + \lambda\,[\operatorname{tr}\varepsilon^1 \operatorname{tr}\overline{\varepsilon}^1 + \operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^2 + \operatorname{tr}\overline{\varepsilon}^0 \operatorname{tr}\varepsilon^2] \\
a_3((u,\ell),(\overline{u},\overline{\ell})) &= 2\mu\,[\varepsilon^1 \cdot\cdot\, \overline{\varepsilon}^2 + \overline{\varepsilon}^1 \cdot\cdot\, \varepsilon^2] + \lambda\,[\operatorname{tr}\varepsilon^1 \operatorname{tr}\overline{\varepsilon}^2 + \operatorname{tr}\overline{\varepsilon}^1 \operatorname{tr}\varepsilon^2] \\
a_4((u,\ell),(\overline{u},\overline{\ell})) &= 2\mu\, \varepsilon^2 \cdot\cdot\, \overline{\varepsilon}^2 + \lambda \operatorname{tr}\varepsilon^2 \operatorname{tr}\overline{\varepsilon}^2
\end{aligned}$$

and

$$\langle \mathcal{B}(f,m),(u,\ell)\rangle_{\mathcal{H}} = \int_\Gamma \alpha_0(h)\,[f\cdot u + m\cdot \ell] + \alpha_1(h)\, f\cdot \ell\, d\Gamma. \tag{41}$$

By construction A is symmetrical and positive and

Lemma 1 *There exists* $\overline{h} > 0$ *and* $\alpha > 0$ *such that for all* $0 < h < \overline{h}$

$$\forall (u,\ell) \in \mathcal{V}, \quad \langle A(u,\ell),(u,\ell)\rangle_{\mathcal{V}} \geq 2\mu\, h\, \alpha \sum_{n=0}^{2} h^{2n} \|\varepsilon^n(u,\ell)\|^2. \tag{42}$$

If the elements of the dual $\mathcal{H}'$ of $\mathcal{H}$ are identified with those of $\mathcal{H}$, then the lemma says that A is a $\mathcal{V}$-$\mathcal{H}$ coercive operator.

Theorem 3 *Given* $\overline{h} > 0$ *as specified in Lemma 1 and assuming that the following condition is verified*

$$\forall (u,\ell) \in \mathcal{N}, \quad \int_\Gamma \alpha_0(h)\,[f\cdot u + m\cdot \ell] + \alpha_1(h)\, f\cdot \ell\, d\Gamma = 0, \tag{43}$$

then for all h*,* $0 < h \leq \overline{h}$*, there exists a unique solution* $(\hat{u},\hat{\ell}) \in \mathcal{V}/\mathcal{N}$ *to the variational equation:*

$$\forall (u,\ell) \in \mathcal{V}, \quad \langle A(\hat{u},\hat{\ell}),(u,\ell)\rangle_{\mathcal{V}} + \langle \mathcal{B}(f,m),(u,\ell)\rangle_{\mathcal{H}} = 0. \tag{44}$$

For a shell with boundary and homogeneous Dirichlet boundary conditions the results are analogous to the ones of Theorem 3 without condition (43) in the space

$$\mathcal{V}_0 \stackrel{\text{def}}{=} \left\{(u,\ell) \in H_0^1(\Gamma)^N \times H_0^1(\Gamma)^N \,:\, \ell \cdot n = 0\right\}. \tag{45}$$

Theorem 4 *Given* $\bar{h} > 0$ *as specified in Lemma 1 and* h, $0 < h \leq \bar{h}$, there exists a unique solution $(\hat{e},\hat{\ell}) \in \mathcal{V}_0$ to the variational equation:

$$\forall (u,\ell) \in \mathcal{V}_0, \quad \langle A(\hat{u},\hat{\ell}),(u,\ell)\rangle_{\mathcal{V}} + \langle \mathcal{B}(f,m),(u,\ell)\rangle_{\mathcal{H}} = 0. \tag{46}$$

5 INTRINSIC TANGENTIAL AND COVARIANT DERIVATIVES

Associate with the space $\mathbb{R}^N$ an orthonormal basis $\{e_1,\ldots,e_N\}$ at the origin. Let Ω be a subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^2 and let Γ be a bounded subset of $\partial\Omega$ with relative boundary $\partial\Gamma \stackrel{\text{def}}{=} \partial_{\partial\Omega}\Gamma$ in the $(N-1)$-submanifold $\partial\Omega$.

5.1 Local coordinates

Assume the existence of a C^2-map

$$\xi' \stackrel{\text{def}}{=} (\xi^1,\ldots,\xi^{N-1}) \mapsto \Phi(\xi') \,:\, \bar{A} \subset \mathbb{R}^{N-1} \to \overline{\Gamma} \subset \mathbb{R}^N, \tag{47}$$

where A is a bounded open connected domain in $\mathbb{R}^{N-1}$ with Lipschitzian boundary ∂A (located on the same side of ∂A). Further assume that in each point of $\bar{A}$ the vectors

$$a_\alpha = \frac{\partial\Phi}{\partial\xi^\alpha}, \qquad 1 \leq \alpha \leq N-1, \tag{48}$$

are linearly independent. In addition assume that Γ is oriented and select a unit normal a_N to Γ. For instance for $N = 3$ the choice is usually

$$a_3 = \frac{a_1 \times a_2}{|a_1 \times a_2|}. \tag{49}$$

But if we want the unit sphere to have a positive curvature, it is necessary to choose for a_N the inward unit normal to the unit ball. We shall follow the usual convention that a Greek index ranges from 1 to $N-1$ and that a Roman index ranges from 1 to N. The contravariant basis is defined as

$$a^i \cdot a_j = \delta_{ij}, \quad \Rightarrow a_N = a^N, \tag{50}$$

where δ_{ij} is the Kronecker index function. *Finally we choose a domain* Ω *such that*

$$a_N = a^N = -\nabla b_\Omega \circ \Phi. \tag{51}$$

5.2 Partial derivatives and fundamental forms

Consider a C^1-function $w : \Gamma \to \mathbb{R}$ and its extension $W = w \circ p$ in a neighbourhood of Γ. By definition the *partial derivative* of w is given by

$$w_{,\alpha} \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi^\alpha}(w \circ \Phi) \quad \text{and} \quad w_{,N} \stackrel{\text{def}}{=} 0, \tag{52}$$

since $w \circ \Phi$ is independent of the normal displacement to the submanifold $\partial\Omega$. Note that

$$w \circ \Phi = (w \circ p) \circ \Phi = W \circ \Phi$$

and consequently

$$\frac{\partial}{\partial \xi^\alpha}(w \circ \Phi) = (\nabla W \circ \Phi) \cdot \frac{\partial \Phi}{\partial \xi^\alpha} = [\nabla(w \circ p) \circ \Phi] \cdot a_\alpha.$$

But $\nabla(w \circ p)|_\Gamma = \nabla_\Gamma w$ and

$$w_{,\alpha} = \frac{\partial}{\partial \xi^\alpha}(w \circ \Phi) = [\nabla_\Gamma w \circ \Phi] \cdot a_\alpha.$$

In the sequel it will be convenient to use the notation a_i for both a_i and $a_i \circ \Phi^{-1}$ and $\nabla_\Gamma w$ for both $\nabla_\Gamma w$ and $\nabla_\Gamma w \circ \Phi$ whenever no confusion arises. Hence

$$w_{,\alpha} = \nabla_\Gamma w \cdot a_\alpha \quad \text{and} \quad w_{,N} = 0 = \nabla_\Gamma w \cdot a_N. \tag{53}$$

This extends to C^1 vector functions $v : \Gamma \to \mathbb{R}^N$ the extension $V = v \circ p$ to a neighbourhood of Γ

$$v_{,\alpha} \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi^\alpha}(v \circ \Phi), \quad v_{,N} \stackrel{\text{def}}{=} 0 \quad \Rightarrow \quad v_{,i} = [D_\Gamma v \circ \Phi]\, a_i. \tag{54}$$

As an application of the above identity, we get the *second and third fundamental forms*

$$b_{\alpha\beta} \stackrel{\text{def}}{=} -a_\beta \cdot a_{N,\alpha} = a_\beta \cdot (D^2 b\, a_\alpha) \quad \text{and} \quad c_{\alpha\beta} \stackrel{\text{def}}{=} b^\lambda_\alpha\, b_{\lambda\beta} = a_\beta \cdot ((D^2 b)^2\, a_\alpha). \tag{55}$$

5.3 Christoffel symbols and covariant derivatives

All the results below readily extend to tensors of higher order. By definition

$$\Gamma^\alpha_{\beta\gamma} \stackrel{\text{def}}{=} a^\alpha \cdot a_{\beta,\gamma} = a^\alpha \cdot D_\Gamma(a_\beta \circ \Phi^{-1}) \circ \Phi\, a_\gamma \tag{56}$$

and for simplicity we use the notation $D_\Gamma a_j$ for $D_\Gamma(a_j \circ \Phi^{-1}) \circ \Phi$. For a vector $v : \Gamma \to \mathbb{R}^N$

$$\begin{aligned} v_\alpha|_\gamma &\stackrel{\text{def}}{=} v_{\alpha,\gamma} - \Gamma^\lambda_{\alpha\gamma} v_\lambda = D_\Gamma v\, a_\gamma \cdot a_\alpha + D^2 b\, a_\alpha \cdot a_\gamma v_N = [{}^*D_\Gamma v + D^2 b\, v_N] a_\alpha \cdot a_\gamma \\ v^\alpha|_\gamma &\stackrel{\text{def}}{=} v^\alpha_{,\gamma} + \Gamma^\alpha_{\lambda\gamma} v^\lambda = D_\Gamma v\, a_\gamma \cdot a^\alpha + D^2 b\, a^\alpha \cdot a_\gamma v_N = [{}^*D_\Gamma v + D^2 b\, v_N] a^\alpha \cdot a_\gamma \\ \Rightarrow &\quad {}^*D_\Gamma v\, a_\alpha \cdot a_\gamma = v_\alpha|_\gamma - b_{\alpha\gamma} v_N \quad \text{and} \quad {}^*D_\Gamma v\, a^\alpha \cdot a_\gamma = v^\alpha|_\gamma - b^\alpha_\gamma v_N \end{aligned} \tag{57}$$

For tangential vector fields ($v \cdot n = 0$), the covariant derivatives coincide with the bilinear form generated by $^*D_\Gamma v$. For non tangent field we have an additional term which arises from the fact that in the definition of $v_\alpha|_\gamma$ and $v^\alpha|_\gamma$ the summation over λ ranges from 1 to $N-1$ missing the normal component v_N.

5.4 A few useful formulas

In the theory of shells some identities will often occur. We summarize some of them below.

Theorem 5 *For all u and v in $H^1(\Gamma)^N$*

$$\begin{aligned} \varepsilon_\Gamma(v)\,a_\alpha \cdot a_\beta &= \tfrac{1}{2}(v_\alpha|_\beta + v_\beta|_\alpha) - b_{\alpha\beta}\,v_N, \\ \varepsilon_\Gamma(v)\,a_\alpha \cdot a^\beta &= \tfrac{1}{2}(v^\alpha|_\beta + v^\beta|_\alpha) - b_\alpha^\beta\,v_N, \\ (D^2b\,D_\Gamma v\,a_\alpha)\cdot a_\beta &= b_\beta^\gamma\,[v_\gamma|_\alpha - b_{\alpha\gamma}\,v_N] = b_\beta^\gamma\,v_\gamma|_\alpha - c_{\alpha\beta}\,v_N, \\ (D_\Gamma v\,D^2b\,a_\alpha)\cdot a_\beta &= b_\alpha^\gamma\,[v_\beta|_\gamma - b_{\beta\gamma}\,v_N] = b_\alpha^\gamma\,v_\beta|_\gamma - c_{\alpha\beta}\,v_N, \end{aligned} \tag{58}$$

where $c_\alpha^\beta = c_{\alpha\gamma}\,a_\beta^\gamma$, $c^{\alpha\beta} = c_{\alpha\gamma}\,a^{\gamma\beta}$. Moreover

$$\begin{aligned} a_\beta \cdot \varepsilon_\Gamma(u)\,a^\beta &= \operatorname{tr}\varepsilon_\Gamma(u) = \operatorname{div}_\Gamma u, \\ a^\alpha \cdot \varepsilon_\Gamma(u)a_\beta\,\varepsilon_\Gamma(v)\,a^\beta \cdot a_\alpha &= \varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\,n)\cdot\cdot\,\varepsilon^0(v, -2\,\varepsilon_\Gamma(v)\,n) \\ &= \varepsilon_\Gamma(u)\cdot\cdot\,\varepsilon_\Gamma(v) - 2\,\varepsilon_\Gamma(u)\,n \cdot \varepsilon_\Gamma(v)\,n \end{aligned} \tag{59}$$

where $\varepsilon^0(u,\ell)$ is given by (29)

6 SOME CLASSICAL LINEAR MODELS

In this section we use the material from §5 to rewrite the linear models of Naghdi and Koiter and the asymptotic model with tangential operators.

6.1 Naghdi's and Koiter's linear models

We use the variational forms and associated definitions from the recent book of Bernadou (1994) (Chapter I, §3). The displacement vector is

$$U = u + \xi^3 \beta_\alpha a^\alpha \tag{60}$$

where $-e < \xi^3 < e$, $2e$ is the thickness of the shell, u and β are maps from Γ to $\mathbb{R}^3$ and

$$\beta = \beta_\alpha\,a^\alpha, \quad \beta \cdot n = 0, \quad \beta_\alpha = \beta \cdot a_\alpha. \tag{61}$$

We use the same notation for a vector $v : \Gamma \to \mathbb{R}^3$ and its 2-dimensional representation $v \circ \Phi$ in the $1-2$ coordinates. We quote the main definitions and give their tangential equivalent

$$
\begin{aligned}
&\varphi_\alpha(u) \stackrel{\text{def}}{=} u_{3,\alpha} + b_\alpha^\lambda u_\lambda = a_\alpha \cdot D^2 b\, u = -D_\Gamma u\, a_\alpha \cdot n,\\
&\gamma_{\alpha\beta}(u) \stackrel{\text{def}}{=} \tfrac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha\right) - b_{\alpha\beta} u_3 \gamma_{\alpha\beta}(u) = \varepsilon_\Gamma(u)\, a_\alpha \cdot a_\beta,\\
&d_{\lambda\alpha}(u) \stackrel{\text{def}}{=} u_\lambda|_\alpha - b_{\lambda\alpha} u_3 = D_\Gamma u\, a_\alpha \cdot a_\lambda,\\
&\chi_{\alpha\mu}(u,\beta) \stackrel{\text{def}}{=} \tfrac{1}{2}\left[\beta_\alpha|_\mu + \beta_\mu|_\alpha - b_\alpha^\lambda d_{\lambda\mu}(u) - b_\mu^\lambda d_{\lambda\alpha}(u)\right]\\
&\qquad = a_\alpha \cdot \left[\varepsilon_\Gamma(\beta) - \tfrac{1}{2}\left(D^2 b\, D_\Gamma u + {}^*D_\Gamma u\, D^2 b\right)\right] a_\mu,\\
&\varepsilon_{\alpha\beta}(u,\beta) \stackrel{\text{def}}{=} \gamma_{\alpha\beta}(u) + \xi^3 \chi_{\alpha\beta}(u,\beta)\\
&\qquad = a_\alpha \cdot \left\{\varepsilon_\Gamma(u) + \xi^3\left[\varepsilon_\Gamma(\beta) - \tfrac{1}{2}\left(D^2 b\, D_\Gamma u + {}^*D_\Gamma u\, D^2 b\right)\right]\right\} a_\beta.
\end{aligned}
\tag{62}
$$

The variational problem for Naghdi's model consists in finding (u,β) such that

$$
\forall (v,\delta), \quad a^S((u,\beta),(v,\delta)) + b^S((u,\beta),(v,\delta)) = f^S(v,\delta), \tag{63}
$$

where the bilinear forms a^S and b^S and the linear form f^S are given by

$$
\begin{aligned}
&a^S((u,\beta),(v,\delta)) \stackrel{\text{def}}{=} \int_\Gamma eE^{\alpha\beta\lambda\mu}\left\{\gamma_{\alpha\beta}(u)\,\gamma_{\lambda\mu}(v) + \frac{e^2}{12}\chi_{\alpha\beta}(u)\,\chi_{\lambda\mu}(v)\right\} d\Gamma\\
&b^S((u,\beta),(v,\delta)) \stackrel{\text{def}}{=} \int_\Gamma \frac{eEa^{\alpha\beta}}{2(1+\nu)}\left(\varphi_\alpha(u) + \beta_\alpha\right)\left(\varphi_\beta(v) + \delta_\beta\right) d\Gamma\\
&f^S(v,\delta) \stackrel{\text{def}}{=} \int_\Gamma p \cdot v\, d\Gamma + \int_{\partial\Gamma} N \cdot v - M \cdot \delta\, d\gamma
\end{aligned}
\tag{64}
$$

and

$$
E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu}\right]. \tag{65}
$$

By using the following identifications

$$
h = e, \quad \ell = -\beta, \quad u = u, \quad \bar{\ell} = -\delta, \quad v = v, \tag{66}
$$

we get an intrinsic reformulation of the variational problem

$$
\begin{aligned}
&\frac{h\,E}{1+\nu}\int_\Gamma \varepsilon^0(u,\ell - \nabla_\Gamma(u\cdot n)) \cdot\cdot\, \varepsilon^0(v,\bar{\ell} - \nabla_\Gamma(v\cdot n)) + \frac{h^2}{12}\varepsilon_t^1(u,\ell) \cdot\cdot\, \varepsilon_t^1(v,\bar{\ell})\\
&\qquad + \frac{1}{1-\nu}\left\{\operatorname{div}_\Gamma u \operatorname{div}_\Gamma v + \frac{h^2}{12}\operatorname{div}_\Gamma \ell \operatorname{div}_\Gamma \bar{\ell}\right\} d\Gamma\\
&= \int_\Gamma p \cdot v\, d\Gamma + \int_{\partial\Gamma} N \cdot v - M \cdot \bar{\ell}\, d\gamma,
\end{aligned}
\tag{67}
$$

where

$$
\begin{aligned}
&\varepsilon^0(u,\ell) \stackrel{\text{def}}{=} \varepsilon_\Gamma(u) + \frac{1}{2}\left[\ell\, {}^*n + n\, {}^*\ell\right],\\
&\varepsilon_t^1(u,\ell) \stackrel{\text{def}}{=} \varepsilon_\Gamma(\ell) + \frac{1}{2}\left[D^2 b\, D_\Gamma u + {}^*D_\Gamma u\, D^2 b\right] - \frac{1}{2}\left[D^2 b\,\ell\, {}^*n + n\, {}^*(D^2 b\,\ell)\right].
\end{aligned}
$$

This model is different from our intrinsic model since it uses the assumption $\sigma \circ T_z\, n \cdot n = 0$. Koiter's linear model is the same model as Naghdi's model with

$$\ell + 2\,\varepsilon_\Gamma(u)\,n = 0 \quad \Rightarrow \quad 0 = \ell + \nabla_\Gamma(u \cdot n) - D^2 b\, u. \tag{68}$$

6.2 Asymptotic membrane equation model

We use the definitions and notation from Ciarlet and Sanchez-Palencia (1993). The model is characterized by the following bilinear and linear forms

$$\begin{aligned}
B(\zeta,\eta) &\stackrel{\text{def}}{=} \int_\Gamma A^{\alpha\beta\rho\delta}\,\gamma_{\rho\sigma}(\zeta)\,\gamma_{\alpha\beta}(\eta)\,d\Gamma \\
A^{\alpha\beta\rho\delta} &\stackrel{\text{def}}{=} \frac{4\lambda\mu}{\lambda+2\,\mu}\,a^{\alpha\beta}\,a^{\rho\delta} + 2\,\mu\,[a^{\alpha\rho}\,a^{\beta\sigma} + a^{\alpha\sigma}\,a^{\beta\rho}] \\
\gamma_{\alpha\beta}(\eta) &\stackrel{\text{def}}{=} \frac{1}{2}\,(\eta_{\alpha,\beta} + \eta_{\beta,\alpha}) - \Gamma^\rho_{\alpha\beta}\,\eta_\beta - b_{\alpha\beta}\,\eta_3.
\end{aligned} \tag{69}$$

Clearly $\gamma_{\alpha\beta}(\eta)$ is the same as the one in Naghdi's model and

$$\begin{aligned}
B(\zeta,\eta) &= \int_\Gamma 4\,\mu\,\varepsilon^0(\zeta, -2\,\varepsilon_\Gamma(\zeta)\,n)\cdot\cdot\,\varepsilon^0(\eta, -2\,\varepsilon_\Gamma(\eta)\,n) \\
&\quad + \frac{4\lambda\mu}{\lambda+2\,\mu}\,\mathrm{tr}\,\varepsilon^0(\zeta, -2\,\varepsilon_\Gamma(\zeta)\,n) + \mathrm{tr}\,\varepsilon^0(\eta, -2\,\varepsilon_\Gamma(\eta)\,n)\,d\Gamma \\
B(\zeta,\eta) &= \int_\Gamma 4\,\mu\,\varepsilon^0(\zeta, -2\,\varepsilon_\Gamma(\zeta)\,n)\cdot\cdot\,\varepsilon^0(\eta, -2\,\varepsilon_\Gamma(\eta)\,n) \\
&\quad + \frac{4\lambda\mu}{\lambda+2\,\mu}\,\mathrm{div}_\Gamma\zeta\,\mathrm{div}_\Gamma\eta\,d\Gamma.
\end{aligned} \tag{70}$$

By making the identification $u = \eta, \quad v = \zeta$, we finally get

$$\begin{aligned}
&\int_\Gamma 4\,\mu\,\varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\,n)\cdot\cdot\,\varepsilon^0(v, -2\,\varepsilon_\Gamma(v)\,n) \\
&+ \frac{4\,\mu\,\lambda}{\lambda+2\,\mu}\,\mathrm{tr}\,\varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\,n)\,\mathrm{tr}\,\varepsilon^0(v, -2\,\varepsilon_\Gamma(v)\,n)\,d\Gamma = \int_\Gamma f\cdot v\,d\Gamma.
\end{aligned} \tag{71}$$

By using the tangential operator $\varepsilon^P_\Gamma(u) = \varepsilon^0(u, -2\,\varepsilon_\Gamma(u)\,n)$, we obtain

$$\int_\Gamma 4\mu\,\varepsilon^P_\Gamma(u)\cdot\cdot\,\varepsilon^P_\Gamma(v) + \frac{4\,\mu\,\lambda}{\lambda+2\,\mu}\,\mathrm{tr}\,\varepsilon^P_\Gamma(u)\,\mathrm{tr}\,\varepsilon^P_\Gamma(v)\,d\Gamma = \int_\Gamma f\cdot v\,d\Gamma. \tag{72}$$

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