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Regularity results for multiphase Stefan-like equations

Giulia Sargenti
Dipartimento di Matematica, Università di Roma La Sapienza
Piazzale Aldo Moro 2, 00185 Roma, Italy. Phone: 06 49913.
Fax: 44701007. E–mail: sargenti@mat.uniroma1.it
and
Vincenzo Vespri
Dipartimento di Matematica Pura e Applicata, Università dell'Aquila
Via Vetoio, 67010 Coppito (AQ), Italy. Phone: 0862 43311.
Fax: 0862 433180. E–mail: vespri@vxscaq.aquila.infn.it

Abstract

In this note the local continuity of any bounded local weak solution of degenerate multiphase Stefan problem is proved. Moreover the modulus of continuity can be determined a priori only in terms of the data.

Keywords

Continuity, multiphase Stefan problem, p-Laplacian

1 INTRODUCTION

In this note we study the partial regularity of bounded weak solutions of nonlinear parabolic equations of the type

$$(\beta(u))_t = \text{ Div } (|\nabla u|^{p-2}\nabla u) \quad \text{in } \; D'(\Omega_T) \tag{1}$$

$$u \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega)) \text{ and } \; p \geq 2 \tag{2}$$

Here Ω is a domain in R^N, $T > 0$ and Ω_T is the cylindrical domain $\Omega_T = \Omega \times (0,T)$.
Assume $\beta(\cdot)$ be a maximal monotone graph in $R \times R$ satisfying:

$$\beta(s_1) - \beta(s_2) \geq c_0(s_1 - s_2) \quad \forall s_1, s_2 \in R \tag{3}$$

for some given constant $c_0 > 0$,

$$\sup_{|u| \leq M} |\beta(u)| < \infty \tag{4}$$

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for every $M > 0$. Lastly we assume that u is the limit of a sequence of local smooth solutions of approximating problems.

Let K be a compact set contained in Ω. In what follows we say that $c = c(data)$ if c is a constant that can be determined a priori only in terms of N, M, p and the distance between $K \times (\varepsilon, T)$ and the parabolic boundary of Ω_T. Now we can state our main result.

Main Theorem 1 *Let (3)-(4) hold and let u be a locally bounded weak solution of (1), (2). Then there exists a continuous nonnegative increasing function ω_{data}, $\omega(0) = 0$ such that*

$|u(P_1) - u(P_2)| \leq \omega(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}})$
for every $P_i = (x_i, t_i) \in K \times (\varepsilon, T)$, $i = 1, 2$.

In the last years several authors studied equations of the type (1) not only for their mathematical interest, but also for their application in many physical phenomena like the transition of phase or the Buckeley Leverett model of two immiscible fluids in a porous media (see Alt and Di Benedetto (1985), Caffarelli and Evans (1983), Chavent and Jaffre (1986), Di Benedetto (1982), Di Benedetto (1993)). In Caffareli and Evans (1983), Di Benedetto (1982) and Ziemer (1982) the case of a β with only a singular point is studied . In Di Benedetto and Vespri (to appear) was faced the case of a general β but the continuity theorem was proved only for local solution of equation of the type (1) with $p = 2$. In this paper we will apply the technique developed in Di Benedetto and Vespri (to appear) ; therefore we will point out only what is really new from this work.

2 NOTATION AND LOCAL ESTIMATES

We denote with B_R a ball of radius R and center in the origin of R^N, with K_ρ a cube centered in the origin of wedge 2ρ, $\rho > 0$ and with $Q(\rho, \rho^p) = K_\rho \times (-\rho^p, 0)$ the cylinder of vertex at the origin, height ρ^p and cross section K_ρ, while for a point $(x_o, t_o) \in R^{N+1}$ we let $(x_o, t_o) + Q(\rho, \rho^p)$ be the cylinder of vertex at (x_o, t_o) and congruent to $Q(\rho, \rho^p)$. In what follows we always assume that u is a locally bounded weak solution of (1) and that ρ is so small that $(x_o, t_o) + Q(\rho, \theta\rho^p)$ is contained in Ω_T, where $\theta > 0$ will be chosen later. For $k \in R$ we define the truncations

$$(u - k)_+ = \max(u - k, 0) \quad (u - k)_- = \max(-u + k, 0),$$

the numbers

$$H_k^\pm = ||(u - k)_\pm||_{\infty, \Omega_T}$$

and the function

$$\Psi(H_k^\pm, (u - k)_\pm, a) = \ln^+\left(\frac{H_k^\pm}{H_k^\pm - (u - k)_\pm + a}\right) \quad a \in (0, H_k^\pm)$$

simply denoted by Ψ. Further we assume ξ is a smooth cut off function defined in $(x_o, t_o) + Q(\rho, \rho^p)$ such that $\xi \in [0, 1]$, $|\nabla\xi| < \infty$, $0 \leq \xi_t < \infty$. We will be working with the smooth approximations to derive a modulus of continuity for smooth solutions of (1). The following two propositions can be proved as in Di Benedetto and Vespri (to appear).

Proposition 2.1 *There exists a constant $c = c(data)$ such that*

$$\begin{cases} sup_{t_o-\theta\rho^p \leq t < t_o} \int_{[x_o+K_\rho]}(u-k)^2_\pm \xi^p(x,t)\,dx \\ +c\int\int_{[(x_o,t_o)+Q(\rho,\theta\rho^p)]} |\nabla(u-k)_\pm \xi|^p\,dx\,dt \\ \leq \int\int_{[(x_o,t_o)+Q(\rho,\theta\rho^p)]}(u-k)_\pm \xi^{p-1}\xi_t\,dx\,dt \\ +\int\int_{[(x_o,t_o)+Q(\rho,\theta\rho^p)]}(u-k)^p_\pm |\nabla\xi|^p\,dx\,dt \\ +\int_{[x_o+K_\rho]}(u-k)_\pm \xi^p(x,t_o-\theta\rho^p)\,dx \end{cases} \tag{5}$$

Proposition 2.2 *There exists a constant c=c(data) such that*

$$\begin{cases} sup_{t_o-\theta\rho^p \leq t < t_o} \int_{[x_o+K_\rho]} \Psi^2(x,t)\xi(x)^p\,dx \leq c\int_{[x_o+K_\rho]} \Psi(x,t_o-\theta\rho^p)\xi^p(x)\,dx \\ +c\int\int_{[(x_o,t_o)+Q(\rho,\theta\rho^p)]} |\Psi'|^{p-2}\Psi|\nabla\xi|^p\,dx\,dt \end{cases} \tag{6}$$

3 SOME BASIC RESULTS

After a translation we may take $(x_o, t_o) = (0,0)$. Set

$\mu^+ = \sup u \ \mu^- = \inf u$ in $Q(\rho,\theta\rho^p)$, $\omega = \mu^+ - \mu^-$

For $\xi^\pm \in (0,1)$ define

$$A^+_{\xi^+,\rho}(t) = \{x \in K_\rho \,:\, u(x,t) > \mu^+ - \xi^+\omega\}$$
$$A^-_{\xi^-,\rho}(t) = \{x \in K_\rho \,:\, u(x,t) < \mu^- + \xi^-\omega\}$$

and set $A^\pm_{\xi^\pm,\rho} = \int^0_{-\theta\rho^p} |A^\pm_{\xi^\pm,\rho}(t)|\,dt$.

Proposition 3.1 *There exists a number $\nu^+ \in (0,1)$ depending upon ω, ρ, data such that if*

$$|A^+_{\xi^+,\rho}| < \nu^+|Q(\rho,\theta\rho^p)| \tag{7}$$

then

$$u(x,t) < \mu^+ - \frac{2}{3}\xi^+\omega \ \forall(x,t) \in Q\Big(\frac{\rho}{2},\theta\Big(\frac{\rho}{2}\Big)^p\Big). \tag{8}$$

If also

$$u(x,-\theta\rho^p) < \mu^+ - \xi^+\omega \ \forall x \in K_\rho \tag{9}$$

there exists a number $\nu^+ \in (0,1)$ depending upon θ, data, independent of ω, ξ^+, such that (8) implies

$$u(x,t) < \mu^+ - \frac{2}{3}\xi^+\omega \ \forall(x,t) \in Q\Big(\frac{\rho}{2},\theta\rho^p\Big). \tag{10}$$

Remark 3.1 *By the proof we get the number* ν^+ *given by*

$$\nu^+ = c(data)\Big(\theta\Big(1+\frac{(\xi^+\omega)^{1-p}}{\theta}\Big)^{\frac{N+p}{p}}\Big)^{-1} \tag{11}$$

Sketch of the proof

For $n = 0,1,2,\dots$ consider the sequence of radii

$$\rho_n = \frac{\rho}{2}+\frac{\rho}{2^{n+1}} \quad \tilde{\rho}_n = \frac{\rho}{2}+\frac{3}{2}\frac{\rho}{2^{n+2}}$$

and the sequence of numbers

$$K_n = \mu^+ - \frac{2}{3}\xi^+\omega - \frac{1}{3}\frac{\xi^+}{2^n}\omega$$

We rewrite (5) over the pair of cylinder $\tilde{Q}_n = K_{\tilde{\rho}_n} \times (-\theta\tilde{\rho}_n^p, 0)$ and $Q_n = K_{\rho_n} \times (-\theta\rho_n^p, 0)$ and using the sequence of cutoff functions ξ_n defined over $Q_n = K_{\rho_n}$ such that $\xi_n = 1$ over $\tilde{Q}_n$, $|\nabla\xi_n| \le \frac{2^{n+1}}{\rho}$, $\partial_t\xi_n \in [0, 2^{p(n+1)}/\theta\rho^p]$. After some calculations (see Di Benedetto and Vespri (to appear) we get

$$Y_{n+1} \le c(data)\Big(1+\frac{(\xi^+\omega)^{1-p}}{\theta}\Big)\theta^{\frac{N+p}{p}} Y_n^{1+\frac{p}{N+p}} \tag{12}$$

where $Y_n = \frac{|A_{\xi_n,\rho_n}|}{|Q_n|}$. To prove (9) we apply Lemma 4.1 of Di Benedetto (1993). For (11) we argue in a similar way using a sequence ξ_n independent of t.

We can state an analogous result for $A_{\xi^-,\rho^-(t)}$, with $\mu^+ - \frac{2}{3}\xi^+\omega$ replaced by $\mu^- + \frac{2}{3}\xi^-\omega$. As the proof is similar to the one in Di Benedetto and Vespri (to appear) we omit it.

Proposition 3.2 *Suppose that for some* $\xi_o^+ \in (0,1)$ *the number* θ *satisfies*

$$\theta \ge (\xi_o^+\omega)^{p-2} \tag{13}$$

and that

$$u(x, -\theta\rho^p) \le \mu^+ - \xi_o^+ \ \forall x \in K_\rho \tag{14}$$

then for any $\nu^+ \in (0,1)$ *there exists a number* $\xi^+ \in (0, \xi_o^+/4)$ *depending upon* θ, ξ_o^+, $c(data)$ *such that*

$$|A_{\xi^+,\frac{\rho}{2}}(t)| \le \nu^+ |K_{\frac{\rho}{2}}| \ \forall t \in (-\theta\rho^p, 0) \tag{15}$$

An analogous proof enables us to get the logarithmic estimate near μ^-.

If (7) holds, then arguing as in Di Benedetto and Vespri (to appear, the first alternative) we get:

$$\operatorname{osc} u \le \eta_o(\omega)\omega \ \forall (x,t) \in Q\Big(\frac{\rho}{8}, \theta\Big(\frac{\rho}{8}\Big)^p\Big) \tag{16}$$

where $\eta_o(\omega) \in (0,1)$.

The unfavourable case, is when both conditions (7) and the analogous one for $A^-_{\xi^-,\rho}$ are violated in the fixed cylinder $Q(\rho, \theta\rho^p)$, thus in particular in the coaxial boxes

$$(0,\tau) + Q(r, \theta r^p)$$

where $\tau \in [-\theta(\rho^p - r^p), 0]$, $r \in (\delta\rho, \rho)$ and $\delta \in (0,1)$ is to be chosen.

Proposition 3.3 *There exists a constant $c = c(data, \omega)$ such that*

$$\theta\omega^p\rho^p \leq c\int_{-\theta\rho^p}^0 \int_{\{\delta\rho<||x||<\rho\}} |\nabla u|^p \, dx \, dt \tag{17}$$

Proof

Arguing as in Di Benedetto and Vespri (to appear), the second alternative] we get the existence of two cylinders Q_i both contained in $\{\delta\rho < ||x|| < \rho\} \times [-\theta\rho^p, 0)$ such that u is near μ^+ in Q_1, u is near μ^- in Q_2 (see Di Benedetto and Vespri (to appear), Proposition 8.1]). Let $\Gamma_{1,2}$ be a path piecewise parallel to the coordinates axes joining P_1 to P_2, P_i belonging to the cross section of Q_i, $i = 1, 2$. Then we obtain

$$\omega < \int_{\Gamma_{1,2}} |\nabla u| \, d\Gamma \tag{18}$$

By integration we get (17).

4 PROOF OF THE MAIN THEOREM

Consider first the case $N = p$. Rewrite (17) for the cylinders

$$Q_n^j = [(0, t_n^j) + Q(\delta^n\rho, \theta\delta^n\rho)] \subset Q(\rho, \theta\rho^p), \tag{19}$$

where $t_n^j = -j\theta\delta^{np}\rho^p$, $j = 0, 1, \ldots, \delta^{-np} - 1$.

We suppose δ^{-1} be an integer number. After adding over n, $n = 0, 1, .., n_o - 1$, with n_o to be chosen, we obtain

$$n_o\theta\rho^p\omega^p \leq \int\int_{Q(\rho,\theta\rho^p)} |\nabla u|^p \, dx \, dt \tag{20}$$

Inequality (5) written for the value $k = \mu^+ - \frac{\omega}{2}$ and using a cutoff function $\xi = 1$ on $Q(\rho/2, \theta\rho^p/2)$, $|\nabla\xi| \leq 2^p/\rho$, $\xi_t \in [0, 2^{p+1}/\rho^p]$ gives

$$\int\int_{Q(\rho,\theta\rho^p)} |\nabla u|^p \, dx \, dt \leq c\omega\theta\rho^N(1 + \omega^{p-1}) \tag{21}$$

Equations (20) and (21) imply

$$n_o \leq c(data, \omega) \tag{22}$$

with $c(data, \omega)$. We choose an n_o so large that

$$n_o > c(data, \omega) \tag{23}$$

This implies a contradiction. Therefore (7) or the analogous one for $A^-_{\xi^-,\rho}$ must hold for the cylinders Q^j_n. If we start from (7), we fix a value $\xi^\pm = \frac{1}{12}$, we use proposition 3.1 to arrive at

$$u(x,t) < \mu^+ - \frac{\omega}{18}\ \forall(x,t) \in [(0, t^j_n) + Q(\delta^n\rho/2, \theta(\delta^n\rho/2)^p] \tag{24}$$

Setting $\delta^n\rho = 8r$, $\sigma = \frac{8^p}{4}(j + 1/2^p)\theta$

$$u(x, -4\sigma r^p) < \mu^+ - \frac{\omega}{18} \forall x \in K_r.$$

We choose ν^+ for the largest value of σ i.e. $\nu^+ = c(data)/\theta\delta^{n_o p}$, n_o as in (23). From Proposition 3.1 we get

$$u(x,t) < \mu^+ - \frac{2}{3}\xi^+\omega\ \forall(x,t) \in Q(\delta^{n_o}\rho/8, \theta(\delta^{n_o}\rho/8)^p)$$

We can repeat the method of the first alternative: there exists $\eta_1(\omega) \in (0,1)$ such that

$$\text{osc}\, u \le \eta_1(\omega)\omega \text{ in } Q(\delta^{n_o}\rho/8, \theta(\delta^{n_o}\rho/8)^p) \tag{25}$$

Therefore arguing as in Di Benedetto and Vespri (to appear), Corollary 12.1, we have that

$$\text{osc}\, u \le \omega_n \text{ in } Q(\rho_n, \theta\rho^p),\ \omega_n \to 0, \text{ as } n \to \infty.$$

The case $N > p$ is more complicated. We follow sections 13-29 of Di Benedetto and Vespri (to appear). We partition the original set into disjointed boxes of the type $(0, t_i) + Q(\rho, \rho^p)$, with $t_i = 0, -\rho^p, .., -(\theta - 1)\rho^p$ and divide each of these boxes in the disjoint cylinders

$$(x_l, t_h) + Q(\rho/m, \rho^p/m),\ l = 1, \dots, m^N,\ h = 1, \dots, m^p.$$

Either the first or the second alternative holds for these sets. Since they are not centered in the origin, we prove that an estimate similar to (24) holds by means of a comparison function. Propositions 24.2, 24.3 of Di Benedetto and Vespri (to appear) show that a number $\theta > 1$ $\lim_{\omega\downarrow 0}\theta(\omega) = \infty$, which can be determined a priori in terms of the data and ω, implies that the first alternative holds.

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