On the use of space invariant imbedding to solve optimal control problems for second order elliptic equations

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Abstract
This paper deals with the application of invariant imbedding to the solution of a control problem of a system governed by a second order elliptic equation. The basic idea is to take advantage of the geometry of the domain (for instance a cylinder or a rectangle) to consider that one space variable plays the role of time for dynamical systems. Then it is possible to decouple the system of optimality in order to get the explicit dependence of the optimal control with respect to the desired state.

Keywords
Optimal control of distributed parameter systems, invariant imbedding.

1 INTRODUCTION

The invariant imbedding technique has been popularized by R. Bellman (Bellman, Dreyfus, 1962) and his coworkers. It allows to decouple systems of coupled first order forward and backward differential equations. It has been widely used to derive the Riccati equation in the context of linear quadratic optimal control problems. It is less known that it can be used to solve second order elliptic boundary value problems with an imbedding variable related to space.
We present here an application of invariant imbedding to solve an optimal control problem governed by an elliptic equation. The method is most easily applied in the case of a cylindrical domain but it can be generalized, for example, to domains derived from cylinders by conformal mapping.
For the sake of simplicity, the method is presented for a model problem of a Laplace equation in a square domain Ω of $\mathbb{R}^2$ for special boundary conditions.

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2 SPACE FACTORIZATION OF THE STATE EQUATION

In order to present the method we start with the space decoupling applied to the state equation, then the same method will be applied to an associated control problem. Let us consider the model problem in $\mathbb{R}^2$:

$$\begin{cases} -\Delta y = f & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad S, \\ y = 0 & \text{on} \quad \Gamma_0, \\ \frac{\partial y}{\partial x_1} = u & \text{on} \quad \Gamma_1, \end{cases} \tag{1}$$

with $\Omega =]0,1[\times]0,a[$, $S = \{\partial\Omega \cap \{x_2 = 0\}\} \cup \{\partial\Omega \cap \{x_2 = a\}\}$, $\Gamma_0 = \partial\Omega \cap \{x_1 = 0\}$ and $\Gamma_1 = \partial\Omega \cap \{x_1 = 1\}$. The function u is given in $H^{-1/2}(\Gamma_1)$ and f is given in $L^2(\Omega)$.

If we pay a special attention to the variable x_1, in a similar way to evolution equations we may consider y as : $y \in L^2(0,1;H_0^1(0,a)) \cap H^1(0,1;L^2(0,a))$. It is solution of the coupled system

$$\begin{cases} \dfrac{\partial y}{\partial x_1} = z, \\ y = 0 \text{ on } \Gamma_0, \\ y = 0 \text{ on } S, \end{cases} \qquad \begin{cases} -\dfrac{\partial z}{\partial x_1} - \dfrac{\partial^2 y}{\partial x_2^2} = f, \\ z = u \text{ on } \Gamma_1. \end{cases} \tag{2}$$

We decouple this system by an invariant imbedding with respect to x_1. Let us consider the family of problems, indexed by s, $0 \leq s \leq 1$, defined on $\Omega_s =]0,s[\times]0,a[$:

$$\begin{cases} -\dfrac{\partial z^s}{\partial x_1} - \dfrac{\partial^2 y^s}{\partial x_2^2} = f, \\ z^s = h \text{ on } \Gamma_s, \end{cases} \tag{3}$$

where $\Gamma_s = \Omega \cap \{x_1 = s\}$ and h is a function in $H^{-1/2}(0,a)$, the left equation of (2) being unchanged but restricted to Ω_s.

One can show Ramos that: $h \mapsto y^s|_{\Gamma_s}$ defines an affine mapping :

$y^s|_{\Gamma_s} = P(s)h + r(s)$,

with $r(s) \in H^{1/2}(0,a)$, $P(s) \in \mathcal{L}(H^{-1/2}(0,a);H^{1/2}(0,a))$. As a consequence of the previous notations one has $y = y^1$ and if we choose h to be $\dfrac{\partial y^1}{\partial x_1}|_{\Gamma_s}$ and dropping the index s one gets :

$$y(s) = P(s)z(s) + r(s), \; \forall s \in [0,1]. \tag{4}$$

Let D^2 be the operator

$$D^2 : y \in H_0^1(0,a)) \mapsto \frac{\partial^2 y^s}{\partial x_2^2} \in H^{-1}(0,a).$$

By differentiating (4) with respect to $x_1 = s$, one obtains the equations satisfied by P and r, and the result is summarized in :

Result 1 *The solution of the state equation (1) given by the solution of the Riccati equations*

$$\begin{cases} \frac{dP}{dx_1} = PD^2P + I \quad , \quad P(0) = 0, \\ \frac{dr}{dx_1} = PD^2r + Pf \quad , \quad r(0) = 0. \end{cases} \tag{5}$$

Then z is solution of

$$-\frac{dz}{dx_1} = D^2 Pz + D^2 r + f, \ \ z(1) = u. \tag{6}$$

and y is obtained by (4). □

Remark 21 So the solution of the second order boundary value problem (1) can be obtained by solving a system of uncoupled first order initial value problems : the system (5) is to be solved forwards from 0 to 1, the first equation of (5) being an operator Riccati differential equation. Then equation (6) is to be solved backwards from 1 to 0. □

It can be shown that the operator P defined by (4) is self-adjoint and positive definite. The same problem can be solved in a symmetric way using invariant imbedding in the domain $\tilde{\Omega}_s :=]s, 1[\times]0, a[$. This leads to a "dual" method of decoupling :

$$z(x_1) = Q(x_1)y(x_1) + t(x_1),$$

with

$$\left\{ \begin{array}{lcll} -\frac{dQ}{dx_1} & = & Q^2 + D^2 & , \ Q(1) = 0, \\ -\frac{dt}{dx_1} & = & Qt + f & , \ \ t(1) = 0, \end{array} \right. \tag{7}$$

the solution y being solution of

$$\frac{dy}{dx_1} = Qy + t, \ \ y(0) = 0. \tag{8}$$

Remark 22 When considering the finite difference discretization of this problem, the same method applied to the resulting linear system leads to the Gauss block factorization of the block tridiagonal matrix representing the Laplace operator on the rectangle.

3 DECOUPLING OF THE CONTROL PROBLEM (I)

Let us consider the optimal control problem associated to the elliptic equation :

$$\left\{ \begin{array}{lll} -\Delta y = f & \text{in} & \Omega, \\ y = 0 & \text{on} & S, \\ y = 0 & \text{on} & \Gamma_0, \\ \frac{\partial y}{\partial x_1} = u & \text{on} & \Gamma_1, \end{array} \right. \tag{9}$$

with the criterion to be minimized :

$$J(u) = \frac{1}{2}\int_0^a |y(1, x_2) - y_d(x_2)|^2 \, dx_2 + \frac{\nu}{2}\int_0^a u^2(x_2) \, dx_2 \ \ (\nu > 0). \tag{10}$$

If we introduce the adjoint state p which is solution of the adjoint equations

$$\begin{cases} -\Delta p = 0 & \text{in} \quad \Omega, \\ p = 0 & \text{on} \quad S, \\ p = 0 & \text{on} \quad \Gamma_0, \\ \frac{\partial p}{\partial x_1} = y(1,.) - y_d & \text{on} \quad \Gamma_1, \end{cases} \tag{11}$$

the optimality condition takes the form (Lions, 1968):

$$p(1,.) + \nu u = 0 \ \text{ on } \ \Gamma_1. \tag{12}$$

The basic idea is to decouple the previous optimality system by an invariant imbedding similar to the one presented in the previous section. The family of optimality systems depends on a parameter s , $0 \leq s \leq 1$ and is defined on $\tilde{\Omega}_s :=]s,1[\times]0,a[$ *. The optimality system takes the form :

$$\begin{cases} -\Delta y^s = f & \text{in} \quad \Omega_s, \\ y^s = 0 & \text{on} \quad S, \\ y^s = h & \text{on} \quad \Gamma_s, \\ \frac{\partial y^s}{\partial x_1} = u & \text{on} \quad \Gamma_1, \end{cases} \qquad \begin{cases} -\Delta p^s = 0 & \text{in} \quad \Omega_s, \\ p^s = 0 & \text{on} \quad S, \\ p^s = g & \text{on} \quad \Gamma_s, \\ \frac{\partial p^s}{\partial x_1} = y^s(1,.) - y_d & \text{on} \quad \Gamma_1, \end{cases} \tag{13}$$

where the function g and h are defined in such a way that y^s and p^s are independent of s. The optimality condition (12) is unchanged. This set of equations is the system of optimality corresponding to the new criterion :

$$J_s(u) = \frac{1}{2}\int_0^a |y^s(1,x_2) - y_d(x_2)|^2\, dx_2 + \frac{\nu}{2}\int_0^a u^2(x_2)\, dx_2 + \int_0^a g(x_2)\frac{\partial y^s}{\partial x_1}(s,x_2)\, dx_2.$$

As previously, considering that the variable x_1 is similar to a "time" variable, equations (13) can be written as first order system with respect to x_1 :

$$\begin{cases} \frac{\partial y^s}{\partial x_1} = z^s & \text{on} \quad]s,1[, \\ y^s(s) = h, \end{cases} \qquad \begin{cases} -\frac{\partial z^s}{\partial x_1} - D^2 y^s = f & \text{on} \quad]s,1[, \\ z^s(1) = u = -\frac{1}{\nu}p^s(1), \end{cases} \tag{14}$$

the adjoint system being :

$$\begin{cases} \frac{\partial p^s}{\partial x_1} = q^s & \text{on} \quad]s,1[, \\ p^s(s) = g, \end{cases} \qquad \begin{cases} -\frac{\partial q^s}{\partial x_1} - D^2 p^s = 0 & \text{on} \quad]s,1[, \\ q^s(1) = y^s(1) - y_d. \end{cases} \tag{15}$$

Then it is possible to express $z^s(s)$ and $q^s(s)$ as an affine function of h and g :

$$\begin{cases} z^s(s) &= P_{11}(s)h + P_{12}(s)g + r(s), \\ q^s(s) &= P_{21}(s)h + P_{22}(s)g + t(s), \end{cases} \tag{16}$$

*We still denote by S the restriction of S to $[s,1]$

where the functions r and t depend upon f and y_d. One can choose $h = y^0|_{\Gamma_s}$ and $g = p^0|_{\Gamma_s}$ and the relation (16) being valid for any s it is possible replace s by x_1. Dropping the superscript s, the formal calculations go along these lines :

$$\begin{cases} \frac{dz}{dx_1} = -D^2 y - f = \frac{dP_{11}}{dx_1} y + P_{11} z + \frac{dP_{12}}{dx_1} p + P_{12} q + \frac{dr}{dx_1}, \\ \frac{dq}{dx_1} = -D^2 p = \frac{dP_{21}}{dx_1} y + P_{21} z + \frac{dP_{22}}{dx_1} p + P_{22} q + \frac{dt}{dx_1}. \end{cases} \tag{17}$$

The elimination of z between (16) and (17) gives

$$-D^2 y - f = \frac{dP_{11}}{dx_1} y + P_{11}\left(P_{11} y + P_{12} p + r\right) + \frac{dP_{12}}{dx_1} p + P_{12}\left(P_{21} y + P_{22} p + t\right) + \frac{dr}{dx_1}.$$

Formally this allows to write three differential equations verified by P_{11}, P_{12} and r :

$$\begin{cases} \frac{dP_{11}}{dx_1} &= -P_{11}^2 - P_{12} P_{21} - D^2, \\ \frac{dP_{12}}{dx_1} &= -P_{11} P_{12} - P_{12} P_{22}, \\ \frac{dr}{dx_1} &= -P_{11} r - P_{12} t - f. \end{cases} \tag{18}$$

Similarly the elimination of q between (16) and (17) leads to

$$-D^2 p = \frac{dP_{21}}{dx_1} y + P_{21}\left(P_{11} y + P_{12} p + r\right) + \frac{dP_{22}}{dx_1} p + P_{22}\left(P_{21} y + P_{22} p + t\right) + \frac{dt}{dx_1}$$

and the differential equations verified by P_{21}, P_{22} and t are :

$$\begin{cases} \frac{dP_{21}}{dx_1} &= -P_{21} P_{11} - P_{22} P_{21}, \\ \frac{dP_{22}}{dx_1} &= -P_{21} P_{12} - P_{22}^2 - D^2, \\ \frac{dt}{dx_1} &= -P_{21} r - P_{22} t. \end{cases} \tag{19}$$

The initial conditions for these previous equations are consequences of

$$z(1) = -\frac{1}{\nu} p(1),$$

which, taking into account (16), gives

$$P_{11}(1) = 0, \;\; P_{12}(1) = -\frac{1}{\nu} I, \;\; r(1) = 0. \tag{20}$$

And the condition

$$q(1) = y(1) - y_d,$$

implies

$$P_{21}(1) = I, \;\; P_{22}(1) = -\frac{1}{\nu} I, \;\; t(1) = y_d. \tag{21}$$

As a consequence of the form of Riccati equations (18)(19) and initial conditions (20)(21), it is clear that $P_{11} = P_{22}$ and $P_{21} = -\nu P_{12}$, then if we set $P = P_{11}$ and $Q = P_{12}$, we obtain the following :

Result 2 *The solution of the control problem is given by the set of equations :*

$$\left\{ \begin{array}{rcl} \frac{dP}{dx_1} &=& -P^2 + \nu Q^2 - D^2 \quad , \quad P(1) = 0, \\ \frac{dQ}{dx_1} &=& -PQ - QP \quad , \quad Q(1) = -\frac{1}{\nu} I, \\ \frac{dr}{dx_1} &=& -Pr - Qt - f \quad , \quad r(1) = 0, \\ \frac{dt}{dx_1} &=& \nu Qr - Pt \quad , \quad t(1) = -y_d, \end{array} \right. \tag{22}$$

$$\left\{ \begin{array}{l} z = Py + Qp + r, \\ q = -\nu Qy + Pp + t, \end{array} \right. \tag{23}$$

the optimal solution $\{y, p\}$ being solution of

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial x_1} = Py + Qp + r, \\ y(0) = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} \frac{\partial p}{\partial x_1} = -\nu Qy + Pp + t, \\ p(0) = 0. \end{array} \right. \tag{24}$$

□

4 DECOUPLING OF THE CONTROL PROBLEM (II)

There is an other way to decouple the optimality system (9), (11), (12) which amounts to taking its restriction on $\Omega_s =]0, s[\times]0, a[$. This situation is more complicated than the previous one because the control acts precisely on the *moving boundary.* In particular one cannot take the family of problems derived from (9), (11), (12) by restriction to Ω_s with arbitrary values of z and q on Γ_s, because it does not represent the optimality system of a family of non trivial control problems. So we consider the family of systems † :

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial x_1} = z \quad \text{on} \quad]0, s[, \\ y(0) = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} -\frac{\partial z}{\partial x_1} - D^2 y = f \quad \text{on} \quad]0, s[, \\ z(s) + \frac{1}{\nu} p(s) = \psi(s), \end{array} \right. \tag{25}$$

with the adjoint system

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial x_1} = q \quad \text{on} \quad]0, s[, \\ p(0) = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} -\frac{\partial q}{\partial x_1} - D^2 p = 0 \quad \text{on} \quad]0, s[, \\ q(s) = y(s) - \varphi(s). \end{array} \right. \tag{26}$$

One can check that it represents the optimality system for the problem with state equation (9) restricted to Ω_s, boundary condition :

$$\frac{\partial y}{\partial x_1} = u + \psi(s) \quad \text{on} \quad \Gamma_s \tag{27}$$

† In fact y, z, ...should have a superscript s which is omitted for the sake of simplicity.

and with the criterion :

$$J(u) = \frac{1}{2}\int_0^a |y(s,x_2) - \varphi(s,x_2)|^2\,dx_2 + \frac{\nu}{2}\int_0^a u^2(x_2)\,dx_2. \tag{28}$$

In particular one checks that $\varphi(1) = y_d(1)$ and $\psi(1) = 0$. Then it is possible to express $y(s)$ and $p(s)$ as an affine function of φ and ψ :

$$\begin{cases} y(s) &= P_{11}(s)\psi + P_{12}(s)\varphi + r(s), \\ p(s) &= P_{21}(s)\psi + P_{22}(s)\varphi + t(s), \end{cases} \tag{29}$$

where the functions r and t depend upon f and y_d. We will not give all the details of the calculations but let us show the starting point. If one takes the derivative with respect to x_1 of y we have

$$z(x_1) = \frac{dy}{dx_1} = \frac{dP_{11}}{dx_1}\psi + P_{11}\left(\frac{1}{\nu}q - D^2y - f\right) + \frac{dP_{12}}{dx_1}\varphi + P_{12}\left(z + D^2p\right) + \frac{dr}{dx_1}.$$

And, on the other hand,

$$q(x_1) = \frac{dp}{dx_1} = \frac{dP_{21}}{dx_1}\psi + P_{21}\left(\frac{1}{\nu}q - D^2y - f\right) + \frac{dP_{22}}{dx_1}\varphi + P_{22}\left(z + D^2p\right) + \frac{dt}{dx_1}.$$

Then, as we have

$$q(x_1) = y(x_1) - \varphi(x_1) = P_{11}\psi + (P_{12} - I)\varphi + r,$$

it is possible to eliminate q in order to get an equation containing only the functions φ, ψ, r and t. A formal identification gives the following set of equations :

$$\begin{cases} \frac{dP_{11}}{dx_1} &= -\frac{1}{\nu}P_{11}^2 + P_{11}D^2P_{11} - P_{12}D^2P_{21} + P_{12}P_{21} - P_{12} - P_{21} + I, \\ \frac{dP_{12}}{dx_1} &= -\frac{1}{\nu}P_{11}P_{12} + \frac{1}{\nu}P_{12}P_{22} + P_{11}D^2P_{12} + P_{12}D^2P_{22} - \frac{1}{\nu}P_{11} - \frac{1}{\nu}P_{22}, \\ \frac{dr}{dx_1} &= -\frac{1}{\nu}P_{11}r + \frac{1}{\nu}P_{12}t + P_{11}D^2r + P_{12}D^2t + P_{11}f. \end{cases} \tag{30}$$

It is possible to derive similar equations verified by P_{21} , P_{22} and t. As previously we observe that $P_{11} = -P_{22} = P$ and $P_{21} = \nu P_{12} = Q$. With these notations the final Riccati system of equations is

$$\begin{cases} \frac{dP}{dx_1} &= -\frac{1}{\nu}P^2 + Q^2 + PD^2P - \nu QD^2Q - 2Q + I \quad , \quad P(0) = 0, \\ \frac{dQ}{dx_1} &= -\frac{1}{\nu}PQ - \frac{1}{\nu}QP + PD^2Q + QD^2P + \frac{2}{\nu}P \quad , \quad Q(0) = 0, \\ \frac{dr}{dx_1} &= PD^2r - QD^2t - \frac{1}{\nu}Pr - \frac{1}{\nu}(I-Q)t + Pf \quad , \quad r(0) = 0, \\ \frac{dt}{dx_1} &= PD^2t + \nu\, QD^2r - (Q-I)r - \frac{1}{\nu}Pt + \nu Qf \quad , \quad t(0) = 0. \end{cases} \tag{31}$$

Unfortunately the solution of system (31) is not sufficient to determine y and p because these two functions are expressed, by means of (29) in term of the unknown functions φ

and ψ. If we replace φ and ψ by their expressions in the initial conditions of (25), (26) we get the following system of equations :

$$\left\{\begin{array}{lcl} y &=& Pz + \frac{1}{\nu}Pp + Qy - Qq + r, \\ p &=& \nu Qz + Qp - P(y-q) + t. \end{array}\right. \tag{32}$$

It is possible to solve this system and we finally obtain :

$$\left\{\begin{array}{lcl} y &=& Az + Bq + Rr + Tt, \\ p &=& \tilde{A}z + \tilde{B}q + \tilde{R}r + \tilde{T}t. \end{array}\right. \tag{33}$$

where

$$\left\{\begin{array}{lcl} A &=& \left(P^{-1}(I-Q) + \frac{1}{\nu}(I-Q)^{-1}P\right)^{-1} + \left(\frac{1}{\nu}Q^{-1}P + (Q^{-1}-I)P^{-1}(I-Q)\right)^{-1}, \\ B &=& \left(I - Q^{-1} - \frac{1}{\nu}Q^{-1}P(I-Q)^{-1}P\right)^{-1} + \left(I + \nu P^{-1}(I-Q)P^{-1}(I-Q)\right)^{-1}, \\ R &=& \left(I - Q + \frac{1}{\nu}P(I-Q)^{-1}P\right)^{-1}, \\ T &=& \left(P + \nu(I-Q)P^{-1}(I-Q)\right)^{-1}, \\ \tilde{A} &=& -\left(\frac{1}{\nu}I + P^{-1}(I-Q)P^{-1}(I-Q)\right)^{-1} + \nu\left(Q^{-1} - I + \frac{1}{\nu}Q^{-1}P(I-Q)^{-1}P\right)^{-1}, \\ \tilde{B} &=& \left(Q^{-1}P + (Q^{-1}-I)P^{-1}(I-Q)\right)^{-1} + \left(P^{-1}(I-Q) + \frac{1}{\nu}(I-Q)^{-1}\right)^{-1}, \\ \tilde{R} &=& -\left(\frac{1}{\nu}P + (I-Q)P^{-1}(I-Q)\right)^{-1}, \\ \tilde{T} &=& \left(I - Q + \frac{1}{\nu}P(I-Q)^{-1}P\right)^{-1}. \end{array}\right. \tag{34}$$

Finally z and q are solution of

$$\left\{\begin{array}{rcl} -\frac{dz}{dx_1} - D^2(Az + Bq) &=& f + D^2(Rr + Tt), \\ z(1) &=& \frac{1}{\nu}(P(1)y_d - t(1)), \end{array}\right. \tag{35}$$

$$\left\{\begin{array}{rcl} -\frac{dq}{dx_1} - D^2(\tilde{A}z + \tilde{B}q) &=& D^2(\tilde{R}r + \tilde{T}t), \\ q(1) &=& (P(1) - I)y_d + r(1)). \end{array}\right. \tag{36}$$

Result 3 *The optimal control u is given by :*

$$u = -\frac{1}{\nu^2}((\tilde{A}(1)P(1) + \nu\tilde{B}(1)(Q(1)-I))y_d + \nu(\tilde{B}(1) + \tilde{R}(1))r(1) + (\nu\tilde{T}(1) - \tilde{A}(1))t(1)), \tag{37}$$

where P, Q, r and t are solution of (31). *As r and t do not depend on y_d this gives explicitly the linear dependence of u on y_d.*

□

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