Stability and Optimality of Continuous Time M-*ary* Rate Control Mechanisms

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Abstract

In feedback rate control mechanisms of both rate and window domains, M-ary control class can be considered as an extension of the widely implemented binary control class. In this paper, the stability properties of the class of continuos time M-ary rate controls are analytically established using dynamic theoretic tools. The dynamic behaviour is shown to exhibit a *Hopf bifurcation* where the stability of the system changes at a critical parameter value. The uniqueness of the optimal controller is established where the optimality criterion is defined as the *convergence rate invariance* property. This contrasts with the non unique and mutually exclusive controls of the discrete time M-ary rate control class. As a consequence of the rate-window duality, performance implications of this analysis for a recently proposed window control algorithm of the M-ary form are discussed. One such implication is that, the systemic stability of a partially distributed implementation of the window controller is guaranteed stable.

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1 INTRODUCTION

The class of feedback flow control mechanisms form a substantial part of overall flow control regimes in computer networks [Agarwala et al., 1992; Bolot et al., 1990; Chiu et al., 1989; Gerla et al., 1980; Jacobson, 1988; Jain, 1989; Keshav, 1991; Mitra et al., 1990; Mukherjee et al, 1991; Shenker, 1990]. The study of these mechanisms is prompted by their intrinsic analytical tractability, implementational simplicity and the relationships that exist between complementary classes of mechanisms. More recently, an attempt has been made to emphasise a precise analytical specification of the flow control problem compared to past efforts. In this respect a few complex flow controls with a sound theoretical basis have been proposed [Keshav, 191; Mitra et al., 1990].

Recently Fendick et al [Fendick et al., 1992] have analysed the window control algorithm of Mitra et al [Mitra et al., 1990], in an attempt to obtain certain performance characteristics of Mitra's algorithm. They have resorted to a direct analysis by mapping the original algorithm in a window domain to an equivalent piece-wise continuous rate control, thus obtaining results pertaining to unilateral stability, an optimal value of a control parameter and obtaining evidence for the existence of a large class of such controls. It is our view that, where a direct analysis of a complex algorithm is appropriate yet difficult, it would be more feasible to analyse a simpler complementary form of algorithm to that of the original, which may yield performance bounds and possibly implementational insight that could generally be valid for the original algorithm.

This study has its origins in two previous works. Bolkot et al [Bolot et al., 1990; Bolot et al., 1992] have studied binary rate control schemes and their dynamic behaviour, and have concluded that the absence of proper modelling tools has hindered the analytic establishment of stability properties. Shenker [Shenker, 1990] has studied a class of controls which we have identified as discrete time M-ary rate controls, where M-ary in our nomenclature refers to virtual M decision levels in the flow controller. The name is synonymous with the term M-ary in digital transmission schemes. The class of continuous time M-ray rate controls is hence identified as the natural counterpart to the discrete time class. We have also observed that Mitra's algorithm generally fits the definition of a M-ary control.

Our investigation in this paper centres on the analysis of the class of continuous time M-ary rate control mechanisms for its performance characteristics as an extension of Shenker's and Bolot et al's work, and as an analysis of Mitra's algorithm through a complementary yet a simpler approach. Our analysis makes the rigorous establishment of the stability of continuous time M-ary rate control possible by dynamic and queuing theoretic techniques. By implication of this stability for all deterministic continuous time M-ay controls, we predict the systemic stability of the partially distributed implementations, among other implementational insight [Ranasinghe, 1994].

In discrete time M-ary rate control, Shenker [Shenker, 1990] established that there are a number of non unique and mutually exclusive controls such as time scale invariant, time scale variant and logistic type forms. In contrast we show that the continuous time M-ary rate control class has a unique optimal controller for each specified equilibrium point. The optimality criterion is defined as the property of *convergence rate invariance*, i.e., the property that the server utilisation remains constant for a range of convergence rates. This property was inherent in the discrete time M-ary rate class, yet it has to be explicitly specified in the optimisation process of the continuous time counterpart.

2 STABILITY OF A HYPOTHETICAL CONTROLLER

2.1 The Network Model

Consider a virtual circuit that traverses a single bottleneck node with a server rate μ (packets/sec). The service policy of the node is FIFO. The non-zero forward and reverse propagation delays are τ_f and τ_r (sec) respectively. The instantaneous source transmission rate is represented by $\lambda(t)$ (packets/sec), and a stochastic packet arrival and service process of general distribution may be assumed. The data flow is assumed to be unidirectional and it is assumed that there exists a reverse acknowledgement flow indicating the bottleneck node queue level $q_B(t)$ to the source. The access line has a rate $\mu_a(>> \mu)$. In order that feedback is enforceable, the source is modelled as an infinite data source. The pipe size, which is the delay-bandwidth product of the path following the bottleneck $\mu\tau_r$, is assumed to be $\mu\tau_r >> 1$. See Figure 1.

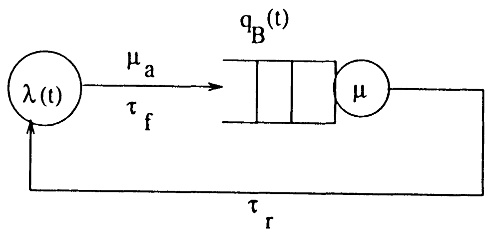

Figure 1: The Physical Model

2.2 Modelling and Analysis

The hypothetical controller we subject to analysis is given as:

$$\dot{\lambda}(t) = \alpha - \lambda(t)(R(t - \tau_r) - 1)/\beta \quad (1)$$

which, as can be seen is a generalised continuous time rate controller consisting of a linear rate increase policy with a convergence rate α (packets/sec) and an exponential rate decrease policy with a backoff time constant β(sec), reacting to a normalised round trip delay R. We shall call this an M-ary extension of the binary rate control of Bolot et al [Bolot et al, 1990]. It has been shown [Bolot et al, 1992] that a discrete space stochastic process at a node approaches a deterministic process in the presence of a large packet population under continuous time control and this enables R(t) to be substituted by $q_B(t)$ as:

$$R(t-\tau_r) = q_B(t-\tau_r)/\mu\tau + 1 \tag{2}$$

where, $\tau = \tau_f + \tau_r$.

The deterministic node process itself is described by the fluid mode [Bolot et al, 1990] as:

$$\dot{q}_B(t) = \left\{ \begin{array}{ll} 0 & \text{if } q_B(t) = 0 \\ & \text{and } \lambda(t-\tau_f) < \mu \\ \lambda(t-\tau_f) - \mu & \text{otherwise} \end{array} \right\} \tag{3}$$

By substitution of (2) into (1), we obtain the rate controller in the form:

$$f_1(\lambda, q_B) = \alpha - \lambda(t)q_B(t)/\beta\mu\tau \tag{4}$$

which in conjunction with (3) describes the dynamic system in delay-differential equation form. The desired queue equilibrium point of the system q_{Be}, can be any chosen value. We shall initially choose q_{Be} to be 1 in line with [Bolot et al., 1990]. The rate equilibrium point remains $\lambda_e = \mu$, as implied by the server rate of the node. The delay-differential model describing the queue and rate dynamics becomes analytically tractable if we convert it to a set of differential equations. Therefore, to obtain a R^n model of the dynamic system which can be subjected to a dynamic theoretic analysis, we make use of a queue model description. The node queue is modelled by an M/M/1 process whose fluid model description in (3) fits the deterministic process required [Filipiak 1988; Ohta, 1988]. The delay segments are modelled by infinite server queues such as $M/G/\infty$. We also choose to model either but not both of the pipe segments for simplicity. This should not affect the qualitative performance expected of the real system as what we essentially need is a non zero propagation delay anywhere in the closed loop. From the two possible equivalent queue model representations we select the following models as the other model introduces unknown functional dependency on state variables into the equations

[Ranasinghe,1994] . See Figure.2

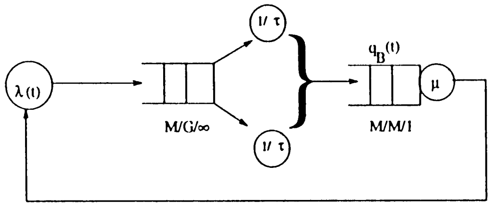

Figure 2: The Queueing Model

$$\dot{q}_p(t) = \lambda(t) - q_p(t)/\tau \tag{5}$$

$$\dot{q}_B(t) = q_p(t)/\tau - \mu \tag{6}$$

$$\dot{\lambda}(t) = \alpha - \lambda(t)q_B(t)/\beta\mu\tau \tag{7}$$

In these equations, $q_p(t)$ is the pipe equivalent queue, $q_B(t)$ is node queue and $\lambda(t)$ is the source rate. τ, μ, α and β are as defined in (1) and (2).

Now the dynamic system is in R^3 , where $\tau_f = \tau$ as $\tau_r = 0$. Given the state space description, the stability properties of the dynamic system at a specified equilibrium point (λ_e, q_{Be}) can be established. A similar proof to that of Fendick et al [Fendick et al, 1992] shows that the system is unstable in the vicinity of the equilibrium point $(\mu, 1)$ for example, and the interested reader is referred to [fendick et al, 1992]. Although the system is locally unstable, it can be proved that it exhibits a stable limit cycle behaviour for a range of parameter values of α (see Appendix) . We have found that there exists a critical convergence rate α_c such that for $0 < \alpha < \alpha_c$, there is a stable limit cycle surrounding the unstable equilibrium point and, for $\alpha > \alpha_c$, the system exhibits asymptotic stability. In dynamic systems theory, this is identified as a ***Hopf bifurcation*** where, as a parameter passes a critical value, the system changes from one state of stability to another [Beltrami 1987] . In contrast, for the discrete time counterpart as shown by Shenker [Shenker,1990] the rate iterations moved from an asymptotic stability position to chaotic instability as a critical convergence rate was passed. For the heuristic controller $f_1(\lambda, q_B)$ in (4) and for $(\lambda_e, q_{Be}) = (\mu, 1)$ this occurs for:

$$\alpha_c = \mu^2(1 - 1/\mu\tau)pkt/sec^2 \tag{8}$$

where $\mu\tau >> 1$. There is also a straightforward way to find the critical convergence rate. This involves the obtaining of the eigenvalue equation for the model in R^n and testing it for asymptotic stability or instability using Routh test. However in the present situation, it only indicates an instability in the region $0 < \alpha < \alpha_c$, which is not in itself a sufficient condition for the existence of a stable limit cycle.

2.3 Simulation Performance

The numerical solution of the delay-differential equation model or the model in R^3 provides an indication of the behaviour of the flow controlled system. In a study of continuous time binary rate control with direct implications to our analysis, Bolot et al [Bolat et al, 1992] have shown that there is a good agreement between the delay-differential model in the analytical domain and the discrete space stochastic model in the physical domain, i.e., the direct simulation of the packet transmission process, at large packet populations corresponding to $\mu\tau >> 1$. This validates the simulation results obtained by numerical solution as representing the packet transmission process. The figures corresponding to the model in R^n were obtained by numerical solution of (5) to (7) using the package. [ISIM 1987], which uses the standard 4th order Runge-Kutta procedure. The rate control form $f_1(\lambda, q_B)$ (9), with the values $\mu = 100$ and $\tau = 1$ have been used throughout. The simulation tests show that each of the models describing the forward or the reverse pipe segments have identical dynamic performance except for a phase shift between the rate variation and the queue variation, which is expected. Figures 3 and 4 show a large discrepancy between the delay-differential model and the model in R^3 at low q_{Be} of O(1). The delay-differential model behaviour further resembles the continuous time binary control of Bolot et al.

From Figures 5 and 6, it can be seen that the delay-differential model and the model in R^3 corresponding to $f_1(\lambda, q_B)$ agree well at large queue equilibria of $O(\mu\tau)$, with a substantial bias towards sinusoidal rate variation.

This observation confirms the applicability of a fluid approximation of he queue models used to describe the non-stationary stochastic behaviour at large packet populations. Therefore to retain equivalence between the model in R^3 and the delay-differential model, we shift the system equilibrium point to $(\lambda_c, q_{Be}) = (\mu, O(\mu\tau))$. However, our simulations also show that the delay-differential model and the model in R^3 differ in their prediction of asymptotic stability criterion, though not consistently. For example, at an $\alpha > \alpha_c$, the delay-differential model of $f_1(\lambda, q_B)$ displays a persistent backlog at the node queue, but the model in R^3 displays asymptotic stability (a.s) (Figures. 7 and 8). It is conjectured that, this discrepancy is the result of the residual modelling differences and provided the working convergence rate does not exceed the critical α by a wide margin, then the delay-differential model exhibits a backlog.

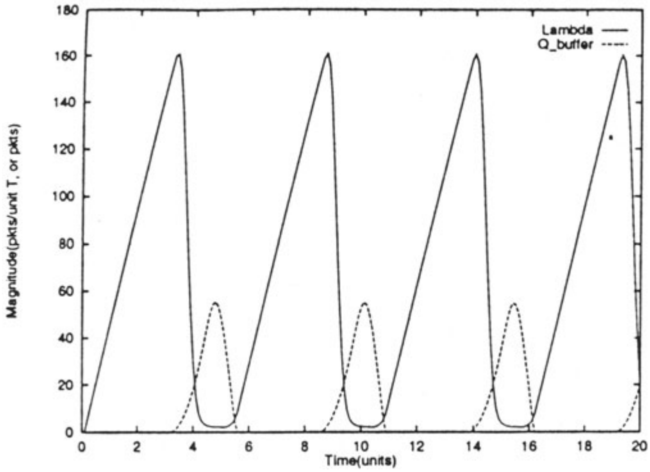

Figure 3: Delay-Differential model
($\alpha = \mu/2\tau, q_{Be} = 1$)

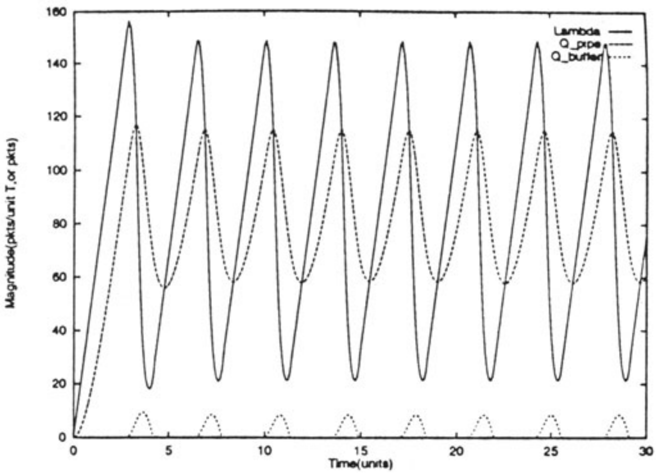

Figure 4: Differential model
($\alpha = \mu/2\tau, q_{Be} = 1$)

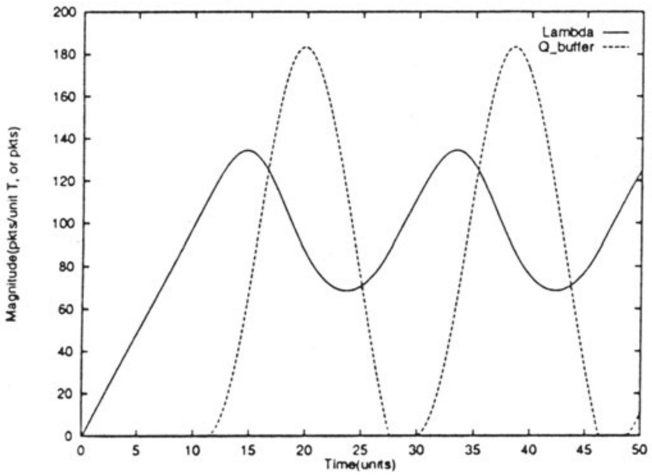

Figure 5: Delay-Differential model
$(\alpha = \mu/10\tau, q_{Be} = 5\mu\tau/2\pi)$

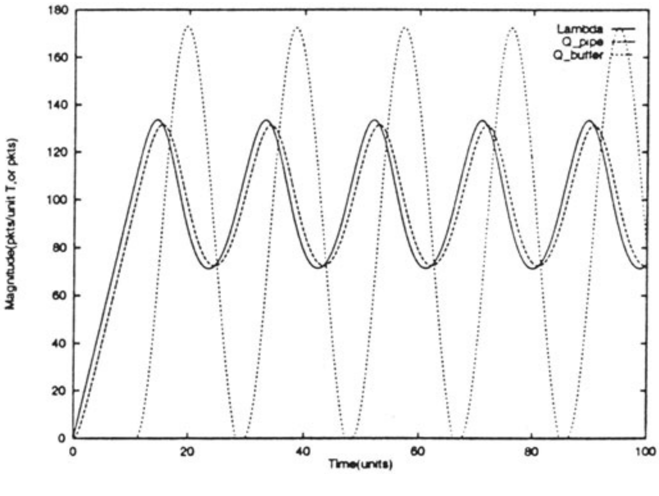

Figure 6: Differential model
$(\alpha = \mu/10\tau, q_{Be} = 5\mu\tau/2\pi)$

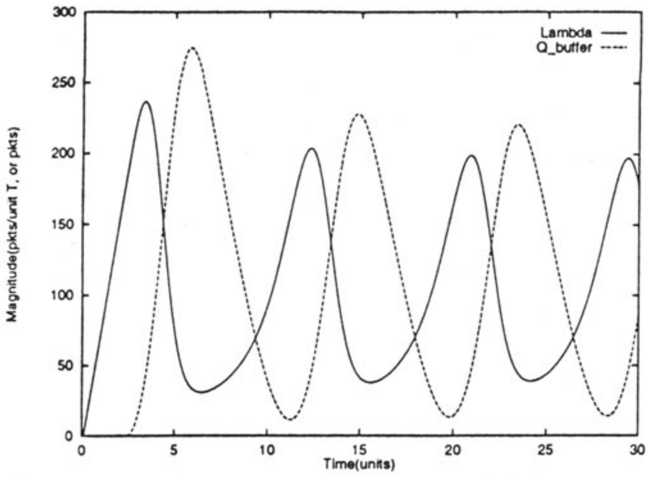

Figure 7: Delay-Differential model
($\alpha = 5\mu/2\pi\tau, q_{Be} = 5\mu\tau/2\pi$)

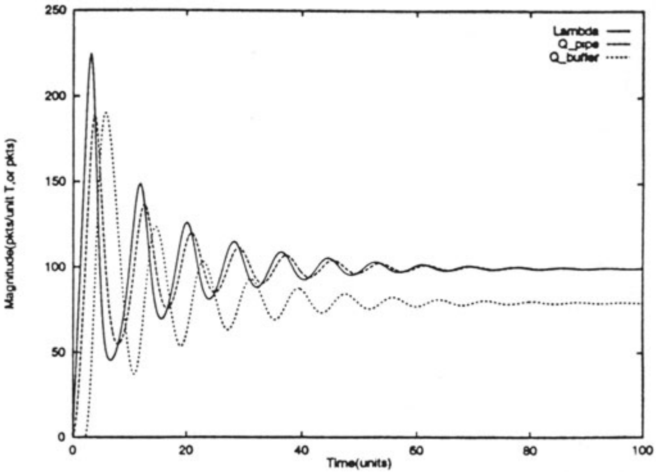

Figure 8: Differential model
($\alpha = 5\mu/2\pi\tau, q_{Be} = 5\mu\tau/2\pi$)

This is confirmed, where a slightly different form of heuristic control, $f_2(\lambda, q_B)$ (see next section) was used, which had a negative critical convergence rate at a specified q_{Be}, a positive α resulted in a.s. for the delay-differential model (Figure not shown). It is reasonable to conclude that a persistent backlog may be the practical reality at an $\alpha > \alpha_c$, for heuristic rate control.

3 OPTIMAL CONTROL

3.1 Optimality Criterion

We now consider two heuristic rate controllers (9) and (10) and compare their critical convergence rates at a specified queue equilibrium point, $q_{Be} = \mu\tau/k$, with k$\geq$ 1 would represent a range of q_{Be} values from $O(1)$ to $O(\mu\tau)$. Let:

$$f_1(\lambda, q_B) = \alpha - \lambda(t)q_B(t)/\gamma \tag{9}$$

which is 4 and

$$f_2(\lambda, q_B) = \alpha - \lambda^2(t)q_B(t)/\gamma \tag{10}$$

where $\gamma = \beta\mu\tau$

The corresponding critical convergence rates are:

$$\alpha_c = \mu(k-1)/\tau \tag{11}$$

for $f_1(\lambda, q_B)$, which is positive for $k > 1$ and

$$\alpha_c = \mu(k-2)/\tau \tag{12}$$

for $f_2(\lambda, q_B)$, which is positive for $k > 2$.

We note that a positive critical convergence rate for practical purposes is achieved at a lower queue equilibrium point for $f_2(\lambda, q_B)$ than for $f_1(\lambda, q_B)$. However , as q_{Be} approaches $O(1)$, and thus operating at a high convergence rate within the critical α_c, the continuous time M-ary rate controller's performance degrades to that of a continuous time binary rate control as observed in section 2.3, with the server utilisation being dependent on the convergence rate. Since our basis for the analysis of the M-ary class of rate controls is Shenker's work [Shenker, 1990], we have noted that the discrete time M-ary rate controls of all forms had the property of *convergence rate invariance* embedded in them. This property implies that for a range of convergence rates, the rate controller maintains a constant server utilisation $\eta(=1)$, in the case of deterministic continuous time control and, $\rho(<1)$ in case of stochastic discrete time controls. We shall define this property as the appropriate optimality criterion. We conclude therefore, any arbitrary heuristic M-ary rate controller is suboptimal in the sense that this property is not observed and, that there exists an optimal controller for any specified operating point (λ_e, q_{Be}).

3.2 On Optimisation

In the specification of the optimal controller, the position of the desired queue equilibrium point has to be defined. In open queuing networks with Markovian nodes, q_{Be} has been usually set to $O(1)$ which subsequently optimised the network power (Stidam, 1985). This approach is equally valid in selecting the operating point for discrete time M-ary rate controls. In continuous time rate controls of both binary and the proposed M-ary type, the network power argument cannot be effectively applied due to the deterministic nature of the node process An alternative approach is required and we raise the queue equilibrium point q_{Be}, to $O(\mu\tau)$ in order to have the widest possible range of convergence rates. For example, in bang-bang rate control, a unit server utilisation was reached at the highest possible convergence rate and at the minimum backoff time at a q_{Be} of $\mu\tau$ [Wang et al. 1991]. Since the idea behind optimal continuous time M-ary rate control is to reach the goal of convergence rate invariance, which would maintain a constant server utilisation η of 1 for $0 < \alpha < \alpha_c$, a sinusoidal rate variation would be the ideal description of such a behaviour due to it inherent symmetry. Therefore, consider the rate variation $\lambda(t)$ of the form:

$$\lambda(t) = \bar{\lambda} + a_0 sin\omega t \tag{13}$$

Where $\bar{\lambda} = \mu$ is the desired average rate, $\omega = 2\pi/T$ is the angular frequency of an oscillation with period T, and $(0 \leq a_0 \leq \mu)$ is the amplitude of the variation.

By application of the flow conversation rule(3) to the node queue and assuming a zero propagation delay between the source and the node, it follows that:

$$q_B(t) = \int_0^t (\lambda(t) - \mu)dt \tag{14}$$

$$= a_0(1 - cos\omega t)/\omega \tag{15}$$

by substitution of (13).

We have argued that the condition $\bar{q_B}$ be equal to q_{Be} , be satisfied when the controller is optimal and it is also seen that $\bar{\lambda} = \lambda_e$ is true [Ranasinghe, 1994] Figures.5 and 6, show $f_1(\lambda, q_B) at \lambda_e = \mu, q_{Be} = 5\mu\tau/(2\pi)$ and with an $\alpha = \mu/(10\tau)$, generating a rate variation that closely tracks the sinusoidal form expected, with average values of rate and node queue being approximately equal to respective equilibrium point values. This does not however mean that $f_1(\lambda, q_B)$ is optimal at the given operating point, as the "real" α_c for the specified q_{Be} may be less than or greater than the "apparent" α_c, that is obtained analytically for a heuristic control. For example, the negative critical α for $f_2(\lambda, q_B)$ clearly proves the case. We can state the optimal rate and node queue variations, taking round trip propagation delay into account as:

$$\lambda_{opt}(t) = \mu + a_0 sin\omega t \tag{16}$$

$$q_{B(opt)}(t) = a_0(1 - cos\omega(t - \tau))/\omega \tag{17}$$

In general, a heuristic controller $f_i(\lambda, q_B)$ can be optimised using established Lagrangian optimisation techniques, However, it has been shown that the closed form solution does not exist for the system in R^3, as it is a non linear servo mechanism problem [Sage er al., 1977] . A numerical solution alternative does exist, but results in an open loop controller , whereas a closed loop controller is desirable. Accordingly, the realisation of the optimal controller is not possible.

3.3 Uniqueness of the Optimal Control

To obtain the predicted variation of the critical convergence rate α_c with q_{Be} confirming the uniqueness of the optimal continuous time M-ary rate controller, we proceed as follows. Since all rate controls must contain a linear rate increase component of the form $\lambda(t) = \alpha$, that includes the convergence rate as a parameter see equation (1), by differentiating (16) and assigning $\omega t = 0$ to represent the highest rate increase effort, we obtain:

$$\alpha = a_0\omega \tag{18}$$

Since $\bar{q_B} = q_{Be}$ and :

$$\bar{q_B} = a_0/\omega \tag{19}$$

which follows from (17), we have:

$$\alpha = a_0^2/q_{Be} \tag{20}$$

As the peak amplitude of $\lambda_{opt}(t)$ is reached at the critical convergence rate α_c, we would then have, $\alpha_c q_{Be} = \mu^2$ as the form of the characteristic curve of $\alpha_c(q_{Be})$ as a function of q_{Be} for the region where a_0 remains constant at μ. The region $a_0 < \mu$ is predicted by reference to qualitative results obtained by simulations [Ranasinghe, 1994]. See Figure 9. These simulation results show that, as q_{Be} decreases, the effective critical convergence rate decreases as well, indicating that there should be an optimal operating point for the q_{Be} . Inferring from bang-bang rate control, which is a continuous time control that maintains a unit server utilisation, we note that the fundamental sinusoidal component of the source rate variation has a peak amplitude equal to μ corresponding to a q_{Be} of $\mu\tau$ [Wang et al., 1991]. It would not be incorrect to model the optimal continuous time M-ary rate control as the one that produces this sinusoidal variation, thus lowering the $\bar{q_B}$ of bang-bang control to the level of q_{Be} . Therefore we let, for the optimal control a value of $q_{Be(opt)} = \mu\tau$.

We reach the conclusion that the optimal controller is unique for a given operating point . In contrast, in the discrete time M-ary rate controls, there were time scale invariant, time scale variant and logistic type rate iterations, and therefore a wide class of controls in reaching and maintaining a stochastic server utilisation of less than one. Alternatively, as the deterministic server utilisation aimed by a controller approaches unity, it becomes increasingly difficult to maintain the time scale invariant property and hence the uniqueness of the continuous time controller.

4 STABILITY REVISITED

4.1 Oscillatory Behaviour

It has been shown that continuous time M-ary rate control exhibits an oscillatory behaviour for $0 < \alpha < \alpha_c(q_{Be})$. This is an artefact of reactive control in the presence of delayed feedback. In the implementation of heuristic controller $f_i(\lambda, q_{Be})$, it uses the a prior knowledge of μ the constant server rate, and τ the round trip propagation delay. Given these two pieces of information, it is possible to design a predictive controller [Keshav,1991], that would converge to the equilibrium state $\lambda_e = \mu$, in one round trip time. Basically, what the controller does is to inject and maintain a number of packets, known as the sliding window size $W(= \mu\tau)$ in the system, filling the pipe and thus reaching the unit server utilisation required.

Extending the analysis to more than one user (or connection), say N users, sharing the bottleneck node, the i-th user individual fair rate can be stated as $\lambda_i = \mu/N$, leading to the window size per user in a predictive controller of $W_i = \mu\tau/N$, Keshav [Keshav, 1991] describes a technique that uses a rate allocating server discipline, which enables the evaluation of μ/N adaptively by each user, which enables the predictive controller to be implemented . However, where a FIFO service discipline exists as in our case, this is not possible and, the way out of the problem is to use a partially distributed method, with an upper bound N_{max} on the number of users.

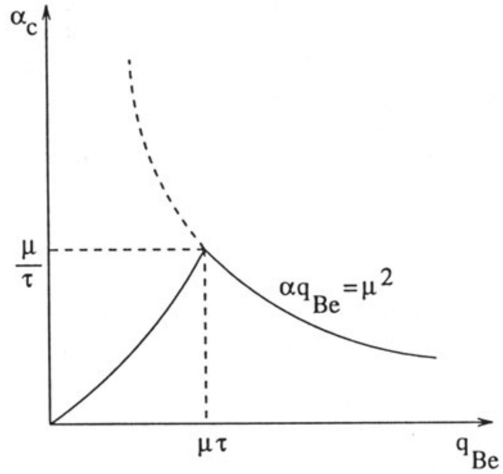

Figure 9: α_c variation with q_{Be} for the optimal controller (not to scale)

4.2 Systemic Stability in a Partially Distributed Implementation

Consider the physical model shown in Figure.1, but with N virtual circuits of equal round trip propagation delay, $\tau_i = \tau_j; \forall i \neq j$, sharing the single bottleneck node and employing the same hypothetical rate control $f_1(\lambda, q_B)$, it is a straightforward task to write the model in R^5 by extending the equation as (5), (6) and (7) (Ranasinghe,1994). Further if $\gamma_i = \gamma_j; \forall i \neq j$, then the model reduces to R^3 , which makes the direct application of unilateral stability conditions to the systemic case possible.

Then by similar reasoning to that in section 2.2, and with the additional condition that $\alpha_i = \alpha_j; \forall i \neq j$ the systemic stability condition is obtained as:

$$N\alpha_i < \alpha_c \tag{21}$$

with equilibrium point condition,

$$N\alpha_i\gamma_i = \lambda_e q_{Be} \tag{22}$$

at the specified equilibrium point(λ_e, q_{Be})

Ideally, as the number of connections vary, each source control should hold its common β constant, and vary the individual convergence rate α_i based on a distributedly known value of N. Where this is not possible, partial distribution with an upper bound value, N_{max}, can be attempted [Jain, 1989 : Mitra et al. , 1990]. Suppose the active number of connections at a given instant N_{active} be greater than N_{max}. Then in satisfying the new equilibrium point condition.

$$N_{active}\alpha_i\gamma_i = \lambda_e q_{Be(active)} \tag{23}$$

where $\lambda_e = \mu$, it becomes $q_{Be(active)} > q_{Be}$, when compared to equation(22).

A controller which had been optimised for q_{Be} , when shifted to a higher $q_{Be(active)}$ without being re-optimised, will produce an "apparent" critical α that may be negative (see section 2.3). As a result, for a positive convergence rate, the system reaches a.s., or will exhibit a stable queue backlog. Similarly, it can be shown that in the case of $N_{active} < N_{max}$, the suboptimal behaviour is reflected in the form of a server under-utilisation which depends on the convergence rate.

Therefore, it can be argued that, the class of continuous time M-ary rate control is robust in the face of a partially distributed implementation. In contrast, a predictive control with $W_i = \mu\tau N_{active}/N_{max}$, is susceptible to a node buffer overflow. The complementariness of the continuous time M-ary rate control to that of Mitra's window algorithm [Mitra et al.. 1990] is further confirmed by the simulation observation that for $N_{active} > N_{max}$, a stable increase in the queue backlog has been observed in the later. This conclusion is valid to all continuous time M-ary controls.

5 CONCLUSIONS

In this paper we have considered the stability and optimality of continuous time M-ary rate control mechanisms. Following a theoretic analysis of the dynamic system, we have shown that the class of continuous time M-ary rate controls exhibit a stable limit cycle that surrounds an unstable equilibrium point which vanishes to asymptotic stability as the convergence rate is raised beyond a critical point. The critical point is a ***Hopf bifurcation*** point, in that the systems stability properties differ either side of this point.

Using a common optimality criterion for both the discrete and continuous time M-ary rate control forms, which is expressed as the ***convergence rate invariance*** property, it has been shown that the continuous time M-ary control produces a unique optimal control for specified operating point, whereas the discrete time counterpart consisted of a wider class of mutually exclusive controls satisfying the optimality criterion. The study of continuous time M-ary rate controls was found to have implications in the performance characterisation and implementational aspects of a complementary class of window controls.

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Appendix

We present the proof of stability of the rate controlled system $f_1(\lambda, q_B)$, described by the model in R^3. We use the Hopf bifurcation theorem [Beltrami 1987] into prove the existence of a stable limit cycle surrounding an unstable equilibrium point.

Consider first, the nonlinear dynamic system described by:

$$\dot{x} = f(x, \xi)$$

where, $x(t) \in R^2$, and has an isolated equilibrium point $x_e(\xi)$, where ξ is a variable parameter. Let the linearised Jacobian of the system $A(\xi)$ have eigenvalues $\lambda_i(\xi) = \alpha(\xi) \pm j\beta(\xi)$ which are differentiable in ξ in a suitable range $|\xi| < \delta$. The Hopf Bifurcation theorem in R^2 states the following [Beltrami 1987].

Suppose that the equilibrium point $x_e(\xi)$ is asymptotically stable for $\xi < 0$ and unstable for $\xi > 0$, and that $\alpha(0) = 0$, i.e., $x_e(0)$ of the linearised system is neutrally stable. If $d\alpha(0)/d\xi > 0$ and $\beta(0) \neq 0$ then for all sufficiently small $|\xi|$, a closed orbit exists for ξ either positive or negative. In particular , if $x_e(0)$ is locally a.s, then there exists a stable limit cycle Γ about $x_e(\xi)$ for all small $\xi > 0$. Moreover, the amplitude of Γ grows as ξ increases.

Since the linearised system is neutrally stable at $x_e(0)$, we have to ensure that $x_e(0)$ is in fact locally a.s., which is a difficult procedure. However, extending the theorem to R^3 we remove this obstacle by observing that two of the three eigenvalues $\lambda_i(\xi); i = 1, 2, 3$ of Jacobian $A(\xi)$ are complex and one is real and negative. At $x_e(0)$ the complex eigenvalues become imaginary but, since $\lambda_3(0)$ is real and negative, the system is locally a.s.

Recall that the model in R^3 corresponding to $f_1(\lambda, q_B)$ as (see (5), (6) and (7)

$$\dot{q}_p(t) = \lambda(t) - q_p(t)/\tau$$

$$\dot{q}_B(t) = q_p(t)/\tau - \mu$$

$$\dot{\lambda}(t) = \alpha - \lambda(t) q_B(t)/\beta\mu\tau$$

with the state vector $x = (q_p, q_B, \lambda)$ and the equilibrium point $x_e = (\mu\tau, 1, \mu)$.

The Jacobian of the linearised system at x_e is obtainable as:

$$A = \begin{pmatrix} -1/\tau & 0 & 1 \\ 1/\tau & 0 & 1 \\ 0 & -\mu/\gamma & -1/\gamma \end{pmatrix}$$

The characteristic equation with eigenvalues λ_i as roots follows from the Jacobian and is given by:

$$\lambda^3 + (1/\tau + \alpha/\mu)\lambda^2 + \lambda\alpha/(\mu\tau) + \alpha/\tau = 0$$

Let there exist two complex conjugate and, one real and negative eigenvalue for this characteristic equation as:

$$\lambda_i(\xi) = \sigma(\xi) \pm j\rho(\xi)$$

for $i = 1, 2$ and λ_3, real.

Therefore from $\prod_{i=1}^{3}(\lambda - \lambda_i)$ we get:

$$\lambda^3 - (2\sigma + \lambda_3)\lambda^2 + (\sigma^2 + \rho^2 + 2\sigma\lambda_3)\lambda - \lambda_3(\rho^2 + \sigma^2) = 0$$

For linearised systems to be neutrally stable at a possible bifurcation point $\xi = 0$ it should be such that:

$$REAL(\lambda_i(0)) = \sigma(0) = 0$$

substitution of which simplifies the characteristic equation, and by comparison with the general equation we get:

$$\lambda_3 = -(1/\tau + \alpha/\mu)$$

$$\rho^2 = \alpha/(\mu\tau)$$

$$\lambda_3\rho^2 = -\alpha/\tau$$

Since $\lambda_3(0)$ is real and negative, the non-linear system is locally a.s. at $x_e(0)$, and since $IMAG(\lambda_i(0)) = \rho \neq 0$, we have satisfied two conditions required by the theorem. The bifurcation point can be determined now. Since;

$$\lambda_3\rho^2 = -\alpha/\tau = -\alpha(1/\tau + \alpha/\mu)/\mu\tau$$

by algebraic simplification we obtain the solution to α, as:

$$\alpha_c = (1 - 1/\mu\tau)\mu^2$$

or, $\xi = (\alpha_c - \alpha)$ with $\xi = 0$ as the possible bifurcation point.

Finally, we need to check that $REAL(d\lambda_i(0)/d\xi) > 0$. Noting that $d\xi = -d\alpha$, by differentiating the characteristic equation with respect to α, followed by the substitution $\xi = 0$, i.e., $\alpha = \alpha_c$ gives:

$$\frac{d\lambda(0)}{d\xi} = \frac{1/\tau + \lambda/\mu\tau + \lambda^2/\mu}{3\lambda^2 + 2\lambda(1/\tau + \alpha_c/\mu) + \alpha_c/\mu\tau}$$

Since $\lambda(0) = \pm j\rho(0)$ in the above equation, a complex fraction results, of which the real part is extractable as:

$$REAL(d\lambda_i(0)/d\xi) = \frac{(1/\tau - \rho^2/\mu)(\alpha_c/\mu\tau - 3\rho^2) + 2\rho^2/\mu\tau(1/\tau + \alpha_c/\mu)}{(\alpha_c/\mu\tau - 3\rho^2)^2 + 4\rho^2(1/\tau + \alpha_c/\mu)^2}$$

of which the denominator is positive, and the numerator further simplifies to give $2\alpha_c(\mu\tau - 1)/\mu\tau^3$ which is positive for $\mu\tau > 1$, thus satisfying the required condition.

Thus for small enough $\xi > 0$, there exists a stable limit cycle about $x_e(\xi)$, by implication of the theorem. Alternatively, from a dynamic theoretic point of view, for $0 < \alpha < \alpha_c$, a local instability gives way to a globally stable closed orbit, and for $\alpha > \alpha_c$, a globally unstable cycle degenerates into a stable point.

6 REFERENCES

[Agarwala et al., 1992] A.K. Agarwala, D.Sanghi. *Design and Evaluation of an Adaptive Flow Control Scheme.* IEEE INFOCOM, pp. 11.A.3.7 - 11.A.3.7, 1992.

[Beltrami 1987] E. Beltrami Mathematics for Dynamic Modelling *Academic Press*, 1987.

[Bolot et al., 1990] J.C. Bolot, A.U. Shankar. *Dynamical Behaviour of Rate Based Flow Control Mechanisms.* Computer Communication Review, Vol.20, No.2, pp.35-90, April 1990.

[Bolot et al., 1992] J.-C. Bolot, A.U.Shankar. *Analysis of a Fluid Approximation to Flow Control Dynamics.* IEEE INFOCOM, pp.11.A.4.1 - 11.A.4.10, 1992.

[Chiu et al., 1989] D.Chiu, R.Jain. *Analysis of the Increased and Decrease Algorithms for Congestion Avoidance in Computer Networks.* Computer Networks and ISDN Systems, Vol.17, No.1, pp.1-14, June 1989.

[Fendick et al., 1992] K.W.Fendick, M.A.Rodrigues, A.Weiss. *Analysis of a Rate Based Control with Delayed Feedback.* ACM SIGCOMM, Vol.22. No.4, pp.136-148, 1992.

[Fernow et al., 1985] j.P.Fernow, M.L.El-Sayed. *Stability of Adaptive Congestion Controls in Packet Networks.* Computer Networks and ISDN Systems, Vol.10, No.?. pp.7-18, 1985.

[Filipiak 1988]. J.Filipiak. Modelling and Control of Dynamic Flows in Computer Networks. *Springer-Verlag*, 1988.

[Gerla et al., 1980] M.Gerla, L. Klienrock. *Flow Control. A Comparative Survey*, IEEE Transaction on Communications, Vol. 28, No.4, pp.553-574, April 1980.

[ISIM 1987] Interactive Simulation Users Manual, *Manchester*, 1987.

[Jacobson, 1988] V.Jacobson. *Congestion Avoidance and Control.* ACM SIGCOMM, Vol.18, No.?, pp.314-329, August 1988.

[Jain, 1989] R.Jain. *A Delay Based Approach for Congestion Avoidance in Interconnected Heterogeneous Computer Networks.* Computer Communication Review, Vol.19, No.5, pp.56-71, October 1989.

[Keshav, 1991] S. Keshav. *A Control Theoretic Approach to Flow Control.* ACM SIGCOMM, Vol.21, No.4, pp.3-15, September 1991.

[Mitra et al., 1990] D. Mitra, J.B.Seery. *Dynamic Adaptive Windows for High Speed Data Networks: Theory and Simulations (extended abstract).* ACM SIGCOMM, Vol.20, No.4, pp.30-40, 1990.

[Mukherjee et al., 1991] A.Mukherjee, J.C.Strikwerda. *Analysis of Dynamic Congestion Control Protocols - A Fokker Plank Approximation.* ACM SIGCOMM, Vol.21, No.4, pp.159-161, 1991.

[Ohta, 1988] M. Ohta. *Effects of Control Delay and Transmission Delay on Congestion Controls.* Proceedings of the 9th International Conference on Communications (ICCC), pp.127-131, 1988.

[Ramakrishnan et al., 1990] K.K.Ramakrishnan, R.Jain. *A Binary Feedback Scheme for Congestion Avoidance in Computer Networks.* ACM Transactions on Computer Systems, Vol.8, No.2, pp.158-181, May 1990.

[Ranasinghe, 1994] D.N.Ranasinghe. On a class of Feedback Rate Control Mechanisms in Computer Networks - Ph.D. thesis (submitted). *University of Wales*, 1994.

[Sage et al., 1977] A.P.Sage, C.White. Optimum Systems Control. *Prentice Hall*, 1977.

[Shenker, 1990] S.Shenker. *A Theoretical Analysis of Feedback Flow Control.* ACM SIGCOMM, Vol.20, No.4, pp.156-165, 1990.

[Stidam, 1985] S. Stidam. *Optimal Control of Admission to a Queuing System.* IEEE Transactions on Automatic Control, Vol.30, No.8, pp.705-713, August 1985.

[Wang et al., 1991]Y.T.Wang, B.T.Sengupta. *Performance Analysis of a Feedback Congestion Control policy Under Non-negligible Propagation Delay.* ACM SIGCOMM, Vo.21, No.4, pp.149-157, 1991.