13

Algorithms for Reliability-Based Optimal Design[†]

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We present a new formulation for a class of structural optimization problems with reliability constraints, which makes it possible to use a new outer approximations algorithm for their solution. Our numerical results support the viability of our approach.

1. INTRODUCTION

This paper deals with the problem of minimizing the initial cost of a structure subject to a minimum reliability requirement. As a measure of reliability we make use of the first-order reliability index, i.e., the minimum distance from the origin to the limit-state surface in the standard normal space [8]. This measure is readily applicable to component reliability problems governed by a single, nearly flat limit-state surface, and this case is the main focus of this paper. However, the proposed formulation is also applicable to series system problems that are governed by a multitude of nearly flat limit-state surfaces, provided the reliability constraint is expressed in terms of separate constraints on each failure mode.

Structural optimization problems with a reliability constraint of the type described above have been studied in numerous previous papers (see [10] for a comprehensive bibliography). However, in virtually all of these previous studies a straightforward approach is used with "off-the-shelf" algorithms.

In most cases a two-level optimization problem is solved (see, e.g., [1]). For a given value of the design parameters, the reliability index is obtained by minimizing the distance from the origin to the limit-state surface, and its gradient with respect to the design parameters is computed. These are used in an outer algorithm to guide the search for a new value of the design parameters that attempts to minimize the cost function while satisfying the reliability constraint. To our knowledge, no proof of convergence of such algorithms has been presented. An alternative approach has been to replace the minimization in the computation of the reliability index with first order optimality conditions, as a constraint. Unfortunately, this approach requires second derivatives of the limit-state functions. Furthermore, as far as we know, no serious attempt has been made to take advantage of the special form of the structural reliability problem.

In this paper we present a new mathematical formulation of the optimization problem with a reliability constraint and propose an ''outer approximations'' algorithm for solving it. The algorithm does not require repeated computation of the reliability index or its sensitivities, nor does it require the second derivatives of the cost or limit-state functions. Most importantly, the algorithm has proven convergence properties, which include an accounting

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for approximations in the various steps of the algorithm.

Initially, the formulation is presented for a component reliability problem. The formulation is then extended to a series system problem with a separate reliability constraint on each of its modes. Numerical results highlight the efficiency and robustness of the proposed algorithm.

Given $x \in \mathbb{R}^n$, $n > 1$, we denote its components by $x^1, x^2, \cdots, x^n$. Subscripts are used to denote elements of a sequence, as in $\{ x_i \}_{i=0}^{\infty}$.

2. STATEMENT OF THE PROBLEM

We begin by developing two alternative mathematical formulations of the simplest structural optimization problems with reliability constraints. Let $f^0 : \mathbb{R}^d \to \mathbb{R}$ be a cost function, and $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ be a limit-state function defined on the n-dimensional standard normal space, obtained by transforming the basic random variables of the problem, and parametrized by a vector $p \in \mathbb{R}^d$ of design parameters. We assume that $f^0(\cdot)$ and $g(\cdot,\cdot)$ are twice continuously differentiable, and that the design variables must be chosen so that they satisfy m constraints of the form

$$f^j(p) \le 0, j \in \mathbf{m}, \tag{2.1}$$

where the $f^j : \mathbb{R}^d \to \mathbb{R}$ are twice continuously differentiable functions, and for any positive integer m, $\mathbf{m} \triangleq \{ 1, \dots, m \}$.

For each design vector $p \in \mathbb{R}^d$, let $\beta(p)$ denote the corresponding first-order reliability index, i.e.,

$$\beta(p) \triangleq \min_{u \in \mathbb{R}^n} \{ \|u\| \mid g(u,p) \le 0 \}. \tag{2.2}$$

Let $r > 0$ be a required lower bound on the first order reliability index. Then the simplest formulation of an optimal design problem, subject to a reliability constraint has the form

$$\mathbf{P}_1 \quad \min \{ f^0(p) \mid \beta(p) \ge r, f^j(p) \le 0, j \in \mathbf{m} \}. \tag{2.3}$$

For each design vector $p \in \mathbb{R}^d$ let $\mathbf{F}_p$ denote the corresponding failure domain, i.e.,

$$\mathbf{F}_p \triangleq \{ u \in \mathbb{R}^n \mid g(u,p) \le 0 \}. \tag{2.4}$$

In view of (2.2), we have that

$$\beta(p) \ge r \text{ if and only if } \mathbf{F}_p \cap \mathbf{B}_r = \emptyset, \tag{2.5a}$$

where $\mathbf{B}_r$ denotes the open ball of radius r and center at the origin. To obtain a mathematical reformulation of $\mathbf{P}_1$ using (2.5a) we introduce the following technical assumption which rules out a most unlikely situation:

Assumption 2.1 For all $p \in \mathbb{R}^d$ and $u \in \mathbb{R}^n$ such that $g(u,p) = 0$, $\nabla_u g(u,p) \ne 0$ (i.e., for all points on the limit-state surface, the gradient of the limit-state function is nonzero). □

When Assumption 2.1 is satisfied, $\mathbf{F}_p \cap \mathbf{B}_r = \emptyset$ if and only if

$$\min_{u \in \overline{\mathbf{B}}_r} g(u,p) \ge 0, \tag{2.5b}$$

where $\overline{\mathbf{B}}_r$ denotes the closed ball of radius r centered at the origin. Hence problem $\mathbf{P}_1$ is equivalent to the following mathematical alternative

$$\mathbf{P}_{1,\infty} \quad \min\{ f^0(p) \mid \max_{u \in \bar{\mathbf{B}}_r} \tilde{g}(u,p) \le 0,\ f^j(p) \le 0,\ j \in \mathbf{m} \}, \tag{2.6}$$

where $\tilde{g}(\cdot,\cdot) \triangleq -g(\cdot,\cdot)$.

3. OUTER APPROXIMATIONS ALGORITHMS

Problem $\mathbf{P}_{1,\infty}$ is a nonlinear programming problem with an infinite number of constraints, since the inequality

$$\max_{u \in \bar{\mathbf{B}}_r} \tilde{g}(u,p) \le 0 \tag{3.1a}$$

is equivalent to the infinite system of inequalities

$$\tilde{g}(u,p) \le 0,\ \forall u \in \bar{\mathbf{B}}_r . \tag{3.1b}$$

Optimization problems with such constraints are known as semi-infinite optimization problems, because the design vector is finite-dimensional, but the number of constraints is infinite. There is a rather large literature dealing with the numerical solution of such problems (see, e.g., the review papers [3,9]). A major source of difficulty in solving $\mathbf{P}_{1,\infty}$ is the fact that in structural design the ball $\bar{\mathbf{B}}_r$ is high dimensional, i.e., $\bar{\mathbf{B}}_r \subset \mathbb{R}^n$ with n large. Hence $\mathbf{P}_{1,\infty}$ cannot be solved using discretization techniques as in [9].

An alternative approach is provided by the method of outer approximations, see, e.g., [2]. First observe that given any set $\mathbf{U}_k \triangleq \{u_1, u_2, \cdots, u_k\} \subset \bar{\mathbf{B}}_r$, the problem

$$\mathbf{P}_{1,k} \quad \min\{ f^0(p) \mid \max_{u \in \mathbf{U}_k} \tilde{g}(u,p) \le 0, f^j(p) \le 0, j \in \mathbf{m} \}, \tag{3.2a}$$

has finitely many constraints. Instead of solving the original problem $\mathbf{P}_{1,\infty}$, we will solve a sequence of problems of the form $\mathbf{P}_{1,k}$. The method is called outer-approximations because the feasible set for each $\mathbf{P}_{1,k}$ contains the feasible set for $\mathbf{P}_{1,\infty}$. Next, assume that $\hat{p}_k$ solves $\mathbf{P}_{1,k}$. Then $\hat{p}_k$ also solves $\mathbf{P}_{1,\infty}$ if

$$\max_{u \in \mathbf{U}_k} \tilde{g}(u,\hat{p}_k) = \max_{u \in \bar{\mathbf{B}}_r} \tilde{g}(u,\hat{p}_k). \tag{3.2b}$$

At the start, we do not know how to choose a set $\mathbf{U}_k$ such that (3.2b) is satisfied. Hence methods of outer approximations accumulate a satisfactory approximation to such a set by using points that satisfy approximately the relationship

$$u_j \in arg \max_{u \in \bar{\mathbf{B}}_r} \tilde{g}(u,p_j), \tag{3.2c}$$

with p_j an approximate solution to $\mathbf{P}_{1,\infty}$, as we will explain more precisely below by stating an algorithm. In practice, methods of outer approximations have been found to be very efficient.

It is easier to grasp the nature of outer approximations algorithms by first viewing them in a simple, "conceptual" form, such as the following algorithm for solving $\mathbf{P}_{1,\infty}$. We call this algorithm conceptual because it contains steps (Step 1 and Step 2) that can be performed only if global optimization techniques are used, which is completely out of the question for most real life structural optimization problems of the form $\mathbf{P}_{1,\infty}$.

Conceptual Algorithm 3.1.

Data. $p_1 \in \mathbb{R}^d$, $\mathbf{U}_0 = \emptyset$.

Step 0. Set $i = 1$.

Step 1. Compute a maximizer u_i for the problem

$$\max_{u \in \bar{\mathbf{B}}_r} \tilde{g}(u, p_i). \tag{3.3a}$$

Step 2. Set $\mathbf{U}_i = \mathbf{U}_{i-1} \cup \{ u_i \}$, and compute a minimizer p_{i+1} for the problem

$$\min_{p \in \mathbb{R}^d} \{ f^0(p) \mid \tilde{g}(u, p) \le 0, \ u \in \mathbf{U}_i, \ f^j(p) \le 0, j \in \mathbf{m} \}. \tag{3.3b}$$

Step 3. Replace i by $i + 1$ and go to Step 1. □

Conceptual Algorithm 3.1 has the following convergence property:

Theorem 3.2 Suppose that Conceptual Algorithm 3.1 has constructed an infinite sequence $\{ p_i \}_{i=1}^{\infty}$. Then every accumulation point of the sequence $\{ p_i \}_{i=1}^{\infty}$ is a solution for $\mathbf{P}_{1,\infty}$. □

The next step is to replace the exact maximization (3.3a) and minimization (3.3b) by appropriate approximations, so as to obtain an implementable algorithm that retains the convergence properties of the Conceptual Algorithm, to the extent that it can be shown to compute stationary points for the problem $\mathbf{P}_{1,\infty}$. We will carry out these modifications in two steps.

First, instead of computing a global minimizer in Step 2 of Conceptual Algorithm 3.1, we will compute an approximate feasible stationary point p_{i+1} for (3.3b). Second, instead of computing a global maximizer in Step 1, we will compute an approximate feasible stationary point u_i for (3.3a). In view of the flatness of the limit-state surface, prevalent in many problems of interest, we introduce the following assumption:

Assumption 3.3. For all $p \in \mathbb{R}^d$ such that $\beta(p) \ge r$, any feasible stationary point of the problem $\min_{u \in \bar{\mathbf{B}}_r} g(u, p)$ is a global minimizer of that problem. □

It is well-known that in the absence of convexity assumptions, nonlinear programming algorithms can only be shown to compute stationary points. Traditionally stationary points are characterized by systems of equations (such as the Kuhn-Tucker or Fritz John conditions). However, for our purposes it is convenient to characterize stationary points as zeros of a continuous, negative-valued optimality function. We will define two families of optimality functions. One for the maximization problems (3.3a), and one for the minimization problems (3.3b). Hence, given $p \in \mathbb{R}^d$, the optimality function $\theta_p : \mathbb{R}^n \to \mathbb{R}$, for (3.3a), is defined by

$$\theta_p(u) \triangleq -\min_{\mu \in \Sigma_2} \{ -\mu^2(\|u\|^2 - r^2) + \tfrac{1}{2}\|\mu^1 \nabla_u g(u, p) + 2\mu^2 u\|^2 \} - \max\{ 0, \|u\|^2 - r^2 \} \tag{3.4a}$$

where for any positive integer q, Σ_q is the q-simplex, i. e.,

$$\Sigma_q \triangleq \{ \mu = (\mu^1, \cdots, \mu^q) \in \mathbb{R}^q \mid \mu^j \geq 0, j \in \mathbf{q}, \text{ and } \sum_{j=1}^{q} \mu^j = 1 \}. \tag{3.4b}$$

Note that at any $u \in \overline{\mathbf{B}}_r$, the minimand in (3.4a) is the sum of two non-negative terms. Hence it is zero if and only if both of these terms are zero, which, in turn, implies that the Fritz John optimality conditions for (3.3a) are satisfied. Next, given a set $\mathbf{U}_i = \{ u_1, \cdots, u_i \} \subset \overline{\mathbf{B}}_r$, let the functions $f^j : \mathbb{R}^d \to \mathbb{R}$, $j = m+1, \cdots, m+i$ be defined by

$$f^{m+j}(p) \triangleq \tilde{g}(u_j, p), \quad j \in \mathbf{i}. \tag{3.5}$$

Then (3.3b) can be rewritten as $\min \{ f^0(p) \mid f^j(p) \leq 0,\ j = 1, \ldots, m+i \}$. We define the optimality function $\Theta_{\mathbf{U}_i} : \mathbb{R}^d \to \mathbb{R}$, for (3.3b), as follows:

$$\Theta_{\mathbf{U}_i}(p) \triangleq - \min_{(\mu^0, \mu) \in \Sigma_{m+i+1}} \{ -\sum_{j=1}^{m+i} \mu^j f^j(p) + \tfrac{1}{2} \| \sum_{j=0}^{m+i} \mu^j \nabla f^j(p) \|^2 \} - \psi(p)_+, \tag{3.6}$$

where

$$\psi(p)_+ \triangleq \max_{1 \leq j \leq m+i} \{ 0, f^j(p) \}. \tag{3.7}$$

Again, note that at any feasible point p, the minimand in (3.6) is the sum of two non-negative terms. Hence it is zero if and only if both of these terms are zero, which, in turn, implies that the Fritz John optimality conditions for (3.3b) are satisfied.

The key properties of $\theta_p(\cdot)$ and $\Theta_{\mathbf{U}_i}(\cdot)$ are given by the following result:

Theorem 3.4 Given any $p \in \mathbb{R}^d$ and $\mathbf{U}_i = \{ u_1, u_2, \ldots, u_i \} \subset \overline{\mathbf{B}}_r$, we have

(a) The functions $\theta_p : \mathbb{R}^n \to \mathbb{R}$ and $\Theta_{\mathbf{U}_i} : \mathbb{R}^d \to \mathbb{R}$ are well-defined, continuous and take values in $(-\infty, 0]$.

(b) $\theta_p(u) = 0$ if and only if $u \in \mathbb{R}^n$ satisfies the Fritz John optimality conditions for problem (3.3a), that is, $\theta_p(u) = 0$ if and only if u is a stationary point for the maximization problem in Step 1 of Conceptual Algorithm 3.1.

(c) $\Theta_{\mathbf{U}_i}(p) = 0$ if and only if $p \in \mathbb{R}^d$ satisfies the Fritz John optimality conditions for problem (3.3b), that is, $\Theta_{\mathbf{U}_i}(p) = 0$ if and only if p is a stationary point for the minimization problem in Step 2 of Conceptual Algorithm 3.1. □

It is clear from Theorem 3.4 that if the value of the optimality function at a point is close to zero, then that point is an approximate stationary point. We use this fact to construct the following implementable algorithm:

Algorithm 3.5.

Parameters. $C_1, C_2 \in (0, \infty)$.

Data. $p_1 \in \mathbb{R}^d$, $\mathbf{U}_0 = \emptyset$.

Step 0. Set $i = 1$.

Step 1. Compute $u_i \in \overline{\mathbf{B}}_r$ such that $\theta_{p_i}(u_i) \geq -C_1/i$.

Step 2. Set $\mathbf{U}_i = \mathbf{U}_{i-1} \cup \{ u_i \}$, and compute $p_{i+1} \in \mathbb{R}^d$ such that $\Theta_{\mathbf{U}_i}(p_{i+1}) \geq -C_2/i$.

Step 3. Replace i by $i + 1$ and go to Step 1. □

The convergence properties of Algorithm 3.5 are given by the following result:

Theorem 3.6. Suppose that Assumption 3.3 is satisfied and that Algorithm 3.5 has constructed an infinite sequence $\{p_i\}_{i=1}^{\infty}$. Then any accumulation point of $\{p_i\}_{i=1}^{\infty}$ is a stationary point for $\mathbf{P}_{1,\infty}$. □

Algorithm 3.5 is an implementable algorithm since there exist mathematical programming algorithms capable of performing Steps 1 and 2 in a finite number of iterations. In Section 5, where we present some computational results, we use CFSQP ([4]) to perform Steps 1 and 2 of Algorithm 3.5.

4. EXTENSION TO MULTIPLE LIMIT-STATE FUNCTIONS

It is possible to extend our implementable algorithm and the corresponding theory to more general problems. In this section we show how to apply the ideas described above to the optimization of a series system with a separate reliability constraint on each of its modes.

Let $g^k : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, $k \in \mathbf{q}$, be twice continuously differentiable limit-state functions defined in the n-dimensional standard normal space and parametrized by $p \in \mathbb{R}^d$. Given $r \in \mathbb{R}^q$, with $r^k > 0$, $k \in \mathbf{q}$, we will consider optimal design problems of the following form:

$$\mathbf{P}_2 \quad \min\{ f^0(p) \mid \beta^k(p) \ge r^k \,,\ k \in \mathbf{q} \,,\ f^j(p) \le 0 \,,\ j \in \mathbf{m} \} \,, \tag{4.1}$$

where, for each $k \in \mathbf{q}$, $\beta^k(p)$ is the first-order reliability index for the limit-state function $g^k(\cdot, p)$, defined as in (2.2). Each of the q reliability constraints in (4.1) can be dealt with using the same ideas applied to deal with the unique reliability constraint in (2.3).

For each $p \in \mathbb{R}^d$, let $\mathbf{F}_p^k$ denote the failure domain of the k-th component, i.e.,

$$\mathbf{F}_p^k \triangleq \{ u \in \mathbb{R}^n \mid g^k(u, p) \le 0 \} \,, \quad k \in \mathbf{q}. \tag{4.2}$$

Then we have

$$\min_{k \in \mathbf{q}} \beta^k(p) \ge r^k \text{ if and only if } \mathbf{F}_p^k \cap \mathbf{B}_{r^k} = \emptyset \,, \text{ for all } k \in \mathbf{q}. \tag{4.3a}$$

If all limit-state functions $g^k(\cdot,\cdot)$, $k \in \mathbf{q}$, satisfy Assumption 2.1, then $\mathbf{F}_p^k \cap \mathbf{B}_{r^k} = \emptyset$, for all $k \in \mathbf{q}$, if and only if

$$\min_{k \in \mathbf{q}} \ \min_{u \in \bar{\mathbf{B}}_{r^k}} g^k(u, p) \ge 0 \tag{4.3b}$$

from which follows that $\mathbf{P}_2$ is equivalent to the following problem:

$$\mathbf{P}_{2,\infty} \quad \min\{ f^0(p) \mid \max_{k \in \mathbf{q}} \ \max_{u \in \bar{\mathbf{B}}_{r^k}} \tilde{g}^k(u, p) \le 0,\ \max_{j \in \mathbf{m}} f^j(p) \le 0 \} \,, \tag{4.4}$$

where $\tilde{g}^k(\cdot,\cdot) \triangleq -g^k(\cdot,\cdot)$, $k \in \mathbf{q}$. □

The following conceptual algorithm solves $\mathbf{P}_{2,\infty}$:

Conceptual Algorithm 4.1.

Data. $p_1 \in \mathbb{R}^d$, $\mathbf{U}_0^k = \emptyset$, $k \in \mathbf{q}$.

Step 0. Set $i = 1$.

Step 1. Compute maximizers u_i^k, $k \in \mathbf{q}$, for the problems

$$\max_{u \in \bar{\mathbf{B}}_{r^k}} \tilde{g}^k(u, p_i), \quad k \in \mathbf{q}. \tag{4.5a}$$

Step 2. Set $\mathbf{U}_i^k = \mathbf{U}_{i-1}^k \cup \{ u_i^k \}$, $\tilde{\mathbf{U}}_i = \bigcup_{k=1}^{q} \mathbf{U}_i^k$ and compute a minimizer p_{i+1} for the problem

$$\min_{p \in \mathbb{R}^d} \{ f^0(p) \mid \tilde{g}^k(u, p) \le 0,\ u \in \tilde{\mathbf{U}}_i,\ f^j(p) \le 0,\ j \in \mathbf{m} \}. \tag{4.5b}$$

Step 3. Replace i by $i + 1$ and go to Step 1. □

It can be shown that any accumulation point of a sequence $\{ p_i \}_{i=1}^{\infty}$ constructed by Conceptual Algorithm 4.1 is a stationary point for $\mathbf{P}_{2,\infty}$.

To obtain an implementable algorithm, we follow the pattern set in Section 3 and replace the exact maximizations in Step 1 and minimization in Step 2 of Conceptual Algorithm 4.1 by appropriate implementable approximations.

Recall that in Section 3 for a given $p \in \mathbb{R}^d$ we defined an optimality function $\theta_p(\cdot)$ for problem (3.3a). In (4.5a) we have q problems of the same form as (3.3a), and hence we define q optimality functions $\theta_{k,p} : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathbf{q}$, similarly to (3.4a).

Next, given a finite set $\tilde{\mathbf{U}}_i \subset \bigcup_{k=1}^{q} \bar{\mathbf{B}}_{r^k}$, we define an optimality function $\bar{\Theta}_{\tilde{\mathbf{U}}_i} : \mathbb{R}^d \to \mathbb{R}$ following the pattern set in (3.6), (3.7). It can be shown that the optimality functions $\theta_{k,p}(\cdot)$ and $\bar{\Theta}_{\tilde{\mathbf{U}}_i}(\cdot)$ have properties similar to those stated in Theorem 3.4 for $\theta_p(\cdot)$ and $\Theta_{\mathbf{U}_i}(\cdot)$ respectively. These optimality functions can be used to construct an implementable algorithm as shown below.

Algorithm 4.3.

Parameters. $C_1, C_2 \in (0, \infty)$.

Data. $p_1 \in \mathbb{R}^d$, $\mathbf{U}_0^k = \emptyset$, $k \in \mathbf{q}$.

Step 0. Set $i = 1$.

Step 1. Compute $u_i^k \in \bar{\mathbf{B}}_{r^k}$, $k \in \mathbf{q}$, such that $\theta_{k,p_i}(u_i^k) \ge -C_1/i$, $k \in \mathbf{q}$,

Step 2. Set $\mathbf{U}_i^k = \mathbf{U}_{i-1}^k \cup \{ u_i^k \}$, $\tilde{\mathbf{U}}_i = \bigcup_{k=1}^{q} \mathbf{U}_i^k$ and compute $p_{i+1} \in \mathbb{R}^d$ such that $\Theta_{\tilde{\mathbf{U}}_i}(p_{i+1}) \ge -C_2/i$,

Step 3. Replace i by $i + 1$ and go to Step 1. □

The convergence properties of Algorithm 4.3 are given by the following result:

Theorem 4.4. Suppose that all $g^k(\cdot,\cdot)$, $k \in \mathbf{q}$, satisfy Assumption 3.3 and that Algorithm 4.3 has constructed an infinite sequence $\{ p_i \}_{i=1}^{\infty}$. Then any accumulation point of $\{ p_i \}_{i=1}^{\infty}$ is a stationary point for $\mathbf{P}_{2,\infty}$. □

5. IMPLEMENTATION AND NUMERICAL RESULTS

We have used Algorithm 3.5 to solve the problem of determining the depth h, and the width b of a short column with rectangular cross section, so that the total mass is minimized, while the first-order reliability index for the fully plastic mode of failure is kept greater than or equal to 2.5. We assumed that the column is made of elastic-perfectly-plastic material with yield stress Y in both directions. Let M and P denote the bending moment and axial force applied to the column. The limit-state function in terms of the variables (P, M, Y) is given by

$$G(x, p) = 1 - \frac{4M}{bh^2Y} - \frac{P^2}{(bhY)^2}. \tag{5.1}$$

where $x \triangleq (P, M, Y)^T$. We assumed that (P, M, Y) has a Nataf-type (see [5]) joint distribution with marginals and correlation coefficients as shown in the table below:

variable	statistics	correlation coefficients		
		P	M	Y
P	N(500,100)	1.0	0.5	0.0
M	N(2000,400)	0.5	1.0	0.0
Y	LN(5,0.5)	0.0	0.0	1.0

We defined the limit-state function on 3-dimensional standard normal space by introducing an invertible transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ (see [5] for details), mapping $x = (P, M, Y)$ into the 3-dimensional standard normal space. The limit-state function $g:\mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ we obtained is of the form

$$g(u, p) \triangleq G(T^{-1}(u), p). \tag{5.2}$$

Finally, we assumed that the width b and the depth h had to satisfy $5 \le b \le 15$ and $15 \le h \le 25$. The mathematical formulation of our example optimal design problem is as follows:

$$\min_{p=(b,h)} \{ bh \mid \beta(p) \ge 2.5,\ 5 \le b \le 15,\ 15 \le h \le 25 \}. \tag{5.3}$$

We used the program CalRel [7] to implement the transformations T and T^{-1} and to evaluate the limit-state function $g(\cdot,\cdot)$ and its derivatives. We carried out the computations, required in Steps 1 and 2 of Algorithm 3.5, by means of the sequential quadratic programming software CFSQP [4].

To evaluate our algorithm, we solved the design problem using a two-level optimization algorithm based on CFSQP, which treats the reliability constraint as nonlinear inequality constraints. Every time CFSQP required the evaluation of the reliability index $\beta(p)$ or its gradient, it called CalRel to perform this computations using the modified HL-RF algorithm [6].

Starting from the initial point $(b_1, h_1) = (5.0,\ 15.0)$, both algorithms converged to $(b, h) = (8.668, 25.0)$. Our algorithm took 14 iterations with 98 evaluations of the limit-state function and 77 evaluations of its gradient. The nested optimization algorithm took 227 evaluations of the limit-state function and 227 evaluations of its gradient.

We note that our implementation of the outer approximations algorithm did not include features such as adaptive precision and constraint dropping schemes (see [2]), which should improve its performance considerably.

6. CONCLUSION

We have presented a new formulation of certain optimal design problems subject to reliability constraints and a new outer approximations algorithm for their solution. Our method does not require the computation of second-derivatives of the limit-state function, nor does it require repeated computation of the first order reliability index. Our preliminary results show that our algorithm outperforms currently used alternatives.

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