Ambiguity and Complementation in Recognizable Two-dimensional Languages

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1 Introduction

The theory of one-dimensional (word) languages is well founded and investigated since fifties. From several years, the increasing interest for pattern recognition and image processing motivated the research on *two-dimensional* or *picture* languages, and nowadays this is a research field of great interest. A first attempt to formalize the concept of finite state recognizability for two-dimensional languages can be attributed to Blum and Hewitt ([7]) who started in 1967 the study of finite state devices that can define two-dimensional languages, with the aim to finding a counterpart of what regular languages are in one dimension. Since then, many approaches have been presented in the literature following all classical ways to define regular languages: finite automata, grammars, logics and regular expressions.

In 1991, a unifying point of view was presented in [13] where the family of *tiling recognizable picture languages* is defined (see also [14]). The definition of recognizable picture language takes as starting point a well known characterization of recognizable word languages in terms of *local* languages and *projection*. Namely, any recognizable word language can be obtained as projection of a local word language defined over a larger alphabet. Such notion can be extended in a natural way to the two-dimensional case: more precisely, local picture languages are defined by means of a set of square arrays of side-length two (called *tiles*) that represents the only allowed blocks of that size in the pictures of the language (with special treatment for border symbols). Then, we say that a picture language is *tiling recognizable* if it can be obtained as a projection of a local picture language. The family of all tiling recognizable picture languages is called REC. Remark that, when we consider words as particular pictures (that is pictures in which one side has length one), this definition of recognizability coincides with the one for the words, i.e. the definition given in terms of finite automata.

The family REC can be characterized by several formalisms such as different variants of tiling systems, on-line tessellation automata, Wang systems, existential monadic second order logic, "special" regular expressions, etc.

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(see [10, 14, 16, 19]). The number of different characterizations indicates that (tiling) recognizable picture languages form a robust a therefore somewhat natural class to study. Further this class inherits most of the important properties from the class of regular word languages (see also [16]). Moreover tiling recognizable picture languages have been considered and appreciated in the image processing and pattern recognition fields (see [9]).

On the other hand, recognizable picture languages do not share some properties that are fundamental in the theory of recognizable word languages. The first big difference regards the complement operation. It be proved (see [14]) that, contrary to the one-dimensional case, the family REC is not closed under complementation. As a consequence, it is interesting to consider the family $REC \cup co-REC$ of picture languages L such that either L itself or its complement ${}^cL$ is tiling recognizable. One has that REC is strictly included in $REC \cup co-REC$. An interesting problem (the *complement problem*) is to search for conditions on a picture language L such that both L and ${}^cL$ are tiling recognizable.

The non closure under complementation is related to the fact that the definition of recognizability in terms of tiling systems, i.e. in terms of local languages and projections, is implicitly non-deterministic. However, contrary to the one-dimensional case, does not exist a unique and clear notion of determinism in two dimensions (see [1]). A notion that indeed can be naturally expressed in terms of tiling systems is the notion of *ambiguity.* Informally, a tiling system is *unambiguous* if every picture has a unique counter-image in its corresponding local language. Observe that an unambiguous tiling system can be viewed as a generalization in two dimensions of the definition of unambiguous automaton that recognizes a word language. A recognizable two-dimensional language is *unambiguous* if it is recognized by a unambiguous tiling system.

We denote by $UREC$ the family of all unambiguous recognizable picture languages. Obviously it holds true that $UREC \subseteq REC$. Remark that, in the one dimensional case, $UREC$ is equal to REC. In [3], it is shown that it is *undecidable* whether a given tiling system is unambiguous. Furthermore some closure properties of $UREC$ are proved. The main result in [3] is that, for pictures, $UREC$ is strictly included in REC. In other words, there exist picture languages in REC that are *inherently ambiguous.*

The aim of this paper is to shed new light on the relations between the complement problem and the unambiguity in the family of recognizable picture languages. Remark that the interest for such relations was also raised by W. Thomas in [24].

Following some ideas in [15], we present a novel general framework to study properties of recognizable picture languages and then use it to study the relations between classes $REC \cup co-REC$, REC and $UREC$. The strict inclusions among these classes have been proved in [8], [20], [3], respectively, using ad-hoc techniques. Here we present again those results in a unified formalism and proof method with the major intent of establishing relations between the complement problem and unambiguity in the family of recognizable picture languages.

We consider some complexity functions on picture languages and combine two main techniques. First, following the approach of O. Matz in [20], we consider, for each positive integer m, the set $L(m)$ of pictures of a language L having one dimension (say the vertical one) of size m. Language $L(m)$ can be viewed as a word language over the alphabet (of the columns) $\Sigma^{m,1}$. The idea is then to measure the complexity of the picture language L by evaluating the grow rate, with respect to m, of some numerical parameters of $L(m)$. In order to specify such numerical parameters we make use, as a second technique, of the Hankel matrix of a word language. The parameters are indeed expressed in terms of some elementary matrix-theoretic notions of the Hankel matrices of the word languages $L(m)$. In particular, we consider here three parameters: the *number of different rows*, the *rank*, and the *maximal size of a permutation sub-matrix*.

We state a main theorem that establishes some bounds on corresponding complexity functions based on those three parameters, respectively. Then, as applications for those bounds we analyze the complexity functions of some examples of picture languages in the case of unary alphabet. By means of those languages we re-prove the strict inclusions of families $REC \cup co-REC$, REC and $UREC$ even in the case of unary alphabet.

Moreover we show an example of a language in REC that does not belong to $UREC$ and whose complement is not in REC. This language introduces further discussions on relations between unambiguity and non-closure under complement.

2 Recognizable Two-dimensional languages

In this section we introduce some definitions about two-dimensional languages and their operations. Then we recall definitions and basic properties of tiling recognizable two-dimensional languages firstly introduced in 1992 in [13] that correspond to family REC. Furthermore, we give the definition of unambiguous recognizable picture languages and of class $UREC$. The notations used together with all the results and proofs mentioned here can be found in [14].

Let Σ be a finite alphabet. A *picture* (or *two-dimensional word*) over Σ is a two-dimensional rectangular array of elements of Σ. Given a picture p, let $p(i,j)$ denote the symbol in p with coordinates (i,j), moreover the *size* of p is given by a pair (m,n) where m and n are the number of rows and columns of p, respectively. The set of all pictures over Σ of size (x,y) for all $x,y \geq 1$ is denoted by Σ^{++} and a *two-dimensional language* over Σ is a subset of Σ^{++}. Very often we will refer to two-dimensional languages as *picture languages*. Remark that in this paper we do not consider the case of empty pictures (i.e. pictures where the number of rows and/or columns can be zero). The set of all pictures over Σ of fixed size (m,n), with $m,n \geq 1$ is denoted by $\Sigma^{m,n}$. We give a first example of a picture language.

Example 1. Let L be the language of square pictures over an alphabet Σ:

$$L = \{\, p \mid p \text{ has size } (n, n),\ n > 0 \,\}.$$

Between pictures and picture languages there are defined two different concatenation operations along the horizontal and vertical directions called *column concatenation* and *row concatenation*, respectively. Notice that they are partial operations because they are defined between pictures with same number of rows (for the column concatenation) or same number of columns (for row concatenation). Furthermore, by iterating the concatenation operations, we obtain the column and row *closure* or *star*.

In order to describe recognizing strategies for pictures, it is needed to identify the symbols on the boundary. Then, for any picture p of size (m, n), we consider picture $\widehat{p}$ of size $(m+2, n+2)$ obtained by surrounding p with a special *boundary symbol* $\# \notin \Sigma$. We call *tile* a square picture of dimension $(2, 2)$ and given a picture p we denote by $B_{2,2}(p)$ the set of all blocks of p of size $(2, 2)$.

Let Γ be a finite alphabet. A two-dimensional language $L \subseteq \Gamma^{++}$ is *local* if there exists a finite set Θ of tiles over the alphabet $\Gamma \cup \{\#\}$ such that $L = \{x \in \Gamma^{++} | B_{2,2}(\widehat{x}) \subseteq \Theta\}$. We will write $L = L(\Theta)$. Therefore tiles in Θ represent all the *allowed blocks* of size $(2, 2)$ for the pictures in L. The family of local picture languages will be denoted by LOC. We now give an example of a local two-dimensional language.

Example 2. Let $\Gamma = \{0, 1\}$ be an alphabet and let Θ be the following set of tiles over Γ.

$$\Theta = \left\{ \begin{array}{cccccc}
\begin{array}{|c|c|}\hline 0 & \# \\ \hline 1 & \# \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & \# \\ \hline 0 & \# \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & 0 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & \# \\ \hline\end{array} \\[2ex]
\begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & 0 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & 0 \\ \hline \# & 0 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & 0 \\ \hline \# & \# \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline \# & \# \\ \hline\end{array} &
\begin{array}{|c|c|}\hline \# & 0 \\ \hline \# & \# \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 1 & \# \\ \hline \# & \# \\ \hline\end{array} \\[2ex]
\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline\end{array} & &
\end{array} \right\}$$

The language $L(\Theta)$ is the language of squares pictures (i.e. pictures of size (n, n) with $n \geq 2$) in which all diagonal positions (i.e. those of the form (i, i)) carry symbol 1, whereas the remaining positions carry symbol 0. That is, pictures as the following:

1	0	0,	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

Notice that the language of squares over a one-letter alphabet is not a local language because there is no "local strategy" to compare the number of rows and columns using only one symbol.

Let Γ and Σ be two finite alphabets. A mapping $\pi : \Gamma \to \Sigma$ will be in the sequel called *projection*. The projection $\pi(p)$ of $p \in \Gamma^{++}$ of size (m, n) is the picture $p' \in \Sigma^{++}$ such that $p'(i, j) = \pi(p(i, j))$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. Similarly, if $L \subseteq \Gamma^{++}$ is a picture language over Γ, we indicate by $\pi(L)$ the projection of language L, i.e. $\pi(L) = \{p' | p' = \pi(p),\, p \in L\} \subseteq \Sigma^{++}$.

A quadruple $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$ is called *tiling system* if Σ and Γ are finite alphabets, Θ is a finite set of tiles over $\Gamma \cup \{\#\}$ and $\pi : \Gamma \to \Sigma$ is a projection. Therefore, a tiling system is composed by a local language over Γ (defined by the set Θ) and a projection $\pi : \Gamma \longrightarrow \Sigma$. A two-dimensional language $L \subseteq \Sigma^{++}$ is *tiling recognizable* if there exists a tiling system $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$ such that $L = \pi(L(\Theta))$. Moreover, we will refer to $L' = L(\Theta)$ as an *underling local language* for L and to Γ as a *local alphabet* for L. Let $p \in L$, if $p' \in L'$ is such that $\pi(p') = p$, we refer to p' as a *counter-image* of p in the underling local language L'.

The family of all two-dimensional languages that are *tiling recognizable* is denoted by *REC*. As first example consider the following.

Example 3. Let L be the language of square pictures (i.e. pictures of size (n, n)) over one-letter alphabet $\Sigma = \{a\}$. Language L is in *REC* because it can be obtained as projection of local language in Example 2 by mean of projection $\pi(0) = \pi(1) = a$.

We remark that a tiling system $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$ for a picture language is in some sense a generalization to the two-dimensional case of an automaton that recognizes a word language. Indeed, in one-dimensional case, the quadruple $(\Sigma, \Gamma, \Theta, \pi)$ corresponds exactly to the state-graph of the automaton: the alphabet Γ is in a one-to-one correspondence with the edges, the set Θ describes the edges adjacency, the mapping π gives the labelling of the edges in the automaton. Then, the set of words of the underlying local language defined by set Θ corresponds to all accepting paths in the state-graph and its projection by π gives the language recognized by the automaton. As consequence, when rectangles degenerate in strings the definition of recognizability coincides with the classical one for strings (cf. [11]).

The family *REC* is closed with respect to different types of operations. In particular: the family *REC* is closed under alphabetic projection, under row and column concatenation, under row and column stars and under union and intersection operations (see [14]).

2.1 Examples of recognizable languages

First family of examples of recognizable picture languages can be obtained as immediate application of closure properties. In fact, as we do in the word case, we can define sort of *picture regular expressions* starting from finite languages and using operations of union, intersection, row and column concatenations and closures and projection.

In this way we can list the following as recognizable two-dimensional languages: languages of pictures with odd number of rows, of pictures with even numbers of as, of pictures with first row equal to the last row, of pictures that contains to equal columns and so on.

In some sense we can consider all the properties of recognizable word languages and "make" the corresponding two-dimensional ones and get a recognizable two-dimensional language. But this does not exhausts the family of all recognizable two-dimensional languages! In fact going from one to two dimensions, such generalization of finite automata can recognize much more properties.

As first example, consider the set of pictures over $\Sigma = \{\{a, b\}$ of size $(n, 2n)$ where the first row is the word $a^n b^n$. The tiling system for this language is quite straightforward. Furthermore, in [26] it is proved that even the language of pictures over $\Sigma = \{\{a, b\}$ where the number of as is equal to the number of bs (providing that the size (m, n) of the pictures is such that $m \leq 2^n$ and $n \leq 2^m$). Therefore in two dimensions we can "count" within a recognizable setting.

Another way to interpret a picture over a two-letters alphabet $\Sigma = \{\{a, b\}$, more in the spirit of pattern recognition, is to consider, for example, the as as background and the bs as the "figure". In [25] it is exhibited a tiling system for the language of connected figures.

Very interesting is the examples of *Chinese boxes* in [9]. Pictures are defined on $\{\{0, 1\}$ alphabet and contain rectangular frames or boxes, placed anywhere. Frames may be nested one inside the other but they may not overlap, touch each other, or touch the border. The perimeter of a frame are encoded by 1 and the background by 0 symbols. It is proved that Chinese boxes are recognizable. Remark that Chinese boxes can be viewed as the two-dimensional version of the "well-formed parenthesis languages" that is not regular in one-dimension.

A family of recognizable two-dimensional languages that is worthwhile to consider are the languages of *pictures on one-letter alphabet*. This corresponds also to consider the *shapes* of the pictures without looking to the inside contents.

Remark that, in this case, a picture is defined by a pair of positive numbers corresponding to its size (m, n) and then a picture language is a set of pairs of natural numbers. Furthermore, given a function f defined on the set of natural numbers, one can consider the set of pictures of sizes $(n, f(n))$ for each n. It can be proved that several families of functions are tiling recognizable like polynomial and exponential functions (see [12] or [14]).

Alternatively, given a set of natural numbers, one can consider the set of square pictures of corresponding sizes. There are some surprising sets of recognizable numbers. One for all, the set of *primes* is proved to be tiling recognizable in [5] where it is also given a characterization involving the Turing Machine.

2.2 Ambiguity and complementation

The examples in previous section indicate that tiling systems are devices having a strong expressive power. Let us observe that, in the one-dimensional case, "well-formed parenthesis" and "counting" are some kind of prototype concepts for non recognizability. On the contrary, examples in the previous section show that the natural extensions of such concepts to two-dimensions define picture languages that are tiling recognizable. So the notion of (tiling) recognizability appears to have, in two dimensions, a stronger expressive power with respect to the one-dimensional case.

At the same time, recognizable picture languages do not share some properties that are fundamental in the theory of recognizable word languages. The first big difference regards the complement operation. In [14], using a combinatorial argument, it is showed that language in Example 6 is not tiling recognizable while it is not difficult to write a picture regular expressions for its complement. This proves the following theorem.

Theorem 1. *REC is* not *closed under complement.*

As consequence of this theorem, it is interesting to consider the family $REC \cup co{-}REC$ of picture languages L such that either L itself or its complement ${}^{c}L$ is tiling recognizable. Previous theorem states that REC is strictly included in $REC \cup co{-}REC$.

Closure by complement for a family of languages is usually related to the existence of a *deterministic* computational model recognizing the languages in the family. Remark that the definition of recognizability in terms of tiling systems, i.e. in terms of local languages and projections, is implicitly non-deterministic. This can be easily understood if we refer to the one-dimensional case: if no particular constraints are given for the tiling system, this corresponds in general to a non-deterministic automaton.

Contrary to the one-dimensional case, there are however some difficulties to define determinism in two dimensions, since tiling systems are not computational models in strict sense. As remarked in [1], they are not effective devices for recognition unless a *scanning strategy* for pictures is fixed (for a word the natural scanning strategy is to read it from left to right). So in [1] is introduced a notion of *tiling automaton* as a tiling system equipped with a scanning strategy and, in this framework, some definitions of determinism are proposed.

Actually, a notion that can be naturally expressed in terms of tiling systems is the notion of *ambiguity*. Informally, a tiling system is *unambiguous* if every picture has a unique counter-image in its corresponding local language. In a more formal way, a tiling system $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$ is *unambiguous* if for any picture $x \in L(\mathcal{T})$ there exists a *unique* local picture $y \in L(\Theta)$ such that $x = \pi(y)$.

An alternative definition for *unambiguous tiling system* is that function π extended to $\Gamma^{++} \rightarrow \Sigma^{++}$ is injective. Observe that an unambiguous tiling system can be viewed as a generalization in two dimensions of the definition of unambiguous automaton that recognizes a word language.

A recognizable two-dimensional language $L \subseteq \Sigma^{++}$ is *unambiguous* if it is recognized by an unambiguous tiling system $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$. We denote by $UREC$ the family of all unambiguous recognizable two-dimensional languages. Obviously it holds true that $UREC \subseteq REC$.

In [3], it is shown that it *undecidable* whether a given tiling system is unambiguous. Furthermore some closure properties of $UREC$ are proved. The main result in [3] is the following theorem.

Theorem 2. *$UREC$ is strictly included in REC.*

This theorem shows that there exist languages in REC that are inherently ambiguous.

In the sequel we will focus on possible relationships between Theorem 1 and Theorem 2, i.e. on the relations between the complement problem and the ambiguity of a picture language. In next section we present a novel general framework to study such a problem, by introducing some complexity functions on picture languages.

3 Hankel matrices and complexity functions

In this section we introduce a novel tool to study picture languages based on combining two main techniques: the Matz's technique (that associates to a given picture language L an infinite sequence $(L(m))_{m\geq 1}$ of word languages) and the technique that describes a word language by means of its Hankel matrix. As results there will be the definitions of some complexity functions for picture languages that will be used to state some necessary conditions on recognizable picture languages.

We first describe a technique, introduced by O. Matz in [20]. Let $L \subseteq \Sigma^{++}$ be a picture language. For any $m \geq 1$, we consider the subset $L(m) \subseteq L$ containing all pictures with exactly m rows. Such language $L(m)$ can be viewed as a word language over the alphabet $\Sigma^{m,1}$ of the columns, i.e. words in $L(m)$ have a "fixed height m". For example, if

$$p = \begin{array}{|ccccc|}\hline a & b & b & a & a \\ a & a & b & b & a \\ b & b & a & b & a \\ a & a & a & a & b \\ \hline\end{array} \in L$$

then the word

$$w = \begin{bmatrix} a \\ a \\ b \\ a \end{bmatrix} \begin{bmatrix} b \\ a \\ b \\ a \end{bmatrix} \begin{bmatrix} b \\ b \\ a \\ a \end{bmatrix} \begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix} \begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ b \end{bmatrix}$$

belongs to the word language $L(4)$ over the alphabet of columns

$$\Sigma^{4,1} = \left\{ \begin{bmatrix} x \\ y \\ s \\ t \end{bmatrix} \mid x, y, s, t \in \Sigma \right\}.$$

Observe that studying the sequence $(L(m))_{m \geq 1}$ of word languages corresponding to a picture languages L does not capture the whole structure of L because in some sense it takes into account only its horizontal dimension. Nevertheless it will be very useful to state some conditions for the recognizability of the picture language L.

We first report a lemma given in [20]. Let L be a recognizable picture languages and let $\mathcal{T} = (\Sigma, \Gamma, \Theta, \pi)$ a tiling system recognizing L.

Lemma 1. *For all $m > 1$ there exists a finite automaton $\mathcal{A}(m)$ with γ^m states that recognizes word language $L(m)$, where $\gamma = |\Gamma \cup \{\#\}|$.*

The proof of the above lemma constructs explicitly such non-deterministic finite automaton $\mathcal{A}(m) = (\Sigma^{1,m}, Q_m, I_m, F_m, \delta_m)$ where $\Sigma^{1,m}$ is the alphabet of the columns of height m over Σ; the set of states Q_m is the set of all possible columns of m. The transitions from a given state p to state q are defined by using the adjacency allowed by the set of local tiles. This construction implies directly the following corollary.

Corollary 1. *If $L \in UREC$, then $\mathcal{A}(m)$ is unambiguous.*

Hankel matrices were firstly introduced in [28] in the context of formal power series (see also [6] and [27]). Moreover they are used under different name in communication complexity (see [18]).

Definition 1. Let $S \subseteq A^*$ be a string language. The Hankel matrix of S is the infinite boolean matrix $H_S = [h_{xy}]_{x \in A^*, y \in A^*}$ where

$$h_{xy} = \begin{cases} 1 & \text{if } xy \in S \\ 0 & \text{if } xy \notin S. \end{cases}$$

Therefore both the rows and the columns of H_S are indexed by the set of strings in A^* and the 1s in the matrix gives the description of language S in the way described above.

Let us observe that, in the case of one letter alphabet, the Hankel matrix of a (string) language is a Hankel matrix in the classical sense, i.e. a matrix, with rows and columns indexed by non negative integers, with constant skew diagonals. In other words it is a matrix in which the (i, j)th entry depends only on the sum $i+j$. Such matrices are sometimes known as persymmetric matrices or, in older literature, orthosymmetric matrices.

Given an Hankel matrix H_S, we call *submatrix of* H_S a matrix K_S specified by a pair of languages (U, V), with $U, V \subseteq A^*$, that is obtained by intersecting all rows and all columns of H_S that are indexed by the strings in U and V, respectively. Moreover, given two Hankel submatrices K_S^1 and K_S^2, their intersection is the submatrix specified by the intersections of the corresponding index sets respectively.

Moreover we recall some further notations on matrices. A *permutation* matrix is a boolean matrix that has exactly one 1 in each row and in each column. Usually when dealing with permutation matrices, one makes a correspondence between a permutation matrix $D = [d_{ij}]$ of size n with a permutation function $\sigma = I\!N \longrightarrow I\!N$ by assuming that $d_{ij} = 1 \Leftrightarrow j = \sigma(i)$.

Finally we recall that the *rank* of a matrix is the size of the biggest submatrix with non-null determinant (with respect to field $\mathbb{Z}$). Alternatively, the rank is defined as the maximum number of row or columns that are linearly independent. Then, observe that, by definition, the rank of a permutation matrix coincides with its size.

Given a picture language L over the alphabet Σ, we can associate to L an infinite sequence $(H_L(m))_{m \geq 1}$ of matrices, where each $H_L(m)$ is the Hankel matrix of string language $L(m)$ associated to L.

We can define the following functions from the set of natural numbers $\mathbb{N}$ to $\mathbb{N} \cup \infty$.

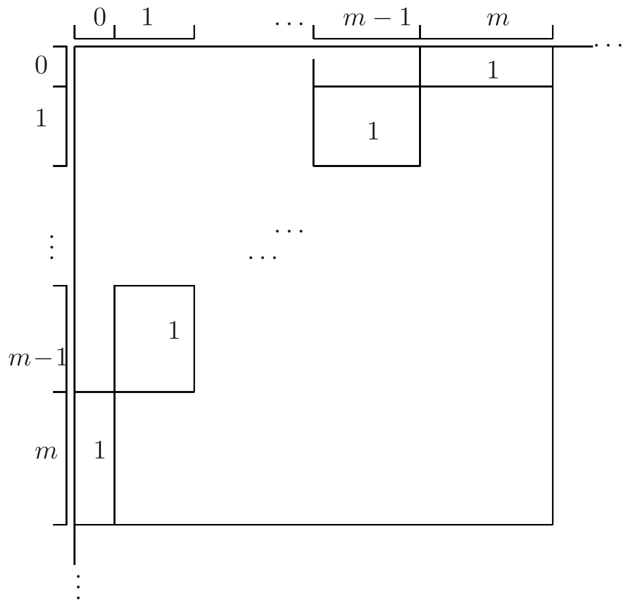
Definition 2. Let L be a picture language.
i) The *row complexity function* $R_L(m)$ gives the number of distinct rows of the matrix $H_L(m)$;
ii) The *permutation complexity function* $P_L(m)$ gives the size of the maximal permutation matrix that is a submatrix of $H_L(m)$;
iii) The *rank complexity function* $K_L(m)$ gives the rank of the matrix $H_L(m)$.

Notice the all the functions $R_L(m)$, $P_L(m)$ and $K_L(m)$ defined above are independent from the order of the rows (columns, resp.) of the Hankel matrix $H_L(m)$. In the sequel we will use any convenient order for the set of strings that index the rows and the columns. We can immediately state the following lemma.

Lemma 2. *Given a picture language L, for each $m \in \mathbb{N}$:*

$$P_L(m) \leq K_L(m) \leq R_L(m).$$

Example 4. Consider the language L of squares over a two-letters alphabet $\Sigma = \{a, b\}$ described in Example 1. Observe that, for each $m \geq 0$, $L(m)$ is the finite language of all possible strings of length m over the alphabet of the columns $\Sigma^{m,1}$. Then consider the Hankel matrix of $L(m)$: it has all its 1s in the positions indexed by pairs (x, y) of strings such that $|x| + |y| = m$. Now assume that the strings that index the rows and the columns of the Hankel matrix are ordered by length: we can have some non-zero positions only in the upper-right portion of $H_L(m)$ that indexed by all possible strings of length $\leq m$ on the alphabet $\Sigma^{m,1}$, included the empty word. More specifically, in this portion the matrix $H_L(m)$ has all 0s with the exception of a chain of rectangles of all 1s from the top-right to the bottom left corner. This is represented in the following figure where the numbers $0, 1, \ldots, m-1, m$ indicate the length of the index words.



It is easy to verify that the number of different rows in $H_L(m)$ is equal to $m+1$ and this is also the number of rows of a permutation submatrix and this is also the rank of $H_L(m)$.

Then for this language it holds that for all positive m:

$$P_L(m) = K_L(m) = R_L(m) = m + 1.$$

Example 5. As generalization of the above Example 4, consider the language L of pictures over an alphabet Σ of size $(n, f(n))$ where $f(n)$ is a non-negative function defined on the set of natural numbers, that is:

$$L = \{\, p \mid p \text{ is of size } (n, f(n)\}.$$

Similar arguments as in the above example show that, for each $m \geq 0$, language $L(m)$ is a finite language (it contains all strings of length $f(m)$ over the alphabet of the columns $\Sigma^{m,1}$) and then, for all positive m: $P_L(m) = K_L(m) = R_L(m) = f(m) + 1$.

Example 6. Consider the language L of pictures over an alphabet Σ of size $(n, 2n)$ such that the two square halves are equal, that is:

$$L = \{\, p \Phi p \mid p \text{ is a square}\}.$$

Again, as in the Example 4, for each $m \geq 0$, language $L(m)$ is a finite language (it contains all strings of length $2m$ over the alphabet of the columns $\Sigma^{m,1}$ of the form ww). Then, doing all the calculations, one obtains that, for all positive m, $P_L(m)$, $K_L(m)$ and $R_L(m)$ are all of the same order of complexity $O(\sigma^{m^2})$, where σ is the number of symbols in the alphabet Σ.

We now state our main theorem that gives necessary conditions for a picture language to be in $REC \cup co-REC$, REC and $UREC$, respectively. Although this is a re-formulation of corresponding three theorems given in [8], [20], [3], respectively, here all the results are given in this unifying matrix-based framework that allows to make connections among these results that before appeared unrelated. A detailed proof can be found in [15].

Theorem 3.

i) If $L \in REC \cup co-REC$ then there exists a positive integer γ such that, for all $m > 0$, $R_L(m) \leq 2^{\gamma^m}$.
ii) If $L \in UREC$ then there exists a positive integer γ such that, for all $m > 0$, $K_L(m) \leq \gamma^m$.
iii) If $L \in REC$ then there exists a positive integer γ such that, for all $m > 0$, $P_L(m) \leq \gamma^m$.

4 Separation results

In this section we state some separation results for the classes of recognizable picture languages here considered. We start by showing that there exist languages L such that are neither L nor ${}^cL$ are recognizable.

Let L_f be a picture language over Σ with $|\Sigma| = \sigma$ of pictures of size $(n, f(n))$ where f is a non-negative function over $\mathbb{N}$. In Example 5 it is remarked that $R_{L_f}(m) = f(m) + 1$. Then, if we choose a function "greater" than the bound in Theorem 3 - *i)*, we obtain the following.

Corollary 2. *Let $f(n)$ be a function that has asymptotical growth rate greater than 2^{γ^n}, then $L_f \notin REC \cup co-REC$.*

We now consider an example of picture language L over one letter alphabet that, together with its complement ${}^{c}L$, will be checked for the inequalities of the Theorem 3. In such a way we show that, even in the case of one letter alphabet, classes $REC \cup co-REC$, REC and $UREC$ are strictly separated.

The proofs of the following results are based on some arithmetic properties of the function $F(n)$ that is introduced below (cf. [21]). Denote by $lcm(x_1, x_2, ..., x_h)$ the lowest common multiple of the integers $x_1, x_2, ..., x_h$. Consider the function

$$G(m) = lcm(m+1, m+2, ..., 2m).$$

It holds the following.

Lemma 3. $G(m) = 2^{\Omega(m)}$.

Consider now the function $F(n) = G(2^n)$ and the language

$$L = \{(n,m) \mid \ m \ is \ not \ multiple \ of \ \ F(n)\,\}.$$

Theorem 4. *$L \in REC$.*

We now calculate our complexity functions for language L. It is not difficult to verify that, for all $n > 0$, the Hankel matrix $H_L(n)$ is such that in its submatrix composed by the first $F(n)$ rows and the first $F(n)$ columns (i.e.the rows and the columns indexed $0, 1, ..., F(n)$) every element in the main skew diagonal is equal to 0 and all other elements are equal to 1. We represent it below.

	0	1	2	3	...	...	...	$F(n)$
1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	0	1
2	1	1	1	1	1	0	1	1
3	1	1	1	1	0	1	1	1
...	1	1	1	0	1	1	1	1
...	1	1	0	1	1	1	1	1
...	1	0	1	1	1	1	1	1
$F(n)$	0	1	1	1	1	1	1	1

On can easily check that, for all $n > 0$:

$$R_L(n) = K_L(n) > F(n)$$

and

$$P_L(n) = 2.$$

Since $L \in REC$ and $F(n) = G(2^n) = 2^{\Omega(2^n)}$, from the inequality $R_L(n) > F(n)$ it holds the following proposition.

Proposition 1. *The bound given in Theorem 3* - i) *is tight.*

From the inequality $K_L(n) > F(n)$ and Theorem 3 - *iii)*, one derives the following result.

Theorem 5. *$UREC$ is strictly included in REC.*

This result was firstly proved in [3] and in the unary case in [4].

Consider now the language ${}^cL$. For all $n > 0$, the Hankel matrix $H_{{}^cL}(n)$ is obtained from the matrix $H_L(n)$ by interchanging the zero's and the one's. It follows that, for all $n > 0$,

$$R_{{}^cL}(n) = K_{{}^cL}(n) > F(n)$$

and

$$P_{{}^cL}(n) > F(n).$$

By the previous inequality and Theorem 3 - *ii)* it follows that ${}^cL \notin REC$ and then one has the following theorem.

Theorem 6. *REC is strictly included in $REC \cup co - REC$.*

Therefore we can conclude that also in the unary case it holds the following hierarchy:

$$UREC \subsetneq REC \subsetneq REC \cup co - REC.$$

5 Final remarks and open questions

We presented an unifying framework based on Hankel matrices to deal with recognizable picture languages. As result, we stated three necessary conditions for the classes $REC \cup co - REC$, REC and $UREC$. The first natural question that arises regards the non-sufficiency of such statements, more specifically the possibility of refining them to get sufficient conditions. Observe that the technique we used of reducing a picture language L in a sequence of string languages $(L(m))_{m>0}$ on the columns alphabets $\Sigma^{m,1}$ allows to take into account the "complexity" of a picture language along only the horizontal dimension. Then the question is whether by combining conditions that use such both techniques along the two dimensions we could get strong conditions for the recognizability of the given picture language.

The novelty of these matrix-based complexity functions gives a common denominator to study relations between the *complement problem* and *unambiguity* in this family of recognizable picture languages. In 1994, in the more general context of graphs, Wolfgang Thomas et. al. had pointed the close relations between these two concepts. In particular, paper [24] ends with the following question formulated specifically for grids graphs and a similar notion of recognizability (here, we report it in our terminology and context).

Question 1. Let $L \subseteq \Sigma^{++}$ be a language in REC such that also ${}^cL \in REC$. Does this imply that $L \in UREC$?

As far as we know, there are no negative examples for this question. On the other hand, we have seen a language L that belongs to REC such that its complement does not and L itself is not in $UREC$. Then we can formulate another question.

Question 2. Let $L \subseteq \Sigma^{++}$ be a language in REC such that ${}^cL \notin REC$. Does this imply that $L \notin UREC$?

Remark that, since our language is on unary alphabet, the above questions are meaningful also in this special case.

As further work we believe that this matrix-based complexity function technique to discriminate class of languages could be refined to study relations between closure under complement and unambiguity. Notice that a positive answer to any of a single question above does not imply that $UREC$ is closed under complement. Moreover observe that the two problems can be rewritten as whether $REC \cap co\text{–}REC \subseteq UREC$ and whether $UREC \subseteq REC \cap co\text{–}REC$, respectively, i.e. they correspond to verify two inverse inclusions. As consequence, if both conjectures were true then we would conclude not only that $UREC$ is closed under complement but also that it the *largest* subset of REC closed under complement.

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