A Short Proof of a Gauss Problem *

H. de Fraysseix and P. Ossona de Mendez

CNRS UMR 0017, EHESS, 54 Boulevard Raspail, 75006, Paris, France.
email : hf@ehess.fr, pom@ehess.fr

Abstract. The traversal of a self crossing closed plane curve, with points of multiplicity at most two, defines a double occurrence sequence.
C.F. Gauss conjectured [2] that such sequences could be characterized by their interlacement properties. This conjecture was proved by P. Rosenstiehl in 1976 [15]. We shall give here a simple self-contained proof of his characterization. This new proof relies on the D-switch operation.

1 Introduction

We first recall and introduce some definitions and notations concerning geometric properties of closed plane curves. For related topics, we refer the reader to the bibliography. P. Rosenstiehl exposed recently a new proof of this theorem, based on patches, that will soon be published.

A *parameterized curve* C is a continuous mapping $C : [0,1] \to \mathbb{R}^2$, such that $C(0) = C(1)$ and such that the *underlying curve* $C([0,1])$ of C has a finite number of multiple points, which all have multiplicity two. Let $P(C)$ denote the set of the points of multiplicity two. To any point $p \in P(C)$, we associate the two parameter values t'_p and t''_p, such that $t'_p < t''_p$ and $C(t'_p) = C(t''_p) = p$. A point $p \in P(C)$ is a *crossing point* if any local deformation of C in a neighborhood of t'_p preserves the existence of a double point. Otherwise, p is a *touching point.* A *touch curve* (resp. a *cross curve*) is a parameterized curve with touching points (resp. crossing points) only.

There are two different types of touching points, depending on the local behavior of the parameterized curve :

Remark. All the touch points of a touch curve are of type 1.

The sequence of the points of $P(C)$ encountered while the parameter t goes from 0 to 1 (excluded) is the *traversal sequence* of C and is denoted by $S(C)$.

In the following, *sequences* are understood to have two occurrences of each symbol and to be defined up to reversal and cyclic permutation. Given a sequence S, two symbols p, q are *interlaced* in S if exactly one occurrence of q appears in S between the two occurrences of p. We shall denote by $\Lambda(S)$ the *interlacement graph* of S defined by the interlacement relation in S.

A sequence S is *realized* by a parameterized curve C if S is the traversal sequence of C. A sequence is *touch realizable* (resp. *cross realizable*) if it can be realized by a touch curve (resp. a cross curve).

* This work was partially supported by the **Esprit LTR Project no 20244-ALCOM IT**.

2 Switches and D-switches

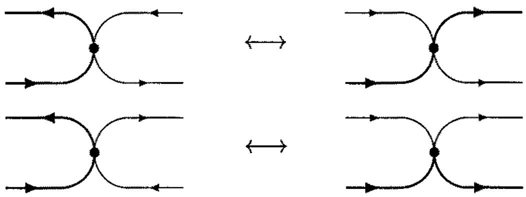
Let us introduce the *switch* operation [4, 8] : Given a point p of $P(C)$, the curve $C' = C \circ p$ is defined by :

$$C'(t) = \begin{cases} C(t), & \text{if } t \notin [t'_p, t''_p] \\ C(t'_p + t''_p - t), & \text{if } t \in [t'_p, t''_p] \end{cases} \tag{1}$$

This curves as the same touching and crossing points as C, with the possible exception of p. The traversal sequence of C' is obtained from the one of C by inverting the order of the points encountered between the two occurrences of p. We shall say that the points that are interlaced with p have been *inverted.* The switch operation on S will be denoted by $S \circ p$, so that : $S(C \circ p) = S(C) \circ p$. Let us remark that these switch operations are involutions : $C \circ p \circ p = C$ and $S \circ p \circ p = S$.

Remark. A switch at a point p of a parameterized curve transforms p in the following way :

- touching point of type 1 $\leftrightarrow$ crossing point,
- touching point of type 2 $\leftrightarrow$ touching point of type 2.



Remark. If q is a touching point of C different from p, then q is a touching point with a type different in C and $C \circ p$ if and only p and q are interlaced (that is if q has been inverted by the switch at p).

The switch of a point p in a sequence S induces a *local complementation* of p in the interlacement graph $\Lambda(S)$: two symbols a, b are adjacent in $\Lambda(S \circ p)$ if and only if

- a or b is not adjacent to p in $\Lambda(S)$ and (a, b) is an edge of $\Lambda(S)$, or
- a and b are both adjacent to p in $\Lambda(S)$ and (a, b) is not an edge of $\Lambda(S)$.

For sake of simplicity, the local complementation of p in $\Lambda(S)$ will be denoted by $\Lambda(S) \circ p$, so that $\Lambda(S \circ p) = \Lambda(S) \circ p$.

Let S be a sequence, a *D-switch* at p consists in a switching at p and in the adding of two occurrences of a new symbol p' (called *twin* of p), one just after the first occurrence of p and one just before the second occurrence of p.

$$S = (\alpha p \beta p \gamma) \mapsto S \circledcirc p = (\alpha p p' \beta^{-1} p' p \gamma)$$

A D-switch of p in S corresponds in $\Lambda(S)$ to a local complementation of p and the addition of a new vertex having the same neighbors as p. This graph operation will be similarly denoted by '$\circledcirc$', so that $\Lambda(S) \circledcirc p = \Lambda(S \circledcirc p)$.

Remark. The sequence obtained from $S \circledcirc p \circledcirc p$ by deleting the two twins of p is equal to S.

3 On realizable sequences

We first state two propositions proved by Dehn, which follow from the remarks of the preceding sections.

Proposition 1. *Consider a cross curve C and any given order $(p_1, \ldots, p_n)$ of the points of C. Then, the parameterized curve $C \circ p_1 \circ \ldots \circ p_n$ obtained from C by switching successively the p_i is a touch curve.* $\square$

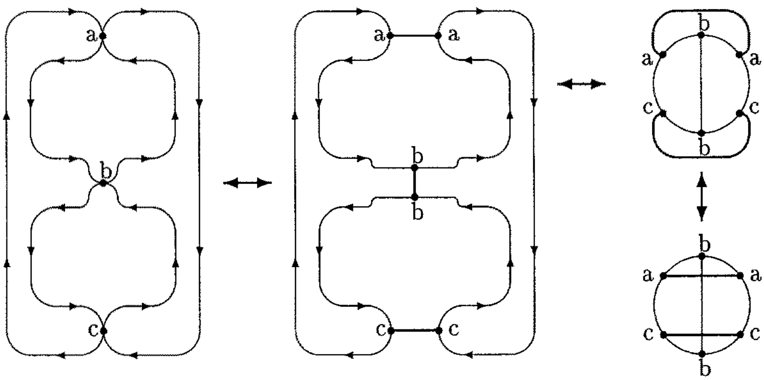
The converse of this proposition is not true (e.g. the sequence $(abab)$ is not cross realizable).

Remarks. Let us note by $S \xrightarrow{\circ} S'$ the existence of an order $(p_1, \ldots, p_n)$ of the symbols of S, such that $S' = S \circ p_1 \circ \ldots \circ p_n$.

- A cross realizable sequence does not determine the cross curve itself up to an homeomorphism : actually, a cross realizable sequence S can be proved to be realized by $2^{c(A(S))-1}$, where $c(A(S))$ is the number of connected components of the interlacement graph of S.
- One may find a cross realizable sequence S_1, a non cross realizable sequence S_2 and a touch realizable sequence S_T, such that $S_1 \xrightarrow{\circ} S_T$ and $S_2 \xrightarrow{\circ} S_T$. Actually, S_1 and S_2 may be proved to have different interlacement graph (using the main theorem).
- Two different cross realizable sequences S_1 and S_2 may have the same interlacement graph (e.g. the sequences $(abcaefdcbefd)$ and $(acbaefdbcefd)$). However, no sequence S_T satisfies $S_1 \xrightarrow{\circ} S_T$ and $S_2 \xrightarrow{\circ} S_T$.

Proposition 2. *A sequence S is touch realizable if and only if its interlacement graph $A(S)$ is bipartite.*

Proof. The figure bellow shows how a touch curve may be transformed into a bipartite chord diagram with the same interlacement (and conversely). $\square$



Theorem 3. *Let S be a sequence, and let $(p_1, \ldots, p_n)$ be any order on its symbols. Then, S is cross realizable if and only if the sequence $S_n = S \circledcirc p_1 \circledcirc \ldots \circledcirc p_n$ obtained by successively D-switching the p_i has a bipartite interlacement graph.*

Proof. – Assume S is realized by a cross curve C. As a D-switch of a crossing point of a parameterized curve gives rise to two touching points (that will never become crossing points again), the curve C is iteratively transformed into a touch curve C_n. The traversal sequence S_n of C_n has hence a bipartite interlacement graph.

- Conversely, assume that S_n has a bipartite interlacement graph.
  Let $S_i = S \circledast p_1 \circledast \ldots \circledast p_i$ denote the sequence obtained after the first i D-switches, we shall inductively construct (for i going from n to 0) a parameterized curve C_i, that realizes S_i, and such that the crossings of C_i are the p_j, with $j > i$. Then, the parameterized curve C_0 will be a cross curve realizing S.
  As $\Lambda(S_n)$ is bipartite, there exists a touching curve C_n whose traversal sequence is S_n. If p_i is of type 1, the suppression of p'_i and the switch of p_i transforms p_i into a crossing point and gives rise to a parameterized curve C_{i-1}, having $p_i, \ldots, p_n$ as crossing points and S_{i-1} as traversal sequence. The recursion is then complete if only this case may occur.
  So, we shall prove that p_i is always of type 1 in C_i, that is that p_i has been inverted an even number of times during the D-switch at $p_i, \ldots, p_n$: The symbol p_i and its twin p'_i are not interlaced in S_i, they are alternatively interlaced and not interlaced after each further inversion and, if p_i has been last inverted by a switch at p_j, p_i (resp. p'_i) and p_j are interlaced in S_n. As $\Lambda(S_n)$ is bipartite, p_i and p'_i are not interlaced in S_n (else p_i, p'_i, p_j would define a triangle of $\Lambda(S_n)$). Hence, the symbol p_i has been inverted an even number of times.

$\square$

Remark. A cross curve realizing the sequence S could be geometrically derived from a touch curve realizing the sequence S' obtained from S_n by suppressing all twined letters by transforming each touching point into a crossing point.

4 Proof of Rosenstiehl's Theorem

Definition 4. Let G be a graph and let (A, B) be a bipartition of its vertex set.

The property $P(G; A, B)$ is satisfied by a pair $\{u, v\}$ of vertices of G whenever the following equivalence holds :

- the vertices u and v have an odd number of common neighbors,
- the vertices u and v are adjacent and belong to the same class (A or B).

Lemma 5. *Let G be a graph with a vertex bipartition A, B and let p be a vertex of G. Let $G' = G \circledast p$ and let A', B' be the vertex bipartition of G' defined by : $A' = A + N(p), B' = B + N(p)$ and assigning p' to the class of p.*

If G is eulerian and any pair $\{u, v\}$ of vertices of G satisfies $P(G; A, B)$, then G' is eulerian and any pair u, v of vertices of G' satisfies $P(G'; A', B')$.

Proof. We have the following relationship between the neighborhood $N_{G'}$ in G' and the neighborhood N_G in G :

- $N_{G'}(u) = N_G(u)$, if u is not adjacent to p (in G or equivalently in G'),
- $N_{G'}(p') = N_{G'}(p) = N_G(p)$,
- $N_{G'}(u) = N_G(u) + u + N_G(p) + p'$, if u is adjacent to p.

In order to prove that G' is eulerian, we only have to check that the neighbors of p have an even degree : the parity of $N_{G'}(u) = N_G(u) + u + N_G(p) + p'$ is the sum of the parities of $N_G(u), \{u\}, N_G(p)$ and $\{p\}$ and hence is even.

Now, we shall prove that any pair $\{u, v\}$ of vertices of G' satisfies $P(G'; A', B')$. If u or v is p', we shall replace it by p as p and p' have the same neighbors, are not adjacent and belong to the same class (A', B').

If two vertices u,v are not adjacent or equal to p, then their adjacencies, their class and their number of common neighbors are the same in G and G'. Thus, the pair u,v satisfies $P(G';A',B')$.

If u is adjacent to p and v is not adjacent or equal to p, then

$$\begin{aligned} <N_{G'}(u),N_{G'}(v)> &= <N_G(u)+u+N_G(p)+p',N_G(v)> \\ &= <N_G(u),N_G(v)>+1 \end{aligned}$$

As u and v belong to the same class (A',B') if and only if they do not belong to the same class (A,B) and as they are adjacent, the pair u,v satisfies $P(G';A',B')$.

If u and v are both adjacent to p, then

$$\begin{aligned} <N_{G'}(u),N_{G'}(v)> &= <N_G(u)+u+N_G(p)+p',N_G(u)+v+N_G(p)+p'> \\ &= <N_G(u),N_G(v)>+<N(u),N(p)>+<N(v),N(p)>+1 \end{aligned}$$

As u is adjacent to p, $<N(u),N(p)> = 1$ if and only if u and p belong to the same class (A,B). So, $<N(u),N(p)>+<N(v),N(p)>+1=1$ if and only if u and v belong to the same class (A,B). As $<N_G(u),N_G(v)>=1$ if and only if u and v are adjacent in G and belong to the same class (A,B), $<N_{G'}(u),N_{G'}(v)>=1$ if and only if u and v are not adjacent in G and belong to the same class (A,B), that is, if and only if they are adjacent in G' and belong to the same class (A',B'). Thus, the pair u,v satisfies $P(G';A',B')$. □

Lemma 6. *Let G be a graph with a vertex bipartition A,B and let p be a vertex of G. Let $G'=G \circledcirc p$ and let A',B' be the vertex bipartition of G' defined by : $A'=A+N(p)$, $B'=B+N(p)$ and assigning p' to the class of p. If G' is eulerian and any pair $\{u,v\}$ of vertices of G' satisfies $P(G';A',B')$,then G is eulerian and any pair $\{u,v\}$ of vertices of G satisfies $P(G;A,B)$.*

Proof. By Lemma 5, $G''=G' \circledcirc p$ has the requested property and this property is still satisfied when deleting the two twins of p. □

Theorem 7 (Rosenstiehl)**[15]**. *A sequence S is cross realizable if and only if its interlacement graph $\Lambda(S)$ satisfies :*

- *the graph is eulerian,*
- *for any non-edge (p,p') of the graph, $N(p)\cap N(p')$ is even,*
- *the set of the of the edges (p,p') of the graph such that $N(p)\cap N(p')$ is even is a cocycle of the graph.*

Proof. The theorem may be restated as follows : A sequence S is cross realizable if and only if its interlacement graph $\Lambda(S)$ is eulerian and if there exists a bipartition A,B of the vertex set of $\Lambda(S)$ such that any pair u,v of vertices of $\Lambda(S)$ satisfies $P(\Lambda(S);A,B)$.

Consider any sequence $S_n = S \circledcirc p_1 \circledcirc \ldots \circledcirc p_n$ obtained by successively D-switching the symbols of S. According to Lemma 5, $\Lambda(S_n)$ is eulerian and has a bipartition A',B' such that any pair of vertices of $\Lambda(S_n)$ satisfies $P(\Lambda(S_n);A',B')$. As all the symbols have been twined and as p and its twin p' have the same neighbors, any two vertices of $\Lambda(S_n)$ have an even number of common vertices. According to property $P(\Lambda(S_n);A',B')$, the graph $\Lambda(S_n)$ is bipartite. Then, from Theorem 3, S is cross realizable.

Conversely, if S is cross realizable, any sequence of D-switches gives rise to a sequence S' having a bipartite interlacement graph. This graph is eulerian (due to the doubling of each symbol) and a bipartition A,B induced by a bicoloration, is such that each pair of vertices satisfies $P(\Lambda(S');A,B)$. The theorem then follows from Lemma 6. □

5 Matroidal Interpretation

As we wanted to give a short self-contained proof, we did not introduce the usual concepts of binary matroids. In such a context, a proof could be done, relying on the following properties : The graphs which satisfy the conditions given for $\Lambda(S)$ in Rosenstiehl's characterization are exactly the principal interlacement graph of some binary matroid M [14]. Any local complementation of the vertices of such a graph gives rise to a bipartite graph, which is the fundamental interlacement graph of M with respect to some base B of M [3]. The further condition that a principal interlacement graph is an interlacement graph (that is a circle graph) implies that the matroid M is planar and then, the principal interlacement graph corresponds to the interlacement of a left-right path of a planar realization of M [3].

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