A Linear-Time Algorithm for Four-Partitioning Four-Connected Planar Graphs

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Abstract. Given a graph $G = (V, E)$, four distinct vertices $u_1, u_2, u_3, u_4 \in V$ and four natural numbers n_1, n_2, n_3, n_4 such that $\sum_{i=1}^{4} n_i = |V|$, we wish to find a partition V_1, V_2, V_3, V_4 of the vertex set V such that $u_i \in V_i$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for each i, $1 \leq i \leq 4$. In this paper we give a simple linear-time algorithm to find such a partition if G is a 4-connected planar graph and u_1, u_2, u_3, u_4 are located on the same face of a plane embedding of G. Our algorithm is based on a "4-canonical decomposition" of G, which is a generalization of an st-numbering and a "canonical 4-ordering" known in the area of graph drawings.

1 Introduction

Given a graph $G = (V, E)$, k distinct vertices $u_1, u_2, \cdots, u_k \in V$ and k natural numbers $n_1, n_2, \cdots, n_k$ such that $\sum_{i=1}^{k} n_i = |V|$, we wish to find a partition $V_1, V_2, \cdots, V_k$ of the vertex set V such that $u_i \in V_i$, $|V_i| = n_i$, and V_i induces a connected subgraph of G for each i, $1 \leq i \leq k$. Such a partition is called a *k-partition* of G. A 4-partition of a graph G is depicted in Fig. 1, where the edges of four connected subgraphs are drawn by solid lines and the remaining edges of G are drawn by dotted lines. The problem of finding a k-partition of a given graph often appears in the load distribution among different power plants and the fault-tolerant routing of communication networks [WK94, WTK95]. The problem is NP-hard in general [DF85], and hence it is very unlikely that there is a polynomial-time algorithm to solve the problem. Although not every graph has a k-partition, Györi and Lovász independently proved that every k-connected graph has a k-partition for any $u_1, u_2, \cdots, u_k$ and $n_1, n_2, \cdots, n_k$ [G78, L77]. However, their proofs do not yield any polynomial-time algorithm for actually finding a k-partition of a k-connected graph. For the case $k = 2$ and 3, the following algorithms have been known:

(i) a linear-time algorithm to find a bipartition of a biconnected graph [STN90, STNMU90];

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(ii) an algorithm to find a tripartition of a triconnected graph in $O(n^2)$ time, where n is the number of vertices of a graph [STNMU90]; and
(iii) a linear-time algorithm to find a tripartition of a triconnected planar graph [JSN94].

On the other hand, polynomial-time algorithms have not been known for the case $k \geq 4$. **

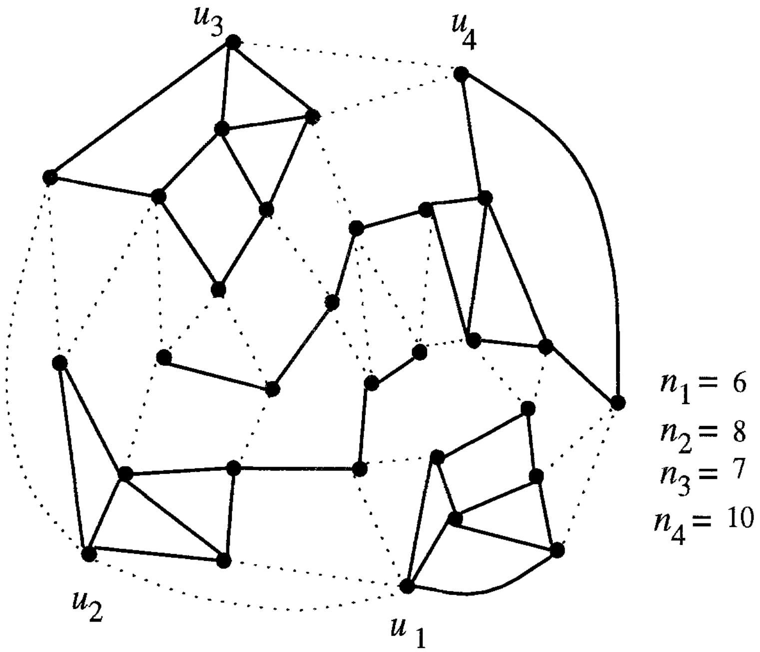

Fig. 1. A 4-partitioning of a 4-connected plane graph G.

In this paper we give a linear-time algorithm to find a 4-partition of a 4-connected plane graph G if u_1, u_2, u_3, u_4 are located on the same face of G, as illustrated in Fig. 1. We first bipartition the 4-connected graph G into two biconnected graphs having about $n_1 + n_2$ and $n_3 + n_4$ vertices respectively, we then bipartition each of them to two connected graphs, and, by adjusting the numbers of vertices in the resulting four graphs, we finally obtain a required 4-partition of G. To bipartition G into two biconnected graphs, we will newly define and use a "4-canonical decomposition" of G, which is a generalization of an st-numbering and a "canonical 4-ordering" known in the area of graph drawings [E79, K94, KH94].

** A polynomial-time algorithm for any k is claimed in [MM94], but is not correct [G96].

The remainder of the paper is organized as follows. In Section 2 we introduce our notations and give a linear-time algorithm to find a 4-canonical decomposition of a 4-connected planar graph. In Section 3 we present a linear-time algorithm to find a 4-partition of a 4-connected planar graph. Finally we put our discussions in Section 4.

2 4-Canonical Decomposition

In this section we introduce some definitions and prove that every 4-connected plane graph has a 4-canonical decomposition and it can be found in linear time.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. Throughout the paper we denote by n the number of vertices in G, that is, $n = |V|$. An edge joining vertices u and v is denoted by (u, v). The *degree* of a vertex v is the number of neighbors of v in G. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. G is called a *k-connected* graph if $\kappa(G) \geq k$. We call a vertex of G a *cut vertex* if its removal results in a disconnected or single-vertex graph. For $W \subseteq V$, we denote by $G - W$ the graph obtained from G by deleting all vertices in W and all edges incident to them.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane* graph is a planar graph with a fixed embedding. The *contour* $C(G)$ of a biconnected plane graph G is the clockwise (simple) cycle on the outer face. We write $C(G) = w_1, w_2, \cdots, w_h, w_1$ if the vertices $w_1, w_2, \cdots, w_h$ on $C(G)$ appear in this order. A *chord* in a biconnected plane graph G is a path P in G satisfying the following (a) - (d):

(a) P connects two vertices w_p and w_q, $p < q$, on $C(G)$;
(b) P does not pass through any vertices on $C(G)$ except the ends w_p and w_q;
(c) P lies on an inner face; and
(d) there is no edge e on $C(G)$ such that P together with e forms an inner face.

The chord is said to be *minimal* if none of $w_{p+1}, w_{p+2}, \cdots, w_{q-1}$ is an end of a chord. Thus the definition of a minimal chord depends on which vertex is considered as the starting vertex w_1 of $C(G)$. Let $\{v_1, v_2, \cdots, v_{p-1}, v_p\}$ be a set of three or more consecutive vertices on $C(G)$ such that the degrees of the first vertex v_1 and the last one v_p are at least three and the degrees of all intermediate vertices $v_2, v_3, \cdots, v_{p-1}$ are two. Then we call the set $\{v_2, v_3, \cdots, v_{p-1}\}$ an *outer chain* of G.

For a cycle C in a plane graph G, we denote by $I(C, G)$ the subgraph of G inside C, that is, the plane subgraph of G induced by the set of vertices inside (or on) the cycle C. Clearly $I(C, G)$ is biconnected if G is biconnected. We have the following lemma.

Lemma 1. *Assume that G is a 4-connected plane graph and that a cycle $C = w_1, w_2, \cdots, w_h, w_1$ in G is not a face of G. Let w_p and w_q be the two ends of any* *minimal chord in $I(C,G)$ if $I(C,G)$ has a chord, and let $w_p = w_1$ and $w_q = w_h$ if $I(C,G)$ has no chord. Then the following* (a) *and* (b) *hold:*

(a) *If $W = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of $I(C,G)$, then $I(C,G) - W$ is biconnected.*
(b) *Otherwise, there is a set $W' = \{w_{p'}, w_{p'+1}, \cdots, w_{q'}\}$ of one or more consecutive vertices on C such that*
   (i) *$p < p' \leq q' < q$, and*
   (ii) *none of the vertices in W' except the first vertex $w_{p'}$ and the last one $w_{q'}$ has a neighbor in the proper outside of C.*

   Moreover, for any of sets W' satisfying (i) and (ii), $I(C,G) - W'$ is biconnected.

Proof. (a) Assume that $W = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of $I(C,G)$. Then $I(C,G)$ has a minimal chord with ends w_p and w_q. Suppose for a contradiction that $I(C,G) - W$ is not biconnected. Then $I(C,G) - W$ has a cut vertex v. Since G is 4-connected, v must be on C. However, the chord above passes through v, and $v \neq w_p, w_q$, contradicting to the condition (b) of the definition of a chord.

(b) Assume that W is not an outer chain of $I(C,G)$. Then $q \geq p+2$ since G is 4-connected. Obviously any singleton set $W' = \{w_{p'}\}, p < p' < q$, satisfies (i) and (ii). Therefore it suffices to prove that $I(C,G) - W'$ is biconnected for any of sets W' satisfying (i) and (ii). Suppose for a contradiction that $I(C,G) - W'$ is not biconnected for a set W' satisfying (i) and (ii). Then $I(C,G) - W'$ has a cut vertex v. If v is not on C, then the removal of v and one or two appropriate vertices in W' disconnects G and hence G would not be 4-connected, a contradiction. If v is on C, then either G would not be 4-connected or a chord with ends w_p and w_q would not be minimal, a contradiction. □

Let $G = (V, E)$ be a connected graph, and let $(s,t) \in E$. We say that an ordering $\pi = v_1, v_2, \cdots, v_n$ of the vertices of G is an *st-numbering* of G if the following conditions are satisfied:

(st1) $v_1 = s$ and $v_n = t$; and
(st2) each $v_i \in V - \{v_1, v_n\}$ has two neighbors v_p and v_q such that $p < i < q$.

Not every connected graph has an st-numbering, but the following lemma holds.

Lemma 2. [E79] *Let G be a biconnected graph, and let (s,t) be any edge of G. Then G has an st-numbering $\pi = v_1, v_2, \cdots, v_n$ such that $v_1 = s$ and $v_n = t$, and π can be found in linear time.*

A bipartition of a biconnected graph can be found by an st-numbering as follows [STNMU90, STN90]. Let $G = (V, E)$ be a biconnected graph, let $u_1, u_2 \in V$ be two designated distinct vertices, and let n_1,n_2 be two natural numbers such that $n_1 + n_2 = n$. We may assume without loss of generality that $(u_1, u_2) \in E$; otherwise, consider as G the graph obtained from G by adding a new edge (u_1, u_2). Since G is biconnected, by Lemma 2 G has an st-numbering $v_1(= u_1), v_2, \cdots, v_n(= u_2)$. Clearly the following fact holds:

(st3) both the subgraphs of G induced by $\{v_1, v_2, \cdots, v_i\}$ and $\{v_{i+1}, v_{i+2}, \cdots, v_n\}$ are connected for each i, $1 \leq i < n$.

Thus, choosing $i = n_1$, one can find a required bipartition of G in linear time.

Generalizing an st-numbering in a sense, we define a "4-canonical decomposition" of a 4-connected plane graph G and in the succeeding section we give an algorithm to find a 4-partition of G by using the "4-canonical decomposition." We now give the definition of a 4-canonical decomposition.

Assume that $G = (V, E)$ is a 4-connected plane graph with four designated distinct vertices u_1, u_2, u_3, u_4 on the same face of G. We may assume that u_1, u_2, u_3, u_4 lie on the contour $C(G)$ of G, since, for any face F of G, we can re-embed G so that F becomes the outer face. We may furthermore assume that the four vertices u_1, u_2, u_3, u_4 appear on $C(G)$ of G in this order. Moreover we may assume that $(u_1, u_2), (u_3, u_4) \in E$; otherwise, consider as G the new graph obtained from G by adding edges (u_1, u_2) and (u_3, u_4). For a set $W_1, W_2, \cdots, W_i$ of pairwise disjoint subsets of V, we denote by G_i the subgraph of G induced by $W_1 \cup W_2 \cup \cdots \cup W_i$, and by $\overline{G_i}$ the subgraph of G induced by $V - W_1 \cup W_2 \cup \cdots \cup W_i$, that is, $\overline{G_i} = G - W_1 \cup W_2 \cup \cdots \cup W_i$. We say that a partition $\Pi = W_1, W_2, \cdots, W_l$ of V is a *4-canonical decomposition* of G if the following three conditions (co1)–(co3) are satisfied:

(co1) W_1 is the set of vertices on the inner face containing edge (u_1, u_2), and W_l is the set of vertices on the inner face containing edge (u_3, u_4);
(co2) for each i, $1 \leq i < l$, both G_i and $\overline{G_i}$ are biconnected; and
(co3) for each i, $1 < i < l$, either W_i consists of exactly one vertex on both $C(G_i)$ and $C(\overline{G_{i-1}})$ or W_i is an outer chain of G_i or $\overline{G_{i-1}}$.

Fig. 2 illustrates the condition (co3); (a) for the case $|W_i| = 1$, and (b) and (c) for the cases w_i is an outer chain of G_i and $\overline{G_{i-1}}$ respectively, where G_i and $\overline{G_{i-1}}$ are indicated by different shading and the vertices in W_i are drawn in black dots.

The 4-canonical decomposition defined for 4-connected plane graphs is a generalization of the "canonical 4-ordering" defined for internally triangulated 4-connected plane graphs [K94, KH94].

We have the following two lemmas.

Lemma 3. *Let $G = (V, E)$ be a 4-connected plane graph with four designated distinct vertices u_1, u_2, u_3, u_4 appearing on $C(G)$ in this order. Then G has a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$. Furthermore Π can be found in linear time.*

Proof. Let W_1 be the set of vertices on the inner face containing edge (u_1, u_2). Clearly G_1 is biconnected, and $u_3, u_4 \notin W_1$. We now claim that $\overline{G_1} = G - W_1$ is also biconnected. Let C be the contour of the biconnected plane graph obtained from G by deleting edge (u_1, u_2). Clearly $I(C, G)$ has neither a chord nor an outer chain; otherwise, G would not be 4-connected. Let cycle C start with u_4,

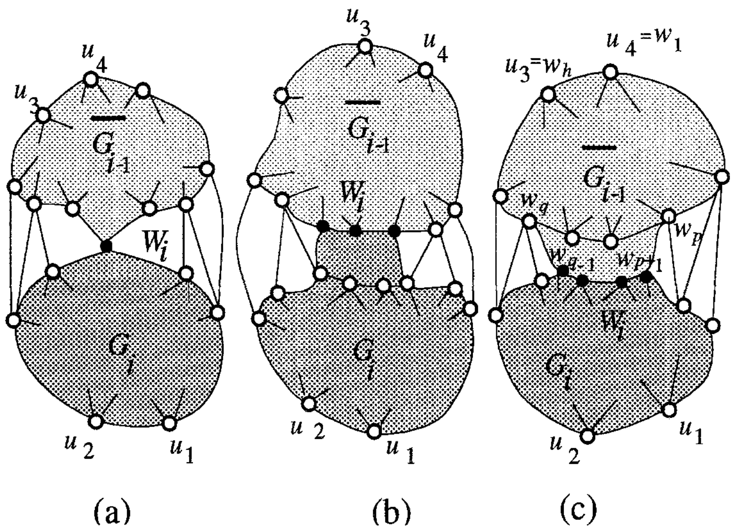

Fig. 2. Illustration of the condition (co3).

then the set W_1 of vertices are consecutive on C and satisfies (i) and (ii) in Lemma 1(b). Therefore $\overline{G_1} = I(C, G) - W_1$ is biconnected.

Assume that we have chosen $W_1, W_2, \cdots .W_{i-1}$, $i \geq 2$, such that the conditions (co2) and (co3) hold for each j, $1 \leq j \leq i-1$, and that $u_3, u_4 \notin W_1 \cup W_2 \cup \cdots \cup W_{i-1}$. Then we show that there is a set W_i ($\subseteq V - W_1 \cup W_2 \cup \cdots \cup W_{i-1}$) such that

(1) G_i is biconnected,
(2) either $u_3, u_4 \notin W_i$ or $u_3, u_4 \in W_i$;
(3) if $u_3, u_4 \notin W_i$, then $\overline{G_i}$ is biconnected and W_i satisfies the condition (co3); and
(4) if $u_3, u_4 \in W_i$, then $l = i$, that is, $V = W_1 \cup W_2 \cup \cdots \cup W_l$, and W_l is the set of vertices on the inner face containing edge (u_3, u_4).

There are the following two cases.
Case 1: graph $\overline{G_{i-1}} = G - W_1 \cup W_2 \cup \cdots \cup W_{i-1}$ is a cycle.

In this case $\overline{G_{i-1}}$ is the inner face of G containing edge (u_3, u_4). We set $l = i$ and $W_l = V - W_1 \cup W_2 \cup \cdots \cup W_{i-1}$. Then $u_3, u_4 \in W_l$, and $V = W_1 \cup W_2 \cup \cdots \cup W_l$. Since $G_i = G$, G_i is biconnected.
Case 2: otherwise.

Let $C(\overline{G_{i-1}}) = w_1, w_2, \cdots, w_h, w_1$ be the contour of $\overline{G_{i-1}}$ with the starting vertex $w_1 = u_4$. Then $\overline{G_{i-1}} = I(C(\overline{G_{i-1}}), G)$. If $\overline{G_{i-1}}$ has a chord then let w_p and w_q be the two ends of a minimal chord, otherwise let $w_p = w_1 = u_4$ and $w_q = w_h = u_3$. Let $W = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$. We now have the following three subcases.

Subcase 2(a): W is an outer chain of $\overline{G_{i-1}}$.

In this subcase we set $W_i = W$. Then $u_3, u_4 \notin W_i$, and W_i satisfies (co3). Since G is 4-connected, each vertex in W_i has at least two neighbors in the biconnected graph G_{i-1} induced by $W_1 \cup W_2 \cup \cdots \cup W_{i-1}$. Therefore the graph G_i induced by $(W_1 \cup W_2 \cup \cdots \cup W_{i-1}) \cup W_i$ is biconnected. By Lemma 1(a), $\overline{G_i} = \overline{G_{i-1}} - W_i$ is biconnected too.

Subcase 2(b): W is not an outer chain of $\overline{G_{i-1}}$, but a vertex w_r in W has two or more neighbors in G_{i-1}.

In this subcase we set $W_i = \{w_r\}$. Then $u_3, u_4 \notin W_i$, and w_r lies on both $C(G_i)$ and $C(\overline{G_{i-1}})$ and hence W_i satisfies (co3). Since w_r has two or more neighbors in G_{i-1}, G_i is biconnected. Since $W_i = \{w_r\}$ satisfies (i) and (ii) in Lemma 1(b), $\overline{G_i} = \overline{G_{i-1}} - W_i$ is biconnected.

Subcase 2(c): otherwise.

In this subcase, W is not an outer chain of $\overline{G_{i-1}}$, and every vertex in W has at most one neighbor in G_{i-1}. Since G is 4-connected, W contains two vertices $w_{p'}$ and $w_{q'}$ such that

(1) $p < p' < q' < q$,
(2) each of $w_{p'}$ and $w_{q'}$ has exactly one neighbor in G_{i-1} and these neighbors are different from each other, and
(3) none of $w_{p'+1}, w_{p'+2}, \cdots, w_{q'-1}$ has a neighbor in G_{i-1}.

We now set $W_i = \{w_{p'}, w_{p'+1}, \cdots, w_{q'}\}$. Clearly $u_3, u_4 \notin W_i$, G_i is biconnected, and W_i is an outer chain of G_i and hence satisfies (co3). Since W_i satisfies (i) and (ii) in Lemma 1(b), $\overline{G_i} = \overline{G_{i-1}} - W_i$ is biconnected.

Thus we have proved that there exists a 4-canonical decomposition.

One can implement an algorithm for finding a 4-canonical decomposition, based on the proof. It maintains a data-structure to keep the outer chains and minimal chords of $\overline{G_i}$. The algorithm traverses every face at most a constant times, and runs in linear time. □

Lemma 4. *Let $W_1, W_2, \cdots, W_l$ be a 4-canonical decomposition of a 4-connected plane graph G. Then the following* (a) *and* (b) *hold for any i, $1 < i < l$:*

(a) *If W_i is an outer chain of G_i as illustrated in Fig. 2(b), then, for any $W_i' \subseteq W_i$, $\overline{G_{i-1}} - W_i'$ is biconnected.*
(b) *If W_i is an outer chain of $\overline{G_{i-1}}$ as illustrated in Fig. 2(c), then, for any $W_i' \subseteq W_i$, $G_i - W_i'$ is biconnected.*

Proof. We give only a proof for (b) since the proof for (a) is similar. Let W_i be an outer chain of $\overline{G_{i-1}}$. The graph G_{i-1} is biconnected. Since G is 4-connected, each vertex in W_i has at least two neighbors in G_{i-1}. Therefore the graph $G_i - W_i'$ induced by $W_1 \cup W_2 \cup \cdots \cup W_{i-1} \cup (W_i - W_i')$ is also biconnected. □

3 4-Partition of 4-Connected Plane Graph

In this section we give our algorithm to find a 4-partition of a 4-connected plane graph G. Assume that the four designated distinct vertices u_1, u_2, u_3, u_4

appear on $C(G)$ in this order and n_1, n_2, n_3, n_4 are natural numbers such that $\sum_{i=1}^{4} n_i = n$.

Algorithm Four-Partition
Find a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$ of G;
Let i be the minimum integer such that $\sum_{j=1}^{i} |W_j| \geq n_1 + n_2$;
Let $r = \sum_{j=1}^{i} |W_j| - (n_1 + n_2)$, that is, r is the excess of the number of vertices in $W_1 \cup W_2 \cup \cdots \cup W_i$ over $n_1 + n_2$;
There are the following two cases (1) $r = 0$, and (2) $r \geq 1$;
Case 1: $r = 0$.
{In this case, G_i contains $n_1 + n_2$ vertices, and $\overline{G_i}$ contains $n_3 + n_4$ vertices.}
Find a bipartition V_1, V_2 of the biconnected graph G_i such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Find a bipartition V_3, V_4 of the biconnected graph $\overline{G_i}$ such that $u_3 \in V_3$, $u_4 \in V_4$, $|V_3| = n_3$, $|V_4| = n_4$, and both V_3 and V_4 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.
Case 2: $r \geq 1$.
{ In this case, G_i contains $n_1 + n_2 + r$ vertices, and $\overline{G_i} = \overline{G_{i-1}} - W_i$ contains $n_3 + n_4 - r$ vertices. Since $r \geq 1$, $|W_i| \geq 2$ and hence W_i is an outer chain of either $\overline{G_{i-1}}$ or G_i.}
Let $C(\overline{G_{i-1}}) = w_1, w_2, ..., w_h, w_1$ where $w_1 = u_4$;
Assume that $W_i = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of $\overline{G_{i-1}}$ as illustrated in Fig 2(c), otherwise, interchange the roles of u_1, u_2 and u_3, u_4;
Find an st-numbering $v_1, v_2, \cdots, v_{n_3+n_4-r}$ of $\overline{G_i}$ such that $s = v_1 = u_4$ and $t = v_{n_3+n_4-r} = u_3$;
Let $w_p = v_{p'}$ and $w_q = v_{q'}$;
Assume that $p' < q'$, otherwise, interchange the roles of u_3 and u_4;
There are the following three subcases (a) $n_4 \leq p'$, (b) $p' + r \leq n_4$, and (c) $p' < n_4 < p' + r$;
Subcase 2(a): $n_4 \leq p'$.
{In this subcase, the last r vertices in the outer chain W_i are added to $\overline{G_i}$ as the deficient r vertices.}
Let $V_4 = \{v_1, v_2, \cdots, v_{n_4}\}$ be the first n_4 vertices in the st-numbering of $\overline{G_i}$;
Let $V_3' = \{v_{n_4+1}, v_{n_4+2}, \cdots, v_{n_4+n_3-r}\}$ be the remaining $n_3 - r$ vertices in $\overline{G_i}$;
{By the fact (st3) of an st-numbering both V_4 and V_3' induce connected graphs.}
Let $W_i' = \{w_{q-1}, w_{q-2}, \cdots, w_{q-r}\}$ be the set of the last r vertices in W_i;
Let $V_3 = V_3' \cup W_i'$;
{Since w_{q-1} is adjacent to w_q in V_3', V_3 induces a connected graph of n_3 vertices.}
Let $G_{12} = G_i - W_i'$;
{ G_{12} is biconnected by Lemma 4(b), and has $n_1 + n_2$ vertices.}
Find a bipartition V_1, V_2 of G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.
Subcase 2(b): $p' + r \leq n_4$.
{In this subcase, the first r vertices in W_i are added to $\overline{G_i}$ as the deficient r vertices.}

Let $V_4' = \{v_1, v_2, \cdots, v_{n_4-r}\}$ be the set of the first $n_4 - r$ vertices of $\overline{G_i}$, where $w_p = v_{p'} \in V_4'$;
Let $V_3 = \{v_{n_4-r+1}, v_{n_4-r+2}, \cdots, v_{n_4+n_3-r}\}$ be the remaining n_3 vertices of $\overline{G_i}$;
Let $W_i' = \{w_{p+1}, w_{p+2}, \cdots, w_{p+r}\}$ be the set of first r vertices in W_i;
Let $V_4 = V_4' \cup W_i'$;
$\{V_3$ and V_4 induce connected graphs having n_3 and n_4 vertices, respectively.$\}$
Let $G_{12} = G_i - W_i'$;
Find a bipartition V_1, V_2 of the biconnected graph G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.
Subcase 2(c): $p' < n_4 < p' + r$.
{In this subcase, the first $n_4 - p'$ and the last $r - (n_4 - p')$ vertices in W_i are added to $\overline{G_i}$ as the deficient r vertices.}
Let $W_{i4}' = \{w_{p+1}, w_{p+2}, \cdots, w_{p+n_4-p'}\}$ be the set of the first $n_4 - p'$ vertices in W_i;
Let $W_{i3}' = \{w_{q-1}, w_{q-2}, \cdots, w_{q-(r-n_4+p')}\}$ be the set of the last $r - (n_4 - p')$ vertices in W_i;
{Since $|W_{i4}'| + |W_{i3}'| = r \leq |W_i|$, $W_{i4}' \cap W_{i3}' = \phi$ and $|W_{i4}' \cup W_{i3}'| = r$};
Let $V_4 = \{v_1, v_2, \cdots, v_{p'}\} \cup W_{i4}'$;
Let $V_3 = \{v_{p'+1}, v_{p'+2}, \cdots, v_{n_4+n_3-r}\} \cup W_{i3}'$;
{ $|V_4| = n_4$, $|V_3| = n_3$, $w_p = v_{p'} \in V_4$, $w_q \in V_3$, and hence both V_4 and V_3 induce connected graphs.}
Let $G_{12} = G_i - W_{i4}' \cup W_{i3}'$;
Find a bipartition V_1, V_2 of the biconnected graph G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.

□

Clearly the running time of the above algorithm is $O(n)$. Thus we have the following theorem.

Theorem 5. *A 4-partition of any 4-connected plane graph G can be found in linear time if the four vertices u_1, u_2, u_3, u_4 are located on the same face of G.*

One can easily derive the following fact from Lemma 3 or directly from the canonical 4-ordering by Kant and He [K94, KH94].

Fact 6. *For any given internally triangulated 4-connected plane graph $G = (V, E)$, two distinct edges (u_1, u_2) and (u_3, u_4) on $C(G)$, and two numbers n_1, n_2 such that $n_1 + n_2 = n$ and $n_1, n_2 \geq 3$, there exists a partition V_1, V_2 of V such that $u_1, u_2 \in V_1$, $u_3, u_4 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce biconnected subgraphs of G.*

Proof. By Lemma 3, G has a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$. Since G is internally triangulated, each $W_i, i = 2, 3, \cdots, l-1$, is not an outer chain of G_i or $\overline{G_{i-1}}$ and hence each W_i consists of exactly one vertex on both $C(G_i)$ and $C(\overline{G_{i-1}})$. Thus all W_i's except W_1 and W_l are singleton sets, each of W_1 and W_l contains exactly three vertices, and hence $l = n-4$. For j, $1 < j < l$, the vertex in W_j has four or more neighbors, two of which are in $W_1 \cup W_2 \cup \cdots \cup W_{j-1}$ and other two of which are in $W_{j+1} \cup W_{j+2} \cdots \cup W_l$. Thus, for j, $1 < j < l$, both $V_1 = W_1 \cup W_2 \cup \cdots \cup W_j$ and $V_2 = V - (W_1 \cup W_2 \cup \cdots \cup W_j)$ induce biconnected graphs. Hence, it suffices to choose $j = n_1 - 2$. □

4 Conclusion

In this paper we give a linear-time algorithm to find a 4-partition of a 4-connected plane graph G in the case four vertices u_1, u_2, u_3, u_4 are located on the same face of G. It is remained as future work to find efficient algorithms for finding a k-partition of a k-connected (not always planar) graph for $k \geq 4$.

References

[DF85] M.E. Dyer and A.M. Frieze, *On the complexity of partitioning graphs into connected subgraphs*, Discrete Applied Mathematics, 10 (1985) 139-153.

[E79] S. Even, *Graph Algorithms*, Computer Science Press, Potomac (1979).

[G78] E. Györi, *On division of connected subgraphs*, Proc. 5th Hungarian Combinational Coll., (1978) 485-494.

[G96] E. Györi, *Private communication*, March 21, 1996.

[JSN94] L. Jou, H. Suzuki and T. Nishizeki, *A linear algorithm for finding a non-separating ear decomposition of triconnected planar graphs*, Tech. Rep. of Information Processing Society of Japan, AL40-3 (1994).

[K94] G. Kant, *A more compact visibility representation*, Proc. of the 19th International Workshop on Graph Theoretic Concepts in Computer Science (WG'93), LNCS 790 (1994) 411-424.

[KH94] G. Kant and X. He, *Two algorithms for finding rectangular duals of planar graphs*, Proc. of the 19th International Workshop on Graph Theoretic Concepts in Computer Science (WG'93), LNCS 790 (1994) 396-410.

[L77] L. Lovász, *A homology theory for spanning trees of a graph*, Acta Math. Acad. Sci. Hunger, 30 (1977) 241-251.

[MM94] J. Ma and S. H. Ma, *An $O(k^2n^2)$ algorithm to find a k-partition in a k-connected graph*, J. of Computer Sci. & Technol., 9, 1 (1994) 86-91.

[STN90] H. Suzuki, N. Takahashi and T. Nishizeki, *A linear algorithm for bipartition of biconnected graphs*, Information Processing Letters 33, 5 (1990) 227-232.

[STNMU90] H. Suzuki, N. Takahashi, T. Nishizeki, H. Miyano and S. Ueno, *An algorithm for tripartitioning 3-connected graphs*, Journal of Information Processing Society of Japan 31, 5 (1990) 584-592.

[WK94] K. Wada and K. Kawaguchi, *Efficient algorithms for triconnected graphs and 3-edge-connected graphs*, Proc. of the 19th International Workshop on Graph Theoretic Concepts in Computer Science (WG'93), LNCS 790 (1994) 132-143.

[WTK95] K. Wada, A. Takaki and K. Kawaguchi, *Efficient algorithms for a mixed k-partition problem of graphs without specifying bases*, Proc. of the 20th International Workshop on Graph Theoretic Concepts in Computer Science (WG'94), LNCS 903 (1995) 319-330.