# Drawing 2-, 3- and 4-colorable Graphs in O(n$^2$) Volume

Tiziana Calamoneri and Andrea Sterbini

University of Rome "La Sapienza", Dept. of Computer Science,
Via Salaria 113, 00198 Roma, Italy
{calamo, andrea}@dsi.uniroma1.it

**Abstract.** A Fary grid drawing of a graph is a drawing on a three-dimensional grid such that vertices are placed at integer coordinates and edges are straight-lines such that no edge crossings are allowed.
In this paper it is proved that each $k$-colorable graph ($k \geq 2$) needs at least $\Omega(n^{3/2})$ volume to be drawn. Furthermore, it is shown how to draw 2-, 3- and 4-colorable graphs in a Fary grid fashion in $O(n^2)$ volume.

**Keywords:** $k$-colorable graphs, Fary grid drawing, 3D drawing.

## 1 Introduction

In the last years three-dimensional drawing of graphs has increased its interest since the cost of 3D circuits has become reasonably low and recent advances in hardware and software technology for computer graphics have opened the possibility of displaying three-dimensional drawings. At present, the related research tends towards theoretical results understanding of 3D drawings. In particular, the most studied models are the orthogonal grid drawing and the Fary grid drawing, both having large application interest [3, 4, 5, 6]. In this paper we point our attention on the Fary grid drawing and we investigate the volume of such a drawing for some particular classes of graphs.

The *Fary grid drawing* [2] of a graph is a drawing on a three-dimensional grid such that vertices are placed at integer coordinates and edges are straight-lines such that no edge-crossings are allowed.

It is suitable to minimize both the volume and the length of the maximum side of the rectangular prism containing the drawing.

In [1] a method for drawing a general graph in a Fary grid fashion in $O(n^3)$ volume is shown, and it is proved that no general drawing method can achieve smaller size, up to a constant. In the same paper it is shown that for every planar graph and upward binary tree there exists a Fary grid drawing with $O(n^2)$ and $O(n \log n)$ volume, respectively, and it is conjectured that some other classes of graphs need a smaller volume than in the general case.

In this paper we partially solve the previous conjecture by showing how to draw 2-, 3- and 4-colorable graphs in a Fary grid fashion in $O(n^2)$ volume.

Furthermore, it is proved that each $k$-colorable graph ($k \geq 2$) needs at least $\Omega(n^{3/2})$ volume to be drawn.

## 2 Fary grid drawing of complete bipartite graphs

In this section we show that $\Omega(n^{3/2})$ volume is required to draw a complete bipartite graph with $n$ vertices even if we conjecture that this lower bound can be improved to $\Omega(n^2)$. Moreover, we show how to draw a complete bipartite graph in $O(n^2)$ volume.

Given any kind of three-dimensional drawing, we call its *rectangular hull* the smallest rectangular prism with sides parallel to the coordinate axes containing the whole drawing; its *size* the length of the longest side and its *volume* the product of its three sides.

Let $K_{r,b} = (R \cup B, E)$ be a complete bipartite graph, that is a graph having $R \cup B$ as vertices set and $E = \{e = (v_1, v_2), \forall v_1 \in R, v_2 \in B\}$ as edges set. From now on, we will call *red* all the vertices belonging to $R$, and *blue* all the vertices belonging to $B$; furthermore $|R| = r$, $|B| = b$ and $r + b = n$.

It is easy to see that the 'worst' case of bipartite graph, in terms of occupied volume, is the complete bipartite graph. Therefore, if we prove that at least $\Omega(n^{3/2})$ volume is needed by a complete bipartite graph in order to be drawn, then this value is a lower bound for the whole class of bipartite graphs.

**Lemma 1.** *Let $\mathcal{D}$ be a three-dimensional Fary grid drawing of the complete bipartite graph $K_{r,b}$, with $n$ vertices, where $r, b \geq 3$, and let $\mathcal{D}$ use volume $V$. Then $V$ is $\Omega(n^{3/2})$.*

*Proof.* Project along the $x$ axis all red vertices and consider their projections on plane $yz$. Then, two cases arise:

- The number of different-coordinates projections are exactly $r$. Then the area of the section of the rectangular hull perpendicular to the $x$ axis is at least $r$.
- The number of different-coordinates projections are less than $r$. Then, at least two different red vertices have the same projection along the $x$ axis. On the other hand, no pairs of blue vertices can lie on a straight-line parallel to the $x$ axis, otherwise a crossing would arise. Therefore, the area of the section of the rectangular hull perpendicular to the $x$ axis is at least $b$.

The same reasoning holds for the sections of the rectangular hull perpendicular to the $y, z$ axes. It is easy to see that the worst case can be obtained when $r = b = \frac{n}{2}$ and when all three sections of the rectangular hull are squares. In such a case each dimension of $V$ is at least $\Omega(n^{1/2})$ and the statement follows. □

Since each graph contains a bipartite graph, then the proof of the previous Lemma can be extended to all $k$-colorable graphs, for any $k \geq 2$. Therefore, the following result holds:

**Theorem 2.** *Let $\mathcal{D}$ be a three-dimensional Fary grid drawing of a k-colorable graph $\mathcal{G}$, with $n = n_1 + \ldots + n_k$ vertices, where $n_i$ is the number of vertices colored with color k. If there exist $i, j \leq k, i \neq j$, such that $n_i, n_j \geq 3$ and $\mathcal{D}$ uses volume V, then V is $\Omega(n^{3/2})$.*

Now we prove that an upper bound of the volume for bipartite graphs is $O(n^2)$ by exhibiting a construction of a Fary grid drawing of a complete bipartite graph.

**Theorem 3.** *Given a bipartite graph having n vertices, it is possible to draw it as a Fary grid drawing of $O(n^2)$ volume.*

*Proof.* We prove the theorem by constructing such a drawing.

Consider two skew straight-lines, with parametric equations ($x = p, y = 0, z = 0$) and ($x = 0, y = q, z = 1$). A Fary grid drawing of $K_{r,b}$ is obtained by placing all red vertices on one straight-line, i.e. at coordinates $R_i = (i, 0, 0), i = 1..r$ and all blue vertices on the other line, i.e. at coordinates $B_j = (0, j, 1), j = 1..b$, as shown in fig. 1.

Notice that every edge-crossing requires four co-planar vertices. Since the lines are skew, no four vertices $r_1, r_2 \in R$ and $b_1, b_2 \in B$ can lie on the same plane. This observation garantees that all crossings are avoided. It is easy to see that $O(n^2)$ volume is enough to fit the drawing. □

# 3 Fary grid drawing of 3- and 4-colorable graphs

The technique used to prove Theorem 3, based on the arrangment of each color on one of some skew straight-lines, can be extended in order to draw 3- and 4-colorable graphs having $n$ vertices in $O(n^2)$ volume.

## 3.1 3-colorable graphs

Let $K_{r,b,g} = (R \cup B \cup G, E)$ be a complete 3-colorable graph ,where $r, b$ and $g$ are the cardinalities of sets $R, B$ and $G$, respectively, and $r + b + g = n$.

It is possible to draw $K_{r,b,g}$ in a Fary grid fashion by using one of the following techniques, both consisting in placing three skew lines in such a way that all vertices of one color can be completely lain on each of them:

1. The three lines lie on three parallel planes ($z = 0, 1, 2$), they pass through the origin of the grid and run along the $x = 0, y = -x, y = x$ lines, respectively. Namely, their parametric equations are ($x = p, y = 0, z = 0$), ($x = -q, y = q, z = 1$) and ($x = s, y = s, z = 2$). The vertices will be placed at coordinates $R_i = (i, 0, 0), i = 1..r$, $B_j = (-j, j, 1), j = 1..b$ and $G_k = (-k, -k, 2), k = 1..g$ (see fig. 2). Observe that, in this way, different portions of space are assigned to the three subsets of edges connecting all possible couples of colors (*"separation property"*).

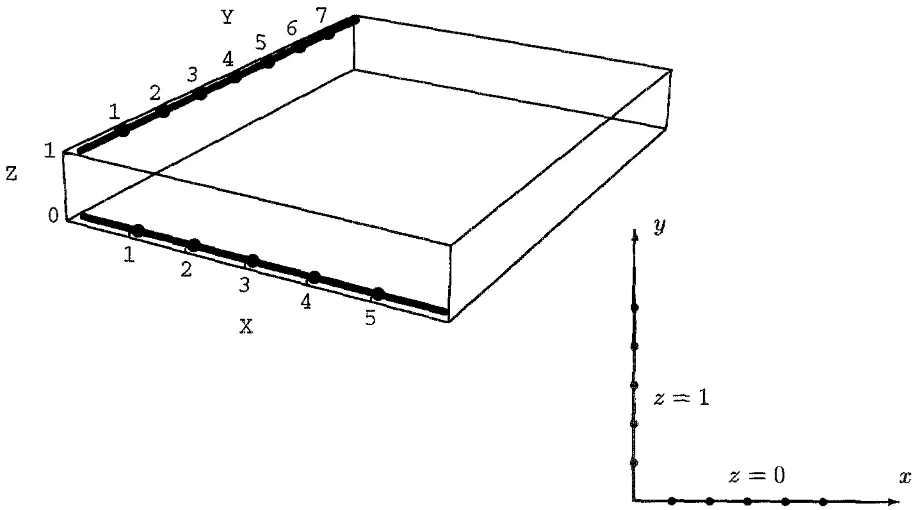


**Fig. 1.** Drawing of a complete bipartite graph in $O(n^2)$ volume and projection of the same drawing on the $xy$ plane.

2. The three lines lie on three parallel planes ($z = 0, 1, 2$) and their projections on the $xy$ plane form a rectangular triangle. Their parametric equations are $(x = 0, y = p, z = 0)$, $(x = q, y = \max(2g, 2r) - q, z = 1)$ and $(x = s, y = 0, z = 2)$, respectively. (see fig. 3). In this second configuration all the edges lie on the same portion of space.

**Theorem 4.** *Given a 3-colorable graph having n vertices, both previously described methods give a Fary grid drawing of $O(n^2)$ volume.*

*Proof.* In both the representations, we will call the three straight-lines by using the capital letters $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{G}$ to mean the lines containing red, blue and green vertices, respectively.

In the first drawing (fig. 2), it is possible to put a vertex on each grid-crossing belonging to one of the three considered straight-lines. No edge-crossings arise because of the *"separation property"* of the space assigned to the edges. Namely, given two edges, if they connect the same couple of colors, they cannot cross because the lines are skew. If the edges connect three colors (let us say, for instance, red-blue and red-green) they cannot cross because they lie in different portions of space.

The volume of such a drawing is $O(n^2)$ because the maximum length of each line is $O(n)$.

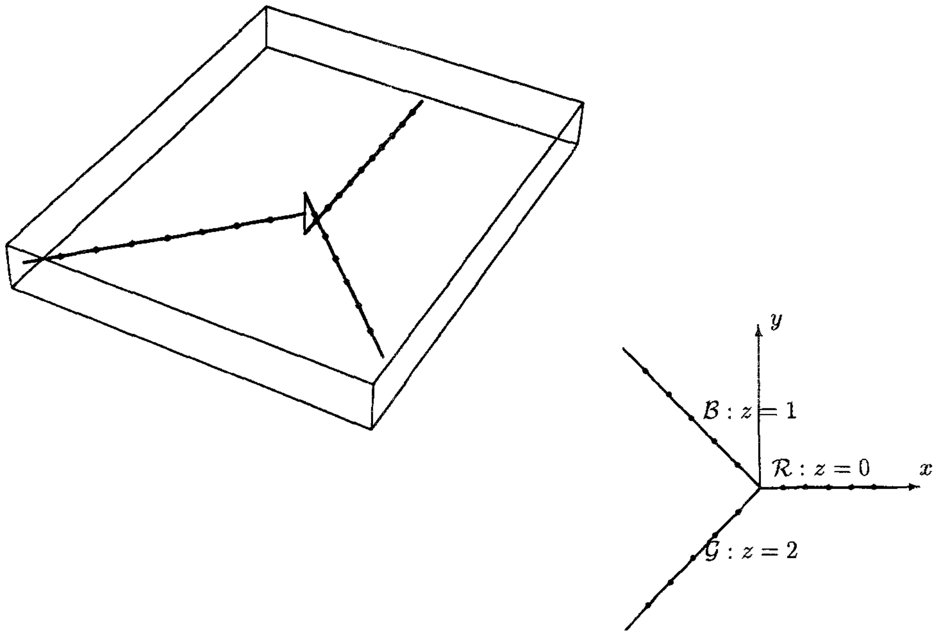


**Fig. 2.** 3-colorable graphs: first configuration. Drawing in $O(n^2)$ volume and projection of the same drawing on the $xy$ plane.

In the second case it is not possible to put a vertex on each grid-crossing belonging to the straight lines, but we must "forbid" some positions, in order to guarantee the absence of crossings, as it is explained in the following.

Consider fig. 3: $\mathcal{R}$ has null $z$-coordinate, $\mathcal{B}$ has $z$-coordinate equal to one and $\mathcal{G}$ has $z$-coordinate equal to two. Consider two arbitrary green vertices $g_1$ and $g_2$, and suppose $g_1$ adjacent to a red vertex $r_1$. The plane passing through the three points $g_1, g_2$ and $r_1$ intersects $\mathcal{B}$ in one point. If this point coincides with a grid-crossing and a blue vertex $b_2$ lies on it, then it is possible that the edges $(g_1, r_1)$ and $(g_2, b_2)$ intersect. In order to avoid this problem, consider all the planes containing straight-line $\mathcal{G}$ and passing through a grid-crossing of $\mathcal{B}$. Each of them intersects $\mathcal{R}$ in one point. The set of all such intersections on $\mathcal{R}$ is the set of gridpoints having even $y$-coordinate on $\mathcal{R}$ (see fig. 4). Then, it is enough to position red vertices only on those grid-crossings of $\mathcal{R}$ having odd $y$-coordinates. A similar reasoning can be done for green vertices by considering

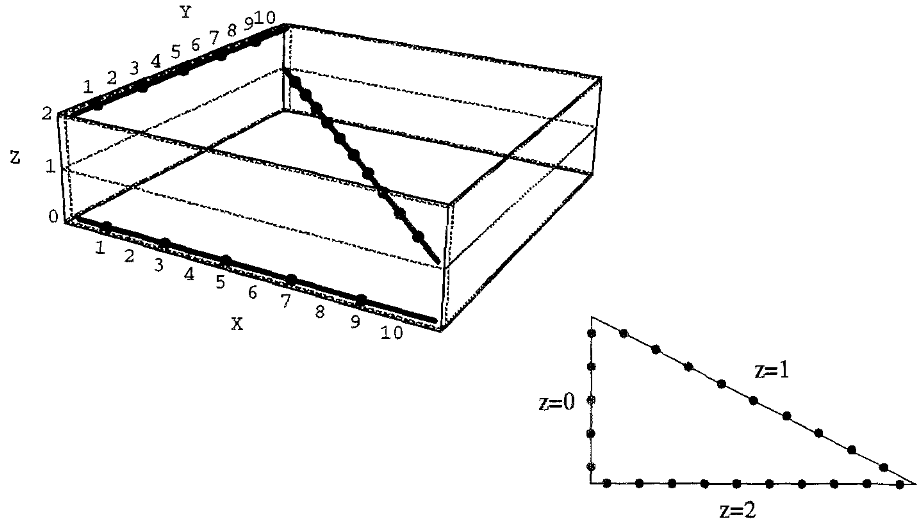


**Fig. 3.** 3-colorable graphs: second configuration. Drawing in $O(n^2)$ volume and projection of the same drawing on the $xy$ plane.

all planes containing straight-line $\mathcal{R}$. It is not possible that any plane containing straight-line $\mathcal{B}$ intersect both $\mathcal{R}$ and $\mathcal{G}$.

Since the maximum length of each line is $O(n)$ and the height of the rectangular hull is constant, the volume of such a drawing is $O(n^2)$. □

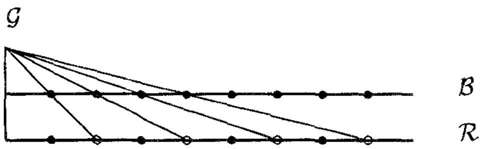


**Fig. 4.** Projection of the second configuration on plane $yz$.

Notice that in both our constructions, one dimension of the rectangular hull is constant, and it allows to bound the volume in $O(n^2)$. It would be interesting to find a method to draw a 3-colorable graph in a Fary grid fashion by using $O(n^2)$ volume, so that all three dimensions are $O(n^{3/2})$. This would guarantee a better bound for the size of the drawing and for the maximum distance between the vertices.

### 3.2 4-colorable graphs

In order to draw in $O(n^2)$ volume a 4-colorable graph, we use again straight-lines lying on four different parallel planes ($z = 0, 1, 2, 3$), such that their projections on the $xy$ plane form a convex quadrilateral (see fig. 5).

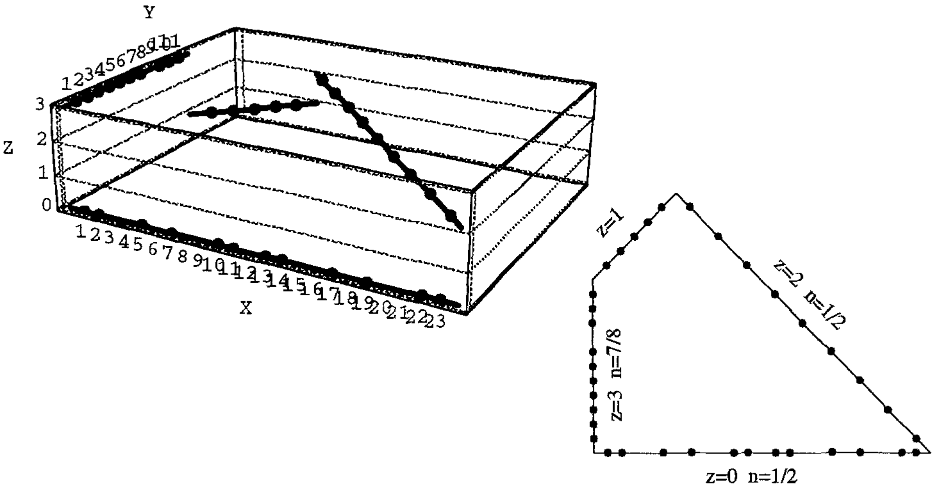


**Fig. 5.** Drawing of a 4-colorable graph in $O(n^2)$ volume and projection of the same drawing on the $xy$ plane.

We call $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{Y}$ the four straight-lines, lying on the planes $z = 0, 3, 1, 2$ and containing red, green, blue and yellow vertices, respectively.

Assume $b, \lfloor \frac{8}{7} g \rfloor$ be odd numbers. If they are not, round them to the next integer odd value.

The parametric equations of the 4 lines are:

line $\mathcal{R}$: $(x = p, y = 0, z = 0)$
line $\mathcal{G}$: $(x = 0, y = q, z = 3)$
line $\mathcal{B}$: $(x = s, y = s + \frac{8}{7} g, z = 1)$
line $\mathcal{Y}$: $(x = t + b, y = \frac{8}{7} g + b - t, z = 2)$

Position the vertices at the following coordinates:

$r$-vertices: $(x = p, y = 0, z = 0)$, where $p = 1..2r$ and $p \bmod 3 \neq 0, p \bmod 4 \neq 0$
$g$-vertices: $(x = 0, y = q, z = 3)$, where $q = 1..\frac{8}{7} g$ and $q \bmod 8 \neq 0$
$b$-vertices: $(x = s, y = s + \frac{8}{7} g, z = 1)$, where $s = 1..b$

$y$-vertices: $(x = t + b, y = \frac{8}{7}g + b - t, z = 2)$, where $t = 1..2y$ and $q \bmod 2 = 0$

**Theorem 5.** *Given a 4-colorable graph having $n$ vertices, the described method is a Fary grid drawing of $O(n^2)$ volume.*

*Proof.* In order to prove that the stated drawing is Fary grid we must show that no crossing of edges can arise. Given a couple of edges, they can connect either 2, or 3 or 4 different colors (see fig. 6). Therefore there are three possible kinds of crossings we need to avoid:

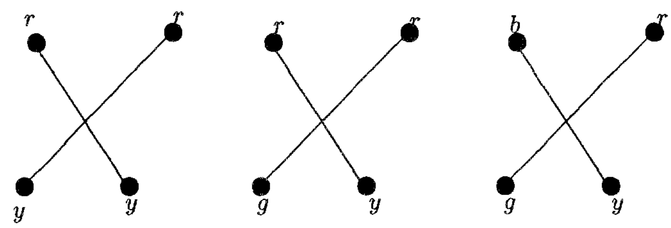


**Fig. 6.** possible crossings in a 4-colorable graph.

**involving 2 colors:** By placing the lines with different directions on different planes, we obtain mutually skew lines and then we are assured that no crossings can arise between two edges involving only 2 colors.

**involving 3 colors:** Using a technique similar to the proof of theorem 4, by removing some vertices along the lines, we assure that no crossings can arise between two edges involving 3 colors.
It can be shown that, on $\mathcal{G}$ we must forbid one grid-crossing out of 8, on $\mathcal{B}$ all the positions remain available, on $\mathcal{Y}$ one out of 2 can be filled up, finally one out of 3 plus one out of 4 (i.e. $(4+3-1)/12 = 1/2$) grid-crossings cannot be used on $\mathcal{R}$ (fig. 5).

**involving 4 colors:** We must check for crossings originated by two edges with their four endpoints all belonging to different colors. In view of the shape of the quadrilateral, the only way to have such a crossing is between edges connecting colors lying on opposite sides of the quadrilateral, that is, between an edge connecting an $r$-vertex to a $b$-vertex and an edge connecting a $g$-vertex to a $y$-vertex. These crossings can be avoided by placing the four lines $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{Y}$ on the planes at levels $z = 0, 3, 1$ and $2$, respectively. In this way, all the edges connecting an $r$-vertex to a $b$-vertex and a $g$-vertex to a $y$-vertex lie in different half-spaces (separated, for example, by the plane $z = 3/2$).

Now we justify the assumption that $b$, $\lfloor \frac{8}{7} \rfloor$ are odd.

Let us call $L_i$ the number of gridpoints of line $i$, where $i \in \{\mathcal{R}, \mathcal{B}, \mathcal{G}, \mathcal{Y}\}$. In view of the shape of the quadrilateral, the following relations hold:

$L_{\mathcal{R}} = L_{\mathcal{G}} + 2L_{\mathcal{B}}$ and $L_{\mathcal{Y}} = L_{\mathcal{G}} + L_{\mathcal{B}}$.

From them, it may be inferred that:

- if $L_{\mathcal{G}}$ is even then $L_{\mathcal{R}}$ is even
- if $L_{\mathcal{G}}$ and $L_{\mathcal{B}}$ have the same parity than $L_{\mathcal{Y}}$ is even

therefore, by choosing both $L_{\mathcal{G}}$ and $L_{\mathcal{B}}$ even, we obtain all four even-length lines. This is desirable, otherwise all the grid-crossings on one line would be forbidden in order to avoid crossings involving 3 colors.

Since the height of the rectangular hull is 3, its volume depends only on the area of the $xy$ projection of the rectangular hull. The worst case arises when all the four lines have $O(n)$ length, therefore the area of the quadrilateral is $O(n^2)$. □

Notice that it is not immediate to extend this construction to 5-colorable graphs for two main reasons. First of all, too many positions should be 'forbidden'; secondly, the 'trick' used to guarantee no crossings involving 4 colors cannot be used anymore. Actually, it is not clear if there is a way to draw 5-colorable graphs in a Fary grid fashion using $O(n^2)$ volume or not.

## 4 Conclusions and Open Problems

In this work we have described some results dealing with the volume of three-dimensional drawing of graphs. Namely, we have established a lower bound of $\Omega(n^{3/2})$ volume for each $k$-colorable graph and an upper bound of $\Omega(n^2)$ volume for 2-, 3- and 4-colorable graphs. Observe that we have made no efforts to compute a drawing with optimal multiplying constant since our aim was to improve the previous bounds for these classes of graphs. It may be that a different arrangment of colors on the skew-lines allows to 'forbid' less positions and therefore to get a smaller constant. Furthermore, it would be interesting to compute a more precise volume by means of a more general bounding convex prism instead of the rectangular hull. Moreover, there are a lot of open problems to solve and of questions to answer. Among them, we will thick those ones encountered in the paper:

- we conjecture that complete bipartite graphs $K_{m,n}$ with $m, n \geq 3$ cannot be drawn in less than $\Omega(n^2)$ volume. This would imply that all $k$-colorable graphs need at least the same volume.
  How is it possible to close the gap between the lower bound and the upper bound?
- Is it possible to draw in a Fary grid fashion 2-, 3- and 4-colorable graphs such that the size is minimum, that is $O(n^{2/3})$?
- Is it possible to extend our construction based on skew lines to $k$-colorable graphs, $k \geq 5$? If the answer is no, is there a way to prove that these graphs need more than $\Omega(n^2)$ volume?
- Do there exist other classes of graphs needing less than $O(n^3)$ volume?

## References


1. R.F.Cohen, P.Eades, T.Lin and F.Ruskey: *Three-dimensional graph drawing*, In Proc.GD '94, Lecture Notes in Comp. Sci. 894, pp. 1-11, 1994.
2. I.Fary: *On straight lines representations of planar graphs*, Acta Sci. Math. Szeged, 11, pp. 229-233 , 1948.
3. G.Frank, D.Hui and C.Ware: *Visualizing object oriented software in three dimensions*, In Proc.CASCON, 1993.
4. J.Mackinley, G.Robertson and S. Card: *Cone trees: Animated 3d visualization of hierarchical information*, In Proc.SIGCHI Conf. on Human Factors in Computing, pp. 189-194, 1991.
5. P.Reid: *Dynamic Interactive Display of Complex Data Structures*, In Graphics Tools for Software Engineers, pp. 62–70, Cambridge, 1989.
6. O.Tversky, S.Snibbe and R.Zeleznik: *Cone trees in the uga graphics system: suggestions of a more robust visualization tool*, Technical Report CS-93-07, Brown University, 1993.