Bandwidth and Cutwidth of the Mesh of *d*-Ary Trees*

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Abstract. We mainly show that the cutwidth of the mesh of d-ary trees $MT(d,n)$ satisfies $\frac{d^{n-2}(d+1)^2}{8} - 1 \leq c(MT(d,n)) \leq \frac{d^{n+3}}{d-1}$; if $d > 2$, we also show that the bandwidth of this graph $b(MT(d,n))$ is in $\theta\left(d^{n+1}\frac{d^n-1}{n(d-1)}\right)$.

1 Introduction and definitions

In this paper, we focus on the bandwidth and the cutwidth of a graph, two well known graph parameters. They are defined as follows. We use graph theory notations of [3]; let G be a graph, $V(G)$ (resp. $E(G)$) the set of vertices (resp. edges) of G. Consider $\mathcal{L}(G)$ the set of all the labelings of $V(G)$; a *labeling* of $V(G)$ is a bijection l between $V(G)$ and $\{0,\dots,|V(G)|-1\}$. The bandwidth of G is defined by $b(G) = \min_{l\in\mathcal{L}(G)}\left(\max_{[X,X']\in E(G)} |l(X)-l(X')|\right)$.
The cutwidth of G is $c(G) = \min_{l\in\mathcal{L}(G)}\left(\max_{X\in V(G)} c_l(X)\right)$, where

$$c_l(X) = |\{[Y,Y'] \in E(G) : l(Y) \leq l(X) < l(Y')\}|.$$

Finding the bandwidth and the cutwidth of a graph are known to be NP-complete problems [5, 7]. These parameters are useful to determine good VLSI designs for interconnection networks, by considering the *Thompson grid model* of VLSI layout [8]. We deal here with the mesh of d-ary trees. This graph is an interesting interconnection network for parallelism, since it uses both tree and grid structures (see section 2). Good parallel algorithms have been developed in it [1, 6], and it has been proposed as a good parallel computer topology for some applications to images analysis [4].
Let us first precise that in all the following, we use some language theory notations. Let $\{0,\dots,d-1\}$ be an alphabet, with $d \geq 2$, and v be a word on it. We denote by $|v|$ the *length* of v, i.e. the number of letters in v; we denote by e the empty word, i.e. $|e| = 0$. Each letter $x \in \{0,\dots,d-1\}$ is also considered as an element of $\mathbb{Z}_d$.

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The mesh of d-ary trees, introduced in [1], is a generalization of the mesh of binary trees (i.e. $MT(2,n)$) [6]. It is defined as follows.

- The vertices of $MT(d,n)$ are all the couples $(u;v)$ of words of $\{0,..,d-1\}^*$, such that $|u|=n$ and $|v|<n$, or $|v|=n$ and $|u|\leq n$.
- There is an edge between two vertices $(u;v)$ and $(w;z)$ in $MT(d,n)$ iff $|u|=n$, $u=w$ and $[v,z]\in E(T(d,n))$, or $|v|=n$, $v=z$ and $[u;w]\in E(T(d,n))$. This edge is denoted by the pair $[(u;v),(w;z)]$.

By definition, for any $u\in\{0,\ldots,d-1\}^n$, the subgraph of $MT(d,n)$ induced by the vertices of the form $(u;v)$ (resp. $(v;u)$), with $v\in\{0,\ldots,d-1\}^*$, is isomorphic to $T(d,n)$. In $MT(d,n)$, the column tree (resp. line tree), isomorphic to $T(d,n)$ with root $(e;u)$ (resp. $(u;e)$), is denoted by $T_{(e;u)}$ (resp. $T_{(u;e)}$).
The number of vertices of $MT(d,n)$ is $|V(MT(d,n))| = d^n\left(d^n+2\frac{d^n-1}{d-1}\right)$.We will also use the following recursive construction of $MT(d,n)$ from $MT(d,n-1)$.
1. Consider first d^2 disjoint copies of $MT(d,n-1)$, each one denoted by $MT_{i,j}(d,n-1)$ with $(i,j)\in\{0,\ldots,d-1\}^2$. A vertex $(u;v)$ in $MT_{i,j}(d,n-1)$ is denoted by $(u;v)_{i,j}$.
2. We add $2d^n$ new vertices : d^n vertices $(w;e)$, with $w\in\{0,\ldots,d-1\}^n$ and d^n vertices $(e;w)$. Then, we add the set of edges

$$\bigcup_{(i,j)\in\{0,\ldots,d-1\}^2}\{[(e;w),(e;u)_{i,j}]:\ w=ui\}\bigcup\{[(w;e),(u;e)_{i,j}]:\ w=ui\}.$$

By associating each vertex $(u;v)_{i,j}$ to the vertex $(ui;vj)$ in $MT(d,n)$, it is easy to see that the graph we obtain is isomorphic to $MT(d,n)$.

We also give the notations and the definitions we use in the next section, similar to the ones of [2]. Let G be a graph and Π be a partition of $V(G)$. For each element $\pi\in\Pi$, we define $\omega(\pi)$ as the *cocycle* of π, i.e. the set $\{[X,X']\in E(G): X\in\pi,\ X'\notin\pi\}$. We note $max_\Pi=\max_{\pi\in\Pi}|\pi|$ and $max_\omega=\max_{\pi\in\Pi}|\omega(\pi)|$. We also denote by $G[\pi]$ the subgraph of G induced by π.
• The *quotient graph of* G *by* Π, denoted by $Q=G/_\Pi$, is defined by
 - $V(Q)=\Pi$,
 - $[\pi,\pi']\in E(Q)\Leftrightarrow(\pi\neq\pi',\exists X\in\pi,\ \exists X'\in\pi':\ [X,X']\in E(G))$.
• Let l_Q be a labeling of $V(Q)$ and l_G be a labeling of $V(G)$. For each $\pi\in\Pi$, we represent by $l_G[\pi]$ the set of all the labels of vertices in π by l_G. We say that l_G is *compatible* with l_Q if for any $\pi\in\Pi$, we have $l_G[\pi]$ is an interval $[m_\pi,..,M_\pi]$ and if for each $\pi'\in\Pi$ such that $l_Q(\pi')<l_Q(\pi)$ (resp. $l_Q(\pi')<l_Q(\pi)$), $M_{\pi'}<m_\pi$ (resp. $m_{\pi'}>M_\pi$).
• For each $\pi\in\Pi$,

$$\delta^+_\omega(\pi)=\min_{X\in\pi}|\{[X,X']\in E(G):\ X'\in\pi',\ \pi' \text{ such that } l_Q(\pi')>l_Q(\pi)\}|$$
$$\delta^-_\omega(\pi)=\min_{X\in\pi}|\{[X,X']\in E(G):\ X'\in\pi',\ \pi' \text{ such that } l_Q(\pi')<l_Q(\pi)\}|.$$

The *edge-bissection* of G is denoted by $bis_e(G)$ (see [3]).
With a general result of [2] and with an original construction, we show in [1] the next result.

Proposition 1. $b(MT(d,n))$ *is in* $\theta\left(d^{n+1}\frac{d^n-1}{n(d-1)}\right)$

In the next section, we also use the following result from [2].

Theorem 2. *Let* l_Q *be a labeling of* $V(Q)$ *achieving the cutwidth of* Q*, and let* l_G *be a labeling of* $V(G)$ *compatible with* l_Q*.*

$$\begin{aligned}(1.)\ c(G) \leq & \left(c(Q) - \left\lceil \frac{\delta(Q)}{2} \right\rceil\right) \cdot max_\omega \\ & + \max_{\pi \in \Pi} \left(c(G[\pi]) + \left(|\omega(\pi)| - |\pi| \cdot \min\left(\delta^+_\omega(\pi), \delta^-_\omega(\pi)\right)\right)\right)\end{aligned}$$

$$(2.)\ c(G) \geq bis_e(G)$$

2 The cutwidth of $MT(d,n)$

Proposition 3. *If* $n \geq 2$*, then* $c(MT(2,n))$ *is in* $\theta(2^n)$ *and if* $d \geq 3$*,*

- $c(MT(d,n)) \geq \frac{d^{n+2}(d^n(d+1)-2)^2}{4(d^{2n}((d^3-d)(d+2)+1)-d^n(d^4+2d^3+3d^2-4d+1)+d^3+3d^2-d))}$
- $c(MT(d,n)) \leq \frac{d^n(d^3+d^2+4)-(d^4-3d+2)}{2(d-1)}$

Corollary 4. *If* $n \geq 2$*, then if* $d \geq 3$*,* $\frac{d^{n-2}(d+1)^2}{8} - 1 \leq c(MT(d,n)) \leq \frac{d^{n+3}}{d-1}$.

Proof of the proposition.

1. *Let us show the upper bound.*

a. Let us first define a partition Π of $V(MT(d,n))$, with $d \geq 2$ and $n > 1$. Π contains d^2+2d parts : d^2 parts $\pi_{i,j}$ with $(i,j) \in \{0,\dots,d-1\}^2$; d other parts, each one denoted by $\pi_{i,e}$, and d last parts $\pi_{e,j}$. They are defined by

- $\pi_{i,j}$ is the set of all the vertices $(m;m') \in V(MT(d,n))$ such that i is the first letter of m and j the first letter of m'.
- $\pi_{i,e}$ (resp $\pi_{e;j}$) is the set of all the vertices $(m;e)$ (resp $(e;m')$) where i (resp. j) is the first letter of m (resp. m')

By definition, $\pi_{i;e}$ and $\pi_{e;j}$ are two independant sets of vertices of $MT(d,1)$. Moreover, it is easy to see that Π is a partition of $V(MT(d,n))$. Let us denote by Q the graph $G/_\Pi$. We now consider a couple $(i,j) \in \{0,\dots,d-1\}^2$.

• Consider $(iu;jv)$ a vertex in a part $\pi_{i,j}$. If X is a vertex in $MT(d,n)$, with $X \notin \pi_{i,j}$, and if $[(iu;jv),X] \in E(MT(d,n))$, then $v=e$ and $X=(iu;e) \in \pi_{i,e}$, or $u=e$ and $X=(e;jv) \in \pi_{e,j}$.

Hence, the edges of Q are pairs $[\pi_{i,j},\pi_{i;e}]$ and $[\pi_{i,j},\pi_{e;j}]$. Then, by associating to each part $\pi_{i,j}$ the couple $(i;j)$, and to each part $\pi_{i,e}$ (resp. $\pi_{e,j}$) the couple $(i;e)$ (resp. $(e;j)$), we can conclude that Q is isomorphic to $MT(d,1)$.

• The subgraph of $MT(d,n)$ induced by $\pi_{i,j}$ is isomorphic to $MT(d,n-1)$. This can be directly deduced from the definition of Π and by following the recursive construction of $MT(d,n)$ from $MT(d,n-1)$ given in section 1: we associate to each vertex $(iu;jv)$ in $\pi_{i,j}$ the vertex $(u;v) \in V(MT(d,n-1))$. Since $|\pi_{i,e}| = |\pi_{e,j}| = d^n$, then $max_\Pi = |V(MT(d,n-1))|$.

• $|\omega(\pi_{i,j})| = |\{(iu;j) \in \pi_{i,j}\} \cup \{(i;vj) \in \pi_{i,j}\}| = 2d^{n-1}$. Moreover, in $MT(d,n)$

the degree of each vertex from $\pi_{i;e}$ and from $\pi_{e,j}$ is equal to d. Hence, since $|\pi_{i,e}| = |\pi_{e,j}| = d^{n-1}$, then $|\omega(\pi_{i,e})| = |\omega(\pi_{e,j})| = d|\pi_{i,e}| = d^n$. So $max_\omega = d^n$.

b. To use Theorem 2, we give an upper bound for $c(MT(d,1))$ by using a labelling of $T(d,1)$ (see [1]), for $d > 2$. Hence, we show that if $d > 2$, $c(MT(d,1)) \leq (d+2)\lceil \frac{d}{2} \rceil - d$. If $d = 2$, $MT(2,1)$ is a cycle of length 8 and so $c(MT(2,1)) = 2$.

c. We can now apply Theorem 2. Assume $d > 2$ and $n > 1$,

$$c(MT(d,n)) \leq \left((d+2)\lceil \tfrac{d}{2} \rceil - d - 1\right) d^n + \\ max\left(c(MT(d,n-1)) + 2d^{n-1};\ d^n - (d^{n-1}.\lfloor \tfrac{d}{2} \rfloor)\right)$$

We then deduce from this inequality an upper bound for $c(MT(d,n))$, i.e. $c(MT(d,n)) \leq \left\lceil \frac{d^n(d^3+d^2+4)-(d^4+3d+2)}{2(d-1)} \right\rceil$. If $d = 2$, we show by the same way that $c(MT(2,n)) \leq 2^{n+2} - 6$.

2. *We now deal with the lower bound.* We give a detailed sketch of the proof. We know that $bis_e(MT(2,n))$ is in $\theta(2^n)$ [6]. To determine $bis_e(MT(d,n))$ with $d > 2$, we give a *routing function* R in $MT(d,n)$. Then, $bis_e(MT(d,n)) \geq \frac{|V(MT(d,n))|^2+1}{2 \cdot cg(R)}$, with $cg(R)$ the congestion of R (see [1]). Thus, we show that $bis_e(MT(d,n)) \geq$

$$\frac{d^{n+2}(d^n(d+1)-2)^2}{4(d^{2n}((d^3-d)(d+2)+1) - d^n(d^4+2d^3+3d^2-4d+1) + d^3+3d^2-d))}$$

We conclude with Theorem 2.2 . □

References

1. D. Barth, Réseaux d'interconnexion : structures et communications, Thesis, Université Bordeaux I, 1994.
2. D. Barth, F. Pellegrini, J. Roman, A. Raspaud, On bandwidth, cutwidth and quotient graphs, LaBRI intern report, RAIRO Theor. Comp. Science, vol. 29, n. 6, pp. 487-508, 1995.
3. C. Berge, Graphes et Hypergraphes, North Holland Publishing, 1971.
4. M.M. Eshaghian, V.K. Prasanna Kumar, Parallel Geometric Algorithms for Digitized Pictures on Mesh of Trees, *Proc. of IEEE FOCS*, pp 270-273, 1986.
5. F. Gavril, Some NP-complete problems on graphs, Proc. 11th Conf. on Inf. sciences and systems, W.H Freeman pub., San Francisco, 1979.
6. F.T. Leighton, Introduction to Parallel Algorithms and Architecture: arrays, trees, hypercubes, Morgan Kaufman Publisher, 1992.
7. C.H. Papadimitriou, The NP-completeness of the bandwidth minimization problem, Computing, vol. 16, pp. 263-270, 1976.
8. C.D. Thompson, Area-time complexity for VLSI, Proc. 11th STOCS, ACM, pp. 81-88, 1979.