The Size Complexity of Strictly Non-blocking Fixed Ratio Concentrators with Constant Depth

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Abstract. Concentrators are interconnection networks that provide vertex-disjoint directed paths to satisfy interconnection requests. An interconnection network is non-blocking in the strict sense if every compatible interconnection request can be satisfied regardless of any existing interconnections. We show a size optimal bound of $\Theta(n^{1+\frac{1}{k}})$ for synchronous strictly non-blocking γn-limited $(\alpha n, \beta n)$-concentrators with non-full capacity and constant depth k, and present a size upper bound of $O(n^{1+\frac{1}{\lceil k/2 \rceil}})$ for synchronous strictly non-blocking βn-limited $(\alpha n, \beta n)$-concentrators with full capacity and constant depth k.

1 Introduction

The interconnection network of an information transmission system provides mediation between a set of inputs (sources of information) and a set of outputs (sinks of information). The graph-theoretic design and analysis of interconnection patterns can idealize such networks. Interconnection networks are classified according to the type of interconnections that they provide. Several generic types of interconnections that have been studied extensively in the literature (see [Clo53], [Ben65], [DDPW83], [FFP88], and [Dai91]) are concentration, superconcentration, connection, expansion, and partition. An orthogonal classification of interconnection networks, based upon the network capability of allowing an interconnection to be destroyed and created dynamically without disturbing any of the other existing interconnections in the network, results in three genera of interconnection capability: rearrangeability, wide-sense non-blockingness, and strictly non-blockingness.

The study of interconnection networks is relevant to theoretical computer science in areas such as parallel computations, graph pebbling, oblivious computations for many naturally occurring functions, modeling circuits with limited depth and unbounded fan-in, and implementation on parallel computers of algorithms for sorting.

In this paper, we study the size-depth complexity tradeoff for strictly non-blocking fixed ratio concentrators. More specifically, we show a size optimal bound of $\Theta(n^{1+\frac{1}{k}})$ for synchronous strictly non-blocking γn-limited $(\alpha n, \beta n)$-concentrators with non-full capacity and constant depth k, and present a size upper bound of $O(n^{1+\frac{1}{\lceil k/2 \rceil}})$ for synchronous strictly non-blocking βn-limited $(\alpha n, \beta n)$-concentrators with full capacity and constant depth k. Our motivation for studying the size-depth tradeoffs for concentrators and generalized-concentrators derives from considering the gap between the lower and upper size bounds versus depth for connectors and generalized-connectors ([Fri88], [FFP88], and [Clo53]) in the strictly non-blocking context.

1.1 Preliminaries

We shall abbreviate "directed graph" to "digraph". For a graph or digraph G, denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. The out-neighborhood of a vertex v in G is denoted by $\Gamma_{out}(v)$. For a directed path P in a digraph G, denote by $term(P)$ the terminus of P.

For positive integers n and m, an (n,m)-network $G = (V, E, I, O)$ is an acyclic digraph with vertex set V and edge set E, a set I of n distinguished vertices called inputs, and a set O of m other distinguished vertices called outputs. We shall use the same denotation G for a network and its underlying acyclic digraph when the context is clear. The size of a network is the size of its underlying digraph.

Two directed paths in a digraph are compatible if their intersection is a common initial directed subpath of them, which may possibly be empty. A route in a network $G = (V, E, I, O)$ is a directed path from an input to an output. The depth of G is the maximum number of edges in a route in G. A state S of G is a set of pairwise compatible routes, or equivalently, S is the set of all routes of a directed forest with its roots at I and leaves at O. A state S saturates a vertex v in G if v appears in a directed path of S.

A concentration request in a network is an input. A concentration request is satisfied by a route if the route is directed from the concentration request. A generalized-concentration assignment is a multi-set of concentration requests. A generalized-concentration assignment A is satisfied by a state S if each concentration request $u \in A$ is satisfied by a number of routes of S equal to the multiplicity of u in A.

Two network parameters c and r that govern respectively the network capacity (an achievable lower bound on the number of possible simultaneous concentration requests allowed in the network) and request multiplicity (an achievable lower bound on the number of possible simultaneous concentration requests with common input allowed in the network) are associated with generalized-concentration. For positive integers c and r with $c \geq r$, a generalized-concentration assignment A is (c,r)-limited if each concentration request in A has multiplicity at most r and the sum of all multiplicities in A is at most c. A state S is (c,r)-limited if the unique generalized-concentration assignment $A(S)$ satisfied by S is (c,r)-limited. A concentration request u is (c,r)-limited in a state S if the multi-set obtained by adjoining the concentration request u to $A(S)$ is a (c,r)-limited generalized-concentration assignment.

A strictly non-blocking (c,r)-limited (n,m)-generalized-concentrator is an (n,m)-network for which the set of all (c,r)-limited states is closed under the (c,r)-limited generalized-concentration extension: for every (c,r)-limited state S_1 and every (c,r)-limited concentration request u in S_1, there exists a (c,r)-limited state S_2 such that $S_1 \subseteq S_2$ and S_2 contains an additional route satisfying the concentration request u.

The notion of generalized-connection is defined analogously (see [Dai91]). For (c,r)-limited (n,m)-generalized-concentrators and (n,m)-generalized-connectors, the case when $c = m$ is referred to as full capacity. The adjective "$(c,1)$-limited (n,m)-generalized-" refers to a non-generalized context and is thus abbreviated to "c-limited (n,m)-" in this paper.

A network is synchronous provided that all the routes in the network have the same length. In a synchronous network $G = (V, E, I, O)$ with depth k, the vertex set V can be partitioned into $k+1$ disjoint ranks $(V_i)_{i=0}^{k}$ and the edge set E can be partitioned into k disjoint stages $(E_i)_{i=1}^{k}$ in an obvious manner such that $V_0 = I$, $V_k = O$, and E_i is the set of edges directed from V_{i-1} into V_i for $i = 1, 2, \ldots, k$.

A special type of concentrators with their parameters, input and output cardinalities and network capacity, in a fixed ratio are called fixed ratio concentrators, that is, they are represented as γn-limited $(\alpha n, \beta n)$-concentrators for some positive integer constants α, β, and γ such that $\alpha > \beta \geq \gamma$.

1.2 Previous Work

Pippenger [Pip74] showed that strictly non-blocking concentrators with capacity c must have size at least $3c \log_3 c - O(c)$. Dai ([Dai93] and [Dai94]; see Theorem 3 below) obtained the size optimal bounds for strictly non-blocking c-limited (n, m)-concentrators and (c, r)-limited (n, m)-generalized-concentrators with depth 1, and lower bound size-depth tradeoffs for their synchronous versions with arbitrary depth k.

For non-blocking connectors, Bassalygo and Pinsker [BP74] proved that strictly non-blocking (n, n)-connectors exist with size $O(n \log n)$; and an explicit construction was obtained through the work of Margulis [Mar75] and Gabber and Galil [GG81]. Friedman [Fri88] showed a lower bound size-depth tradeoff, n^2 and $\Omega(n^{1+\frac{1}{k-1}})$ for synchronous strictly non-blocking (n, n)-connectors with depth 2 and depth $k \geq 3$, respectively. Clos [Clo53] gave a size upper bound versus depth for these networks, which is $O(n^{1+\frac{1}{j}})$ for depth $2j - 1$. For strictly non-blocking (n, n)-generalized-connectors, Bassalygo and Pinsker [BP80] proved that such networks must have size $\Omega(n^2)$ for any depth.

2 Construction of Expanding Networks

We present an upper bound on the size of synchronous strictly non-blocking fixed ratio concentrators with non-full capacity versus constant depth, which turns out to be optimal. The size bound is achieved via an explicit construction of interconnection networks with strong expanding property and appropriate size.

Suppose that G is a synchronous network with depth k, and rank partition $(V_i)_{i=0}^{k}$ and stage partition $(E_i)_{i=1}^{k}$. Construct a synchronous network G' with depth $k+1$, and rank partition $(V_i')_{i=0}^{k+1}$ and stage partition $(E_i')_{i=1}^{k+1}$ such that:

1. $V_i' = V_i$ for $i = 0, 1, \ldots, k$, V_{k+1}' is disjoint from $V(G)$ with $|V_{k+1}'| \geq |V_k'|$, and V_{k+1}' is partitioned into $|V_k'|$ pairwise disjoint subsets of vertices with cardinalities either $\lfloor \frac{|V_{k+1}'|}{|V_k'|} \rfloor$ or $\lceil \frac{|V_{k+1}'|}{|V_k'|} \rceil$; and these $|V_k'|$ subsets are indexed by vertices in V_k' as $\{S_v \mid v \in V_k'\}$, and
2. $E_i' = E_i$ for $i = 1, 2, \ldots, k$, and $E_{k+1}' = \cup\{\{v\} \times S_v \mid v \in V_k'\}$.

That is, the network G' evolves from G with its last stage E_{k+1}' being the uniform "projection" of V_k' into V_{k+1}'. We call G' the projection of G onto V_{k+1}', denoted by $proj(G, V_{k+1}')$.

Let $\mathcal{G}$ be a set of pairwise vertex-disjoint synchronous networks with depth k. Denote by $proj(\mathcal{G}, V_{k+1})$ the synchronous network with depth $k+1$ that is the graph-theoretic union of the projections $proj(G, V_{k+1})$ for $G \in \mathcal{G}$. The projections created in this progressive manner are magnified as the depth increases, and this allows each vertex in the input rank to access a sufficiently large number of vertices in the projected ranks. The following theorem shows an evolution of a synchronous network with depth $k+1$ from a set of pairwise vertex-disjoint synchronous networks with depth k and expanding capability via projection, which results in a network with stronger expansion by properly manipulating the cardinalities of inputs and outputs.

The expansion capability of a network is measured by the quantity of accessible output vertices from an input vertex regardless of the existing interconnections. Formally, for a network $G = (V, E, I, O)$ and a positive integer Δ, G is said to satisfy the

Δ-expanding property provided that for every set S of pairwise vertex-disjoint routes of G and every input vertex v that is not saturated by S, there exist at least Δ output vertices that are not saturated by S to which v can be directed, via routes that are vertex-disjoint from each route in S.

Theorem 1. *For integer constants k, α_i and β_i, $i = 1, 2, \ldots, k$, let*

$$\Delta_i = \beta_i - \alpha_i - \Big(\sum_{\eta=1}^{i-1} \frac{\alpha_\eta}{\beta_\eta}\Big)\beta_i - (i-1)\frac{\beta_i}{\beta_1}$$

for $i = 1, 2, \ldots, k$ (note that $\sum_{\eta=1}^{i-1} \frac{\alpha_\eta}{\beta_\eta} = 0$ if $i = 1$).

If $\Delta_i > 0$ for $i = 1, 2, \ldots, k$, then, for sufficiently large n ($\geq N_k$, a constant), there exists a synchronous $(\alpha_k n, \beta_k n)$-network G_k with depth k and size $O(n^{1+\frac{1}{k}})$ that satisfies the $\Delta_k n$-expanding property (the implied constants depend on k, α_i and β_i for $i = 1, 2, \ldots, k$).

Proof. We prove the theorem by induction on the depth k. For the basis of induction that $k = 1$, suppose that α_1 and β_1 are positive integer constants such that $\Delta_1 = \beta_1 - \alpha_1 > 0$. Let G_1 be the digraph whose underlying graph is the complete bipartite graph $K_{\alpha_1 n, \beta_1 n}$ with bipartition (V_0, V_1) where $|V_0| = \alpha_1 n$ and $|V_1| = \beta_1 n$, and $V(G_1) = V(K_{\alpha_1 n, \beta_1 n})$ and $E(G_1) = V_0 \times V_1$, that is, the rank and stage partitions of G_1 are respectively (V_0, V_1) and $(V_0 \times V_1)$. For every set S of pairwise vertex-disjoint directed edges from V_0 into V_1, and for every input vertex $v \in V_0 - S(V_0)$, $|\Gamma_{out}(v) - S(V_1)| = |V_1 - S(V_1)| = |V_1| - |S(V_1)| \geq \beta_1 n - (\alpha_1 n - 1)$ since $|S(V_1)| \leq \alpha_1 n - 1$. Thus, the digraph G_1 satisfies the desired $\Delta_1 n$-expanding property.

For the induction step, assume that the statement in the theorem is true for all depths less than k where $k > 1$. Suppose that α_i and β_i, $i = 1, 2, \ldots, k$, are positive integer constants such that $\Delta_i > 0$ for $i = 1, 2, \ldots, k$. The positiveness of $\Delta_1, \Delta_2, \ldots, \Delta_{k-1}$ enables us to apply the induction hypothesis in the case of depth $k-1$. Let n be sufficiently large that $\lfloor n^{\frac{k-1}{k}} \rfloor \geq N_{k-1}$. Then there exists a synchronous $(\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor, \beta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor)$-network G_{k-1} with depth $k-1$ and size $O((\lfloor n^{\frac{k-1}{k}} \rfloor)^{1+\frac{1}{k-1}})$ that satisfies the corresponding $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor$-expanding property. For sufficiently large n, let $\tau = \lceil \frac{\alpha_k n}{\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} \rceil$ and $\mathcal{G}$ be a set of τ pairwise vertex-disjoint copies of G_{k-1}, say $\mathcal{G} = \{G_{k-1}^{(1)}, G_{k-1}^{(2)}, \ldots, G_{k-1}^{(\tau)}\}$ where $G_{k-1}^{(i)}$ has rank partition $(V_j^{(i)})_{j=0}^{k-1}$ and stage partition $(E_j^{(i)})_{j=1}^{k-1}$, and consider the synchronous network $G = proj(\mathcal{G}, V_k)$ where $|V_k| = \beta_k n$.

Clearly, G is a synchronous $(\tau\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor, \beta_k n)$-network (note that $\tau\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor = \lceil \frac{\alpha_k n}{\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} \rceil \alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor \geq \alpha_k n$) with depth k. We can obtain the size of G by noting that the projection of $G_{k-1}^{(i)}$ onto V_k, $proj(G_{k-1}^{(i)}, V_k)$, has size $|E(G_{k-1}^{(i)})| + \beta_k n = |E(G_{k-1})| + \beta_k n$, and therefore $|E(G)| = \sum_{\eta=1}^{\tau}(|E(G_{k-1}^{(\eta)})| + \beta_k n) = \tau(|E(G_{k-1})| + \beta_k n)$. Observe that $|E(G_{k-1})|$ is $O((\lfloor n^{\frac{k-1}{k}} \rfloor)^{1+\frac{1}{k-1}})$, and $(\lfloor n^{\frac{k-1}{k}} \rfloor)^{1+\frac{1}{k-1}} \leq (n^{\frac{k-1}{k}})^{\frac{k}{k-1}} = n$, therefore $|E(G)|$ is $O(\tau n)$. We note that $\tau n = \lceil \frac{\alpha_k n}{\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} \rceil n$, and thus $|E(G)|$ is $O(n^{1+\frac{1}{k}})$.

We now show that the synchronous network G satisfies the $\Delta_k n$-expanding property. Let S be a set of pairwise vertex-disjoint routes of G, and v be an input vertex of G that is not saturated by S (i.e., $v \in (\cup_{\eta=1}^{\tau} V_0^{(\eta)}) - S(\cup_{\eta=1}^{\tau} V_0^{(\eta)})$). Observe that,

$$(\cup_{\eta=1}^{\tau} V_0^{(\eta)}) - S(\cup_{\eta=1}^{\tau} V_0^{(\eta)}) = \cup_{\eta=1}^{\tau}(V_0^{(\eta)} - (\cup_{\xi=1}^{\tau} S(V_0^{(\xi)}))) = \cup_{\eta=1}^{\tau}(V_0^{(\eta)} - S(V_0^{(\eta)}))$$

by the pairwise disjointedness of $V_0^{(i)}$, $i = 1, 2, \ldots, r$. Therefore, $v \in V_0^{(i)} - S(V_0^{(i)})$ for some $i \in \{1, 2, \ldots, r\}$. Note that, by the pairwise vertex-disjointedness in $\mathcal{G} = \{G_{k-1}^{(i)} \mid i = 1, 2, \ldots, r\}$ and the definition of $proj(\mathcal{G}, V_k)$, the set $S^- = \{P - \{term(P)\} \mid P \in S\}$ can be partitioned into $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ where $S^{(i)}$ is a set of pairwise vertex-disjoint directed paths in $G_{k-1}^{(i)}$ that are directed from $V_0^{(i)}$ into $V_{k-1}^{(i)}$ for $i = 1, 2, \ldots, r$.

Since $v \in V_0^{(i)} - S(V_0^{(i)})$ $(= V_0^{(i)} - S^{(i)}(V_0^{(i)}))$, then by the $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor$-expanding property of $G_{k-1}^{(i)}$, there exist at least $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor$ vertices in $V_{k-1}^{(i)} - S^{(i)}(V_{k-1}^{(i)})$ to which v can be directed via directed paths that are vertex-disjoint from each directed path in $S^{(i)}$, and therefore from each directed path in S^-. Thus, by the definition of projection in $proj(\mathcal{G}, V_k)$, there exist at least $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor\lfloor\frac{|V_k|}{|V_{k-1}^{(i)}|}\rfloor$ vertices in V_k to which v can be directed via directed paths that are vertex-disjoint from each directed path in S^-, that is, there exist at least $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor\lfloor\frac{|V_k|}{|V_{k-1}^{(i)}|}\rfloor - |S^- - S^{(i)}|$ vertices in V_k to which v can be directed via routes of G that are vertex-disjoint from each route in S. Consider that

$$\begin{aligned}\delta_k &= \Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor\lfloor\frac{|V_k|}{|V_{k-1}^{(i)}|}\rfloor - |S^- - S^{(i)}| \geq \Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor\lfloor\frac{|V_k|}{|V_{k-1}^{(i)}|}\rfloor - |S^-| \\ &\geq \Delta_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor(\frac{|V_k|}{|V_{k-1}^{(i)}|} - 1) - (r\alpha_{k-1}\lfloor n^{\frac{k-1}{k}}\rfloor - 1) \\ &\geq (\Delta_{k-1}\frac{\beta_k}{\beta_{k-1}} - \alpha_k)n - (\Delta_{k-1} + \alpha_{k-1})\lfloor n^{\frac{k-1}{k}}\rfloor + 1.\end{aligned}$$

The coefficient of n in δ_k is

$$\begin{aligned}\Delta_{k-1}\frac{\beta_k}{\beta_{k-1}} - \alpha_k &= (\beta_{k-1} - \alpha_{k-1} - (\sum_{\eta=1}^{k-2}\frac{\alpha_\eta}{\beta_\eta})\beta_{k-1} - (k-2)\frac{\beta_{k-1}}{\beta_1})\frac{\beta_k}{\beta_{k-1}} - \alpha_k \\ &= \beta_k - \alpha_k - (\sum_{\eta=1}^{k-1}\frac{\alpha_\eta}{\beta_\eta})\beta_k - (k-1)\frac{\beta_k}{\beta_1} + \frac{\beta_k}{\beta_1} = \Delta_k + \frac{\beta_k}{\beta_1}\end{aligned}$$

(note that $\sum_{\eta=1}^{k-2}\frac{\alpha_\eta}{\beta_\eta} = 0$ if $k \leq 2$). Thus, $\delta_k \geq (\Delta_k + \frac{\beta_k}{\beta_1})n - (\Delta_{k-1} + \alpha_k)\lfloor n^{\frac{k-1}{k}}\rfloor + 1 \geq \Delta_k n$ for sufficiently large n.

We observe that the $\Delta_k n$-expanding property of G is preserved if any set of input vertices together with the edges incident with them are deleted. Therefore, the digraph G_k, obtained by deleting a set of input vertices from G such that the input rank has cardinality $\alpha_k n$, satisfies the statement in the theorem. This completes the induction step, and the proof of the theorem. □

3 Size Optimal Bound versus Constant Depth

Our scheme for constructing synchronous strictly non-blocking fixed ratio concentrators with non-full capacity and constant depth k is to employ the above-mentioned expanding networks as principal components. The construction is composed of two facets of interconnection, in which the first $k-1$ stages yield a set of pairwise vertex-disjoint synchronous expanding networks and the last stage unites the projections of moderate expansion from the components onto the output rank. Hence the concern is the actual accessibility of any input of a given component to the outputs, which is limited by the existing concentration effect created by the other components.

Theorem 2. *For positive integer constants α, β, γ, and k with $\alpha > \beta > \gamma$, and for sufficiently large n, there exists a synchronous strictly non-blocking γn-limited $(\alpha n, \beta n)$-concentrator with non-full capacity γn and depth k, and size $O(n^{1+\frac{1}{k}})$ (the implied constants depend on α, β, γ, and k).*

Proof. Since $\beta > \gamma > 0$, we can see that there exists a sufficiently large constant t such that $t \geq 2k-3$ and $1 - \frac{\gamma}{\beta} \geq \frac{2k-2}{2^t}$. Let α_i and β_i be positive integer constants such that $\beta_1 = 2^t$ and $\frac{\alpha_i}{\beta_i} = \frac{1}{2^t}$ for $i = 1, 2, \ldots, k-1$.

We show that the positive integer parameters α_i and β_i, $i = 1, 2, \ldots, k-1$ satisfy the hypothesis of Theorem 1. For $i = 1, 2, \ldots, k-1$,

$$\Delta_i = \beta_i - \alpha_i - (\sum_{\eta=1}^{i-1} \frac{\alpha_\eta}{\beta_\eta})\beta_i - (i-1)\frac{\beta_i}{\beta_1} = 2^t\alpha_i - \alpha_i - (\sum_{\eta=1}^{i-1} \frac{1}{2^t})2^t\alpha_i - (i-1)\frac{2^t\alpha_i}{2^t}$$
$$= (2^t - 2i + 1)\alpha_i \geq (2^t - 2(k-1) + 1)\alpha_i > (t - (2k-3))\alpha_i \geq 0.$$

Thus, for sufficiently large n such that $\lfloor n^{\frac{k-1}{k}} \rfloor \geq N_{k-1}$ (a constant), there exists a synchronous $(\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor, \beta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor)$-network G_{k-1} with depth $k-1$ and size $O((\lfloor n^{\frac{k-1}{k}} \rfloor)^{1+\frac{1}{k-1}})$ that satisfies the $\Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor$-expanding property stated in Theorem 1.

For sufficiently large n, let $\tau = \lceil \frac{\alpha n}{\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} \rceil$. Define $\mathcal{G}$ as in the proof of Theorem 1, and $G = proj(\mathcal{G}, V_k)$ where $|V_k| = \beta n$.

The derivation in the proof of Theorem 1 can also show that G is a synchronous $(\tau\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor, \beta n)$-network (note that $\tau\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor = \lceil \frac{\alpha n}{\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} \rceil \alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor \geq \alpha n$) with depth k and size $O(n^{1+\frac{1}{k}})$. It must be shown that the network G is a synchronous strictly non-blocking γn-limited $(\tau\alpha_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor, \beta n)$-concentrator with depth k, that is, for a state S of G and a γn-limited concentration request v in S, there exists a state S_1 of G that contains S and an additional route satisfying the concentration request v. We proceed as in showing the $\Delta_k n$-expanding property of G in Theorem 1, and we can see that the desired state S_1 exists provided that $\delta = \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor \lfloor \frac{|V_k|}{|V_{k-1}^{(i)}|} \rfloor - (|S^-| - |S^{(i)}|) > 0$ where $|S^-| = |S| \leq \gamma n - 1$. Noting that $\Delta_{k-1} = (2^t - (2k-3))\alpha_{k-1}$, we have

$$\delta \geq \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor (\frac{|V_k|}{|V_{k-1}^{(i)}|} - 1) - |S^-| \geq \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor (\frac{\beta n}{\beta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor} - 1) - (\gamma n - 1)$$
$$= (\frac{\Delta_{k-1}}{\beta_{k-1}}\beta - \gamma)n - \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor + 1 = ((2^t - (2k-3))\frac{\alpha_{k-1}}{\beta_{k-1}}\beta - \gamma)n - \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor + 1$$
$$= (1 - \frac{\gamma}{\beta} - \frac{2k-2}{2^t})\beta n + \frac{1}{2^t}\beta n - \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor + 1 \geq \frac{1}{2^t}\beta n - \Delta_{k-1}\lfloor n^{\frac{k-1}{k}} \rfloor + 1 > 0$$

for sufficiently large n.

Since the strictly non-blocking concentration property and its capacity in a network are preserved if any set of input vertices together with the edges incident with them are deleted, the digraph obtained by deleting a set of input vertices from G such that the input rank has cardinality αn is the desired synchronous strictly non-blocking γn-limited $(\alpha n, \beta n)$-concentrator with depth k and size $O(n^{1+\frac{1}{k}})$. This completes the proof of the theorem.

□

The following theorem, which gives lower bound size-depth tradeoffs for synchronous strictly non-blocking concentrators and generalized-concentrators with arbitrary depth, together with the size upper bound in Theorem 2, provide the size optimal bound for synchronous strictly non-blocking fixed ratio concentrators with non-full capacity and constant depth.

Theorem 3.

1. *[Dai93] For positive integers n, m, and c with $n > m \geq c$, the optimal size of strictly non-blocking c-limited (n,m)-concentrators with depth 1 is $(n-m+c)c+(m-c)$. For positive integers n, m, c, and k such that $n > m \geq c$, a synchronous strictly non-blocking c-limited (n,m)-concentrator with depth k has size at least $k(n-m+\frac{k}{k+1}c)c^{\frac{1}{k}}-(k-1)(n-c)$.*
2. *[Dai94] For positive integers n, m, c, and r with $n > m \geq c \geq r$, the optimal size of strictly non-blocking (c,r)-limited (n,m)-generalized-concentrators with depth 1 is $nc - \lfloor \frac{m-c}{r} \rfloor (c-r)$.*
   For positive integers n, m, c, r, and k such that $n > m \geq c \geq r$, a synchronous strictly non-blocking (c,r)-limited (n,m)-generalized-concentrator with depth $k \geq 2$ has size at least $k(n-m+\frac{k}{k+1}c)c^{\frac{1}{k}}-(k-1)(n-c)$ if $r < k-1$, and $\alpha_k(n-\frac{m}{r}+\beta_k\frac{k}{k+1}\frac{c}{r})r^{\frac{k-1}{k}}c^{\frac{1}{k}}-\frac{1}{2}(n-\frac{m}{r})r$ otherwise, where $\alpha_k = \frac{1}{2}\frac{k}{(k-1)^{\frac{k-1}{k}}}$ $(> \frac{1}{2})$ and $\beta_k = 1-\frac{1}{2^{1+\frac{1}{k}}}$ $(> \frac{1}{2})$.

Theorem 4. *For positive integer constants α, β, γ, and k with $\alpha > \beta > \gamma$, the optimal bound on the size of synchronous strictly non-blocking γn-limited $(\alpha n, \beta n)$-concentrator with non-full capacity γn and depth k is $\Theta(n^{1+\frac{1}{k}})$ (the implied constants depend on α, β, γ, and k).*

4 Size Upper Bound for Full Capacity

Dai [Dai91] showed a folklore result how the capacity parameter of networks promotes the interconnection property possessed by the networks from concentration to connection in both wide-sense and strictly non-blocking contexts, and thus increases the size complexity of the networks.

Theorem 5. *For positive integers n and m with $n > m$, if G is a strictly (wide-sense) non-blocking (n,m)-concentrator with full capacity, then G is a strictly (respectively, wide-sense) non-blocking (n,m)-connector with full capacity.*

The best known upper bound on the size of synchronous strictly non-blocking (n,n)-connectors with full capacity and depth k is $O(n^{1+\frac{1}{\lceil k/2 \rceil}})$, via an explicit recursive construction due to Clos [Clo53]. We show below the same size upper bound for these concentrators versus constant depth, but it employs the explicit construction of expanding networks detailed in Theorem 1.

Theorem 6. *For positive integer constants α, β, and k with $\alpha \geq \beta$, there exists a synchronous strictly non-blocking βn-limited $(\alpha n, \beta n)$-concentrator with full capacity and depth k (that is, connector in the same context), and size $O(n^{1+\frac{1}{\lceil k/2 \rceil}})$ (the implied constants depend on α, β, and k).*

Proof. It suffices to prove the theorem for the case of $k = 2j-1$. Let t be a sufficiently large integer constant such that $2^{t-1} > 2j-3$. Consider the sequences of positive integer constants α_i and β_i such that $\beta_1 = 2^t$ and $\frac{\alpha_i}{\beta_i} = \frac{1}{2^i}$ for $i = 1, 2, \ldots, j-1$, as in the proof of Theorem 2, which satisfy the hypothesis of Theorem 1, i.e., $\Delta_i = (2^t - 2i + 1)\alpha_i > 0$ for $i = 1, 2, \ldots, j-1$. Thus, for sufficiently large n, there exists a synchronous $(\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor, \beta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor)$-network G_0 with depth $j-1$ and size $O((\lfloor n^{\frac{j-1}{j}}\rfloor)^{1+\frac{1}{j-1}})$, which satisfies the $\Delta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$-expanding property stated in Theorem 1. We also note that in this case, $\Delta_{j-1} > \frac{1}{2}\beta_{j-1}$ since

$$\begin{aligned}\Delta_{j-1} &= (2^t - (2j-3))\alpha_{j-1} = 2^{t-1}\alpha_{j-1} + (2^{t-1} - (2j-3))\alpha_{j-1} \\ &= \frac{1}{2}\beta_{j-1} + (2^{t-1} - (2j-3))\alpha_{j-1} > \frac{1}{2}\beta_{j-1}.\end{aligned}$$

For a digraph $H = (V, E)$, its mirror image denoted by H^r is defined to be the digraph (V, E^r) where $E^r = \{(u, v) \mid (v, u) \in E\}$. For sufficiently large n, let $a = \lceil \frac{\alpha n}{\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor} \rceil$ and $b = \lceil \frac{\beta n}{\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor} \rceil$. Let G be a synchronous $(a\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor, b\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor)$-network with depth k, together with rank partition $(V_i)_{i=0}^k$ and stage partition $(E_i)_{i=1}^k$, defined as follows (note that $a\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor = \lceil \frac{\alpha n}{\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor} \rceil \alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor \geq \alpha n$, and similarly, $b\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor \geq \beta n$). The subdigraph G_{in} of G induced by its first j ranks $V_0, V_1, \ldots, V_{j-1}$ is the graph-theoretic union of $G_1, G_2, \ldots, G_a$, which is a sequence of a pairwise vertex-disjoint copies of above-mentioned synchronous network G_0, while the subdigraph G_{out} of G induced by its last j ranks $V_j, V_{j+1}, \ldots, V_k$ is the graph-theoretic union of $G_1^r, G_2^r, \ldots, G_b^r$, which is a sequence of b pairwise vertex-disjoint copies of G_0^r; and the center stage E_j is composed of all the edges directed from the output rank of G_p into the input rank of G_q^r in any one-to-one correspondence manner for $p = 1, 2, \ldots, a$, and $q = 1, 2, \ldots, b$.

Computation can show that $|E(G_{in})|$ and $|E(G_{out})|$ are $O(n^{1+\frac{1}{j}})$, and $|E_j| = ab\beta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$ that is $O(n^{1+\frac{1}{j}})$. Thus $|E(G)|$ is $O(n^{1+\frac{1}{j}})$. To show that the network G is a synchronous strictly non-blocking $(a\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor, b\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor)$-connector with depth $k = 2j-1$, let S be a state of G and (u, v) be a $b\alpha_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$-limited connection request in S. Therefore there exist $p \in \{1, 2, \ldots, a\}$ and $q \in \{1, 2, \ldots, b\}$ such that u is an input of G_p and v is an output of G_q^r. Since G_p satisfies the $\Delta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$-expanding property stated in Theorem 1, there exist at least $\Delta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$ ($> \frac{1}{2}\beta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$) vertices in the output rank of G_p to which u can be directed via directed paths that are vertex-disjoint from each route in S. By symmetry, a similar argument gives that there exist $\Delta_{j-1}\lfloor n^{\frac{j-1}{j}}\rfloor$ vertices in the input rank of G_q^r from which v can be directed via directed paths that are vertex-disjoint from each route in S. Then, since the directed edges in E_j provide an one-to-one correspondence between the outputs of G_p and the inputs of G_q^r, and $2\Delta_{j-1} > \beta_{j-1}$, there exists a route that satisfies the connection request (u, v) and is vertex-disjoint from each route in S. This shows the strictly non-blocking connection property of G. A desired synchronous strictly non-blocking fixed ratio $(\alpha n, \beta n)$-connector can be obtained by deleting all but αn inputs and all but βn outputs of G; and the theorem is proved. □

5 Concluding Remarks

The equivalences between the full-capacity generalized-concentration (concentration) and the full-capacity generalized-connection (respectively, connection) in the strictly and wide-sense non-blocking contexts allow us to apply the known results on size lower bound and upper bound versus depth ($\Omega(n^{1+\frac{1}{k-1}})$ and $O(n^{1+\frac{1}{\lceil k/2 \rceil}})$, respectively) for synchronous strictly non-blocking (n,n)-connectors with full capacity and depth $k \geq 2$ to synchronous strictly non-blocking concentrators with full capacity. These two size bounds show an optimality result for depth $k \leq 3$, but there is a considerable gap between them for depth $k \geq 4$. An improvement in narrowing this gap is desirable.

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