GEOMETRY-DRIVEN CURVE EVOLUTION

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Abstract. In this paper we show how geometry-driven diffusion can be used to develop a system of curve-evolution that is able to preserve salient features of closed curves (such as corners and straight line segments), while simultaneously suppressing noise and irrelevant details. The idea is to characterise the curve by means of its angle function (i.e. the angle between the tangent and a fixed axis) and to apply geometry-driven diffusion to this one-dimensional representation.

1 Introduction

Curve-evolutions have been widely studied in the mathematical literature [2, 3, 4, 5] and have recently been introduced in computer vision by Kimia et al. [6]. The idea is to capture the essence of a complex contour by tracking its appearance as it evolves across different scales. To be more precise: suppose a closed contour is given as a mapping $\gamma_0 : I \subset \mathbb{R} \longrightarrow \mathbb{R}^2 : u \longmapsto \gamma_0(u) = (x(u), y(u))$, where u is an arbitrary parameter that runs along the contour. To describe the evolution of this contour across different scales, one introduces a new parameter t (which one can think of as *time* or *scale*) and one studies the family of curves $\gamma(t)$ which is obtained by allowing a specified dynamical system to act on the original curve $\gamma_0 = \gamma(0)$. In most cases the dynamics is determined by specifying the time-derivative $d\gamma/dt$.

One such system that has been thoroughly studied is the so-called *Euclidean curve shortening flow*:

$$\frac{\partial \gamma}{\partial t} = -\kappa \mathbf{N} \tag{1}$$

where $\mathbf{N}$ is the outward unit normal and κ the Euclidean curvature.

Gage, Hamilton and Grayson proved that a planar embedded curve evolving according to eq.(1) converges to a circle. They also showed that no corners can be created during the evolution. When the coordinates are expressed as a function of arc length, this system is actually equivalent to a simple diffusion equation (again cfr. Kimia *et al.* [6]).

In [9] the authors generalised these results to the case of affine invariance and introduced the so-called *affine shortening flow*.

Although all these theorems are very satisfactory from a mathematical point of view, it could be argued that it is exactly the attractive simplicity of the limit curves that turns into a disadvantage when it comes to practical applications. For one thing, it seems to us that a uniform limit (be it circle or ellipse) is a

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serious drawback for recognition purposes since it implies that salient features (such as corners or line-segments) are destroyed in the course of the evolution. Consequently, in order to retrieve such features one must halt the evolution at a time when a reasonable balance between smoothness and saliency has been struck. But this requires the intervention of a supervisor and it would be more satisfactory if the process converged autonomously to a meaningful but non-trivial limit.

It is for this reason that we have turned our attention to *anisotropic* or *geometry-driven diffusion* because such processes offer the possibility to produce a limit that captures the essential geometry of the original contour. In this paper we propose a curve-evolution scheme that attempts to draw a "caricature" or sketch of the original curve. By this we mean that the curve of its own accord evolves to a less noisy limit, while at the same time keeping or even enhancing its salient features. These caricatures can then be used to yield a more efficient classification or characterisation the original curve.

2 Geometry-driven diffusion of the angle function

Let us assume that we are given a closed curve γ_0 (of length ℓ) which is parametrised by its (Euclidean) arc length s:

$$\gamma_0 : [0, \ell] \subset \mathbb{R} \longrightarrow \mathbb{R}^2 : s \longmapsto \gamma_0(s) = (x(s), y(s)), \tag{2}$$

where the fact that γ_0 is closed implies that $\gamma_0(0) = \gamma_0(\ell)$. Since we are interested in the form of the contour and not in its position, it stands to reason to use a description which is invariant under Euclidean motions. A standard result in geometry tells us that for a smooth curve such a description is provided by expressing the Gaussian curvature κ as a function of the arc length s: $\kappa = \kappa(s)$. But if we denote by θ the (oriented) angle between the velocity vector $\dot{\gamma}_0 = d\gamma_0/ds$ and the x-axis, then the curvature is equal to the derivative of θ with respect to the arc length ($\kappa = d\theta/ds$). Hence, we can just as well give a description of the curve by expressing θ as a function of s: $\theta = \theta(s)$.

The basic idea is then simple: first we obtain a characterisation of the curve in terms of its angle versus the arc length $\theta = \theta(s)$ (in what follows we will call the function $\theta(s)$ the *angle function* for the contour). This description reflects not only the salient features present in the curve, but also all the noise and minor irrelevant variations that happen to be present. To rid ourselves of these non-essential features and to restore the "basic" or "characteristic" form of the curve, we apply diffusion to the function $\theta = \theta(s)$. This means that we should interpret θ as a function of both time t and arc lenght s: $\theta = \theta(s, t)$. Of course, ordinary diffusion will smooth out all variations (irrespective of their saliency) and we will end up with a rather trivial limit. We will therefore investigate the curve-dynamics when the angle function evolves according to the Nordström equation [7]:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial s}\left(c(\theta')\frac{\partial \theta}{\partial s}\right) - \mu(\theta - \theta_0), \qquad \text{where } \theta' \equiv \frac{\partial \theta}{\partial s}. \tag{3}$$

Here $\theta_0 \equiv \theta_0(s) \equiv \theta(0, s)$ is the representation of the original curve and μ is a parameter that determines the coupling between the original and the evolving curve. This coupling parameter μ plays an important role in the convergence of the diffusion. Notice that if we put $\mu = 0$ we recover the original Perona-Malik equation [8]. Furthermore, we used the non-linear diffusion coefficient

$$c(p) = \frac{a}{1 + (p/b)^2} = \frac{ab^2}{b^2 + p^2},$$

so that a is indicative of the overall magnitude of the diffusion, whereas b determines the switching-threshold for the boosting of local gradients. Notice that we can recover the constant diffusion coefficient by letting $b \longrightarrow \infty$. If at the same time we put $\mu = 0$, then we obtain the Euclidean curve shortening flow eq.(1) (cfr. [3]).

In the case of anisotropic evolution of grey-level values, this equation suffices to determine the subsequent evolution, but for the problem at hand there is an additional constraint: we must make sure that at each time t, the angle-representation $\theta = \theta_t(s)$ actually represents a *closed* curve. So we will have to modify the process in such a way that at each iteration step closedness of the curve is guaranteed. It is well known that a planar curve γ of length l is closed if its angle function $\theta(s)$ satisfies

$$\int_0^\ell \cos\theta(s)\; ds = 0 \quad \text{and} \quad \int_0^\ell \sin\theta(s)\; ds = 0. \tag{4}$$

For the curve to be smooth we have to insist that $\theta(0) = \theta(\ell) \pmod{2\pi}$. But it is clear that we can always enforce this condition through a simple multiplication of the angle function by a global factor.

Let us assume that at time t the closed curve $\gamma(t)$ characterised by the function $\theta = \theta(s)$, and that a small time step $\triangle t$ later the evolution equation produces a "small" increment $\psi(s)$ such that $\gamma(t + \triangle t)$ is determined by the description $\theta = \theta(s) + \psi(s)$. In order for this new curve to be closed, we have to make sure that the two conditions (4) are satisfied when using $(\theta(s)+\psi(s))$ as the angle function. But since we assume that the increment ψ is small in comparison to θ, it follows that $\cos(\theta+\psi) \approx \cos\theta - \psi\sin\theta$ and similarly $\sin(\theta+\psi) \approx \sin\theta + \psi\cos\theta$ and the closedness conditions reduce to (up to a first approximation)

$$\int_0^\ell \psi(s)\cos\theta(s)\; ds = 0 \quad \text{and} \quad \int_0^\ell \psi(s)\sin\theta(s)\; ds = 0. \tag{5}$$

Let us recap for a moment: given a closed curve we compute its form at a slightly later time by computing the corresponding increment $\psi(s)$ to the angle function $\theta(s)$. However, to ensure that the resulting curve is still closed, we must modify $\psi(s)$ in such a way that it will satisfy eq.(5).

This problem is most easily tackled when expressed in abstract terms. Let us use the abbreviations $b_1(s) \equiv \cos\theta(s)$ and $b_2(s) \equiv \sin\theta(s)$; furthermore, recall that the integral along the curve

$$< f, g > \equiv \int_0^{\ell} f(s)g(s)\ ds$$

defines an inner-product (on the vectorspace of square integrable functions $f : [a,b] \longrightarrow \mathbb{R}$). We then recognize that the conditions (5) amount to an orthogonality requirement of ψ with respect to b_1 and b_2. Since $\gamma(t)$ is a simple closed curve, the function $\theta(s)$ varies over a total of 2π, from which it follows that the two vectors b_1 and b_2 are orthogonal

$$< b_1, b_2 > \equiv \int_0^{\ell} \cos\theta(s)\sin\theta(s)\ ds = 0.$$

This suggest that we use the Gram-Schmidt orthogonalization scheme: starting from an original increment ψ (supplied by the evolution equation) we compute a new increment $\tilde{\psi}$ using

$$\tilde{\psi} = \psi - \frac{< \psi, b_1 >}{< b_1, b_1 >} b_1 - \frac{< \psi, b_2 >}{< b_2, b_2 >} b_2. \tag{6}$$

Since $\tilde{\psi}$ is orthogonal to both b_1 and b_2 it follows that it satisfies condition (5) and therefore (to a good approximation) condition (4).

As a final step we will reconstruct $\gamma(t + \Delta t)$ using the angle function $\theta = \theta(s) + \tilde{\psi}(s)$ (instead of $\theta = \theta(s) + \psi(s)$), thus ensuring the closedness of the result.

3 Implementation results

As explained at the outset, the main aim of the evolution process expounded in this paper is to simplify a contour while keeping the characteristic and salient features. In this section we will illustrate the potential of this technique.

In figure 1 we consider a "daisy" with and without noise. In this case the 5 petals are the salient features. Choosing $a = 15$, $b = 0.15$ and $\mu = 1$ the contours evolve to a 5-pointed star, capturing the essential geometry. Notice that the evolution drives the contours to an almost identical limit, underlining the robustness of the process.

Figure 2 is another illustration of the robustness of the proposed systems. A diamond-shaped contour (the corners of which have been rounded) is shown with (bottom) and without (top) noise. In both cases the contour evolves to diamonds with sharp corners which are virtually indistinguishable.

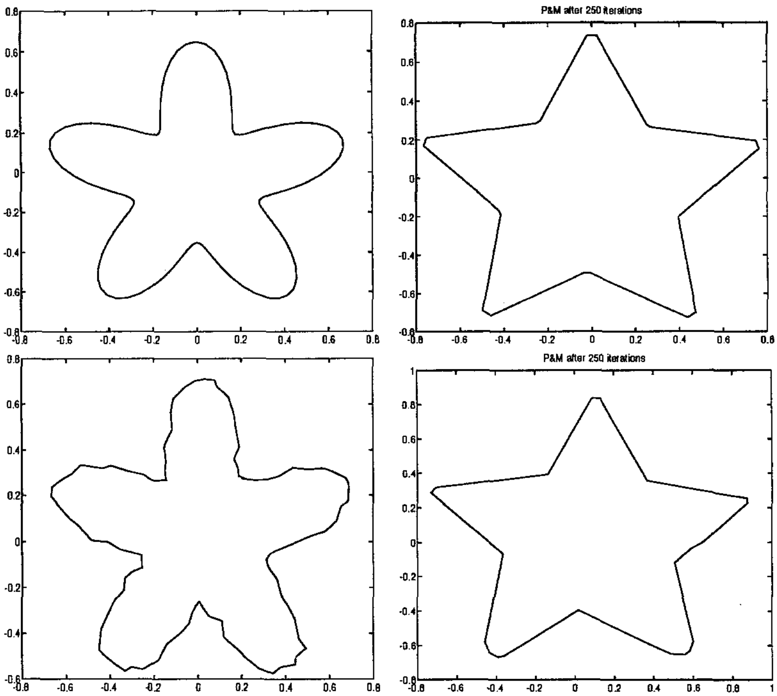

Fig. 1. Top:(left) original input curve ("daisy") and processed curve after 250 iterations (right). Bottom:(left) daisy with noise and processed curve after 250 iterations (right).

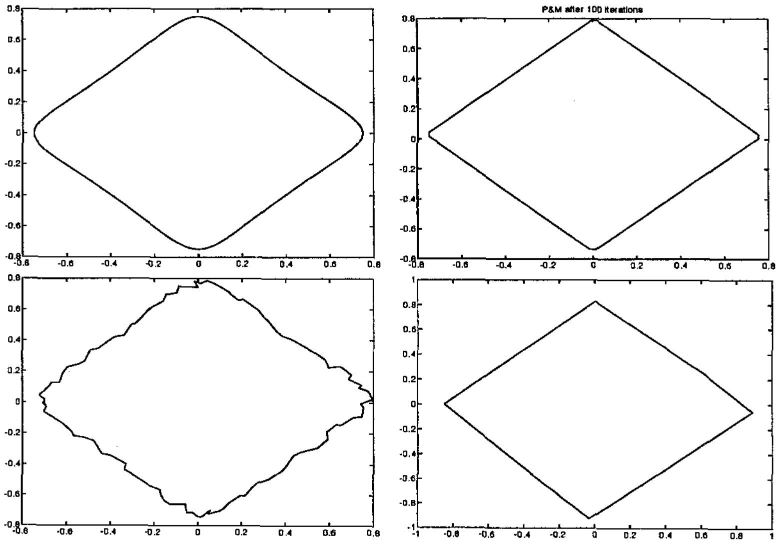

Fig. 2. Diamond-shaped contour without (top left) and with (bottom left) noise. On the right the result after 100 iterations is shown. Notice how both contours converge to the same limit, illustrating the robustness of the approach.

4 Conclusion

In this paper we have shown how geometry-driven diffusion can be used to develop a system of curve-evolution that is able to preserve salient features of closed curves (such as corners and straight line segments), while simultaneously suppressing noise and irrelevant details. The idea is to characterise the curve by means of its angle function (i.e. the angle between the tangent and a fixed axis) and to apply the dynamics of the Nordström diffusion equation to this one-dimensional representation. Elementary algebra provides a way to keep the corresponding curve closed at all times.

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