On the Ehrenfeucht-Fraïssé Game in Theoretical Computer Science *

(Extended Abstract)

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Abstract. An introduction to (first-order) Ehrenfeucht-Fraïssé games is presented, and three applications in theoretical computer science are discussed. These are concerned with the expressive power of first-order logic over graphs, formal languages definable in first-order logic, and modal logic over labelled transition systems.

1 Introduction

The Ehrenfeucht-Fraïssé game is a convenient and flexible method to determine the expressive power of logical formalisms (involving boolean connectives and quantifiers). It was first introduced for first-order logic, but exists now in a multitude of variants covering other logics occurring in computer science, such as process logics, query languages, logics that capture complexity classes, and regular-like expressions.

In this short note, we are not able to survey the recent developments in sufficient detail. Instead we give an introduction to the nonspecialist, explaining the basic case of first-order logic, and discuss three applications within first-order logic that are relevant to computer science. Some selected references concerning extensions of first-order logic are also mentioned. (The applications and the subject of extended logics are treated in more depth in the full paper.)

The expressive power of a logic is measured by its ability to distinguish between structures (of a form admitted by the respective semantics). Thus, evaluating the expressive power of a logic means to describe the equivalence between structures that holds if they are indistinguishable by formulas of this logic. Instead of keeping track of all the formulas that could play a role in this equivalence, one looks for a description of it which refers directly to the "algebraic" properties of the structures and thus is easier to handle. Since the logical systems to be considered here cannot distinguish between isomorphic structures, an algebraic formulation of logical indistinguishability will lead to a weakening of isomorphism (or to isomorphism itself).

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It was R. Fraïssé [Fr54] who introduced an algebraic notion (weakening isomorphism) which captures indistinguishability by first-order formulas of a relational signature. A. Ehrenfeucht [Ehr61] reformulated Fraïssé's algebraic treatment in game theoretic terminology and extended the method to weak monadic second-order logic (analyzing its power to distinguish between countable ordinals). Ehrenfeucht's paper helped much to spread the idea. Today we speak of "Ehrenfeucht-Fraïssé games". The game theoretic formulation is more intuitive, but in many concrete applications it is useful to work in an algebraic framework as Fraïssé originally developed.

The Ehrenfeucht-Fraïssé technique is one of the few methods from model theory which is applicable to finite structures, hence to many definability questions in computer science. Presently, in the rapidly developing area of *finite model theory*, Ehrenfeucht-Fraïssé games play a central role. Perhaps the method is used so frequently in theoretical computer science because it is applicable in a very transparent way over relational structures with relations of arity 1 and 2 only, like graphs, linear orderings, and partially ordered structures. Structures of this type are predominant in many fields of computer science (e.g., formal language theory, data base theory, semantics of concurrency).

In classical model theory, the emphasis is different: Its cornerstones are the the Löwenheim-Skolem Theorem and the Compactness Theorem, both meaningful only when infinite structures are admitted, and the algebraic applications (to groups, fields, etc.) require relational signatures involving higher arities. This may be a reason why there are relatively few textbooks where Ehrenfeucht-Fraïssé games are treated. We mention [EFT84, Chapter 11], [Ro82, Chapter 13], [Mo76, Chapter 26]; for (model theoretic) extensions of first-order logic see the survey volume [BF85].

In Section 2 we summarize basic facts on first-order Ehrenfeucht-Fraïssé games, guided by the exposition in [EFT84]. In Section 3, applications are outlined on the expressive power of first-order logic over graphs, on logical definability of formal languages, and on a system of modal logic over labelled transition systems.

2 Basics

2.1 m-Equivalence

In the sequel we consider a first-order language with equality and a simple signature S, consisting of unary relation symbols $P_1, \ldots, P_k$ and binary relation symbols $R_1, \ldots, R_l$ only. The restriction to unary and binary relations is inessential for the results but saves notation and covers all applications to be discussed below. Letters P and R will indicate unary, resp. binary relation symbols from S. Relational structures for this signature (*S-structures*) are of the form $\mathcal{A} = (A, P_1^A, \ldots, P_k^A, R_1^A, \ldots, R_l^A)$ where A is the structure's universe, $P_i^A \subseteq A$ for $1 \leq i \leq k$ and $R_j^A \subseteq A \times A$ for $1 \leq j \leq l$. Sometimes we expand such a structure by designated elements from its universe.

First-order formulas for the signature S (*S-formulas*) involve variables x_1, $x_2, \ldots$, and are built up from atomic formulas of the form $x_i = x_j$, Px_i, and Rx_ix_j by applying the boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ and the quantifiers $\exists$, $\forall$. For a tuple $\overline{x} = (x_1, \ldots, x_n)$ of variables, the notation $\varphi(\overline{x})$ indicates that φ is a formula in which at most $x_1, \ldots, x_n$ occur free. As a measure of complexity for formulas we use *quantifier-depth*: Define $qd(\varphi)$ inductively by setting

- $qd(\varphi) = 0$ for atomic φ, $qd(\neg\varphi) = qd(\varphi)$,
- $qd(\varphi \vee \psi) = qd(\varphi \wedge \psi) = qd(\varphi \rightarrow \psi) = qd(\varphi \leftrightarrow \psi) = max(qd(\varphi), qd(\psi))$
- $qd(\exists x\varphi) = qd(\forall x\varphi) = qd(\varphi) + 1$

Given an n-tuple $\overline{a} = (a_1, \ldots, a_n)$ of elements from the universe A of the S-structure $\mathcal{A}$ and a formula $\varphi(\overline{x})$, one writes $(\mathcal{A}, \overline{a}) \models \varphi(\overline{x})$ if φ holds in $\mathcal{A}$ when interpreting x_i by a_i for $1 \leq i \leq n$ (as well as symbols P and R by P^A and R^A, respectively).

Let $\mathcal{A}$, $\mathcal{B}$ be S-structures with universes A, B, and let $\overline{a}$, $\overline{b}$ be n-tuples of elements from A, B, respectively. Given $m \geq 0$ we say that $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ are *m-equivalent* (short: $(\mathcal{A}, \overline{a}) \equiv_m (\mathcal{B}, \overline{b})$) if

$$(\mathcal{A}, \overline{a}) \models \varphi(\overline{x}) \iff (\mathcal{B}, \overline{b}) \models \varphi(\overline{x})$$

for all S-formulas $\varphi(\overline{x})$ of quantifier-depth $\leq m$. For the case of empty sequences $\overline{a}$ and $\overline{b}$ this means that the two structures satisfy the same sentences (formulas without free variables) of quantifier-depth m.

2.2 m-Isomorphism

Our aim is to describe $\equiv_m$ as a weakening of isomorphism. First we do this for $m = 0$, where the notion of "partial isomorphism" turns out appropriate. Given S-structures $\mathcal{A}$ and $\mathcal{B}$ with universes A and B, we indicate a finite relation $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq A \times B$ by $\overline{a} \mapsto \overline{b}$. Such a relation is called a *partial isomorphism* if the assignment $a_i \mapsto b_i$ determines an injective (partial) function p from A to B (whose domain consists of the elements in $\overline{a}$), which moreover preserves all relations P^A, R^A under consideration, in the sense that

$$P^A a \iff P^B p(a) \quad \text{and} \quad R^A aa' \iff R^B p(a)p(a')$$

for all symbols P, R from S and all a, a' in the domain of p.

Let us verify that partial isomorphisms characterize $\equiv_0$-equivalence: We have

$(\mathcal{A}, \overline{a}) \equiv_0 (\mathcal{B}, \overline{b})$

iff any boolean combination of formulas $x_i = x_j$, Px_i, and Rx_ix_j is satisfied in $(\mathcal{A}, \overline{a})$ iff it is satisfied in $(\mathcal{B}, \overline{b})$

iff (by boolean logic) any of the atomic formulas $x_i = x_j$, Px_i, and Rx_ix_j is satisfied in $(\mathcal{A}, \overline{a})$ iff it is satisfied in $(\mathcal{B}, \overline{b})$

iff $a_i = a_j \Leftrightarrow b_i = b_j$ and $P^A a_i \Leftrightarrow P^B b_i$ and $R^A a_i a_j \Leftrightarrow R^B b_i b_j$ $(1 \leq i, j \leq n)$.

Hence $(\mathcal{A}, \overline{a}) \equiv_0 (\mathcal{B}, \overline{b})$ iff $\overline{a} \mapsto \overline{b}$ is a partial isomorphism. As we may expect, this characterization does not extend to $\equiv_m$ for $m > 0$. Consider, for example, the linear orderings $(\mathbb{R}, <^{\mathbb{R}})$ and $(\mathbb{Z}, <^{\mathbb{Z}})$ of the integers, resp. the real numbers. Then $p_0 : (2,3) \mapsto (3,4)$ is a partial isomorphism (i.e., order preserving) but does not preserve truth of the formula $\exists x_3(x_1 < x_3 \wedge x_3 < x_2)$, which states that between the two considered elements there is a third one. In the terminology of partial isomorphisms, this means that $p_0 : (2,3) \mapsto (3,4)$ cannot be extended to a new partial isomorphism having for example $2\frac{1}{2}$ in its domain. The idea in Fraïssé's Theorem is that the possibility of extending partial isomorphisms m times (in both directions) characterizes the m-equivalence $\equiv_m$.

To describe this extension property, we introduce sets $I_1, I_2, \ldots, I_m$ of partial isomorphisms such that I_k contains partial isomorphisms which allow k-fold extension. Call $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ *m-isomorphic* (short: $(\mathcal{A}, \overline{a}) \cong_m (\mathcal{B}, \overline{b})$) if there are nonempty sets $I_0, \ldots, I_m$ of partial isomorphisms, each of them extending $\overline{a} \mapsto \overline{b}$, such that for all $k = 1, \ldots, m$

- *(back property)* $\forall p \in I_k\ \forall b \in B\ \exists a \in A$ such that $p \cup \{(a,b)\} \in I_{k-1}$
- *(forth property)* $\forall p \in I_k\ \forall a \in A\ \exists b \in B$ such that $p \cup \{(a,b)\} \in I_{k-1}$.

Fraïssé's Theorem. *For $m \geq 0$: $(\mathcal{A}, \overline{a}) \equiv_m (\mathcal{B}, \overline{b})$ iff $(\mathcal{A}, \overline{a}) \cong_m (\mathcal{B}, \overline{b})$.*

2.3 The Game Theoretic View

In the game theoretic view due to Ehrenfeucht, relations (such as partial isomorphisms) are configurations in a two-person game, and moves in this game perform extensions of relations. Consider two structures $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$. A play of the associated *Ehrenfeucht-Fraïssé Game* $G_m((\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b}))$ consists of m rounds and is carried out as follows: The initial configuration is $\overline{a} \mapsto \overline{b}$. Given a configuration r, a round is composed of two moves: first player I picks an element a from A or b from B, and then player II reacts by choosing an element in the other structure, i.e. some b from B, resp. some a from A. The new configuration is $r \cup \{(a,b)\}$. After m rounds, player II has won if the final configuration is a partial isomorphism (otherwise player I has won). Instead of asking for a partial isomorphism at the end of the play, one may as well require partial isomorphisms during the whole play (because the final configuration is a partial isomorphism iff all configurations during the play are). While player II aims at a partial isomorphism at the end, player I tries to avoid this. (To emphasize this, [FSV92] introduce the suggestive names "spoiler" and "duplicator" instead of Ehrenfeucht's names "I" and "II".) We say that II *wins* the game $G_m((\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b}))$ if II has a strategy to win each play (we skip a formal definition of "strategy").

Ehrenfeucht's Theorem. *For $m \geq 0$:*

$$(\mathcal{A}, \overline{a}) \cong_m (\mathcal{B}, \overline{b}) \quad \textit{iff} \quad \textit{II wins } G_m((\mathcal{A}, \overline{a}), (\mathcal{B}, \overline{b})).$$

The proof is straightforward, because sets of configurations which allow player II to win with k rounds ahead correspond to sets I_k in the definition of m-isomorphism, and the possibility for player II to stay within "winning" configurations corresponds to the assumption that the two extension properties (back and forth) hold.

2.4 Distributive Normal Form

The proof of Fraïssé's theorem proceeds by induction on m in both directions. The proof of m-equivalence given m-isomorphism is not difficult. For the converse direction

$$(\mathcal{A}, \overline{a}) \equiv_m (\mathcal{B}, \overline{b}) \ \Rightarrow \ (\mathcal{A}, \overline{a}) \cong_m (\mathcal{B}, \overline{b})$$

it is sufficient to describe the $\cong_m$-classes by formulas of quantifier-depth m. So we have to present for any structure $(\mathcal{C}, \overline{c})$ a formula $\varphi^m_{(\mathcal{C},\overline{c})}(\overline{x})$ of quantifier-depth m which is satisfied exactly by the structures which are m-isomorphic to $(\mathcal{C}, \overline{c})$. The suitable inductive definition is built on a formalization of $\equiv_0$-equivalence and of the two extension properties. Let, for a structure $(\mathcal{C}, \overline{c})$ with $\overline{c} = (c_1, \ldots, c_n)$

$$\varphi^0_{(\mathcal{C},\overline{c})}(\overline{x}) := \bigwedge_{\varphi(\overline{x}) \text{ atomic, } (\mathcal{C},\overline{c}) \models \varphi(\overline{x})} \varphi(\overline{x}) \wedge \bigwedge_{\varphi(\overline{x}) \text{ atomic, } (\mathcal{C},\overline{c}) \models \neg\varphi(\overline{x})} \neg\varphi(\overline{x})$$

$$\varphi^{m+1}_{(\mathcal{C},\overline{c})}(\overline{x}) := \bigwedge_{c \in C} \exists x_{n+1} \varphi^m_{(\mathcal{C},\overline{c},c)}(\overline{x}, x_{n+1}) \wedge \forall x_{n+1} \bigvee_{c \in C} \varphi^m_{(\mathcal{C},\overline{c},c)}(\overline{x}, x_{n+1})$$

To justify this definition in case the structure $\mathcal{C}$ is infinite, one has to observe that (due to the finite signature) there are only finitely many atomic formulas involving variables from $x_1, \ldots, x_n$, and that (as verified by induction on m) the number of logically nonequivalent formulas $\varphi^m_{(\mathcal{C},\overline{c})}(\overline{x})$ is finite (for any given length of tuples $\overline{c}$). Thus the disjunction and the conjunction (over $c \in C$) in the definition of $\varphi^{m+1}_{(\mathcal{C},\overline{c})}(\overline{x})$ both range only over finitely many formulas $\varphi^m_{(\mathcal{C},\overline{c},c)}(\overline{x}, x_{n+1})$ and thus specify first-order formulas.

The formulas $\varphi^m_{(\mathcal{C},\overline{c})}(\overline{x})$ go back to Hintikka [Hi53] and are sometimes called "Hintikka formulas". They are the basis of [illegible] l form for first-order formulas. Obviously, the class of structures which sati [illegible] en formula $\varphi(\overline{x})$ of quantifier-depth m must be a union of $\equiv_m$-classes, i [illegible] raïssé's Theorem, a union of $\cong_m$-classes. Each of these is defined by a Hi [illegible] rmula. Thus $\varphi(\overline{x})$ is logically equivalent to the (finite!) disjunction of tl [illegible] tikka formulas which define these $\cong_m$-classes. This representation is cal [illegible] *distributive normal form* for first-order logic. For more details, variants ol [illegible] a formulas, and applications see e.g. [Fl74], [Sc79].

3 Three Applications

3.1 Directed Graphs of Bounded Degr[illegible]

An S-structure $\mathcal{A} = (A, P_1^A, \ldots, P_k^A, R_1^A, \ldots, R_l^A)$, where the P_i^A form a partition of A and the R_j^A are disjoint relations, may be considered as a graph with labelled vertices and edges. The edge relation is $E = \bigcup_j R_j^{\mathcal{A}}$, and the indices i, j represent the labels for the vertices, resp. edges. In the sequel, "graphs" are meant to be S-structures of this kind. A graph is of degree $\leq d$ if for any vertex a there are at most d vertices b with Eab or Eba. We want to determine the expressive power of first-order logic over graphs of bounded degree.

For $\mathcal{A}$ as above, $a \in A$, and $r \in \mathbb{N}$, the "sphere with radius r around a in A" is the induced subgraph of $\mathcal{A}$ with vertices of distance $\leq r$ from a. (Here we assume that edges may be traversed in both directions.) This subgraph with designated center a is denoted r–sph$(\mathcal{A}, a)$. Since we consider graphs of degree $\leq d$, there are, for any $r > 0$, only finitely many possible isomorphism types of r-spheres. For an isomorphism type σ of r-spheres, let $occ(\sigma, \mathcal{A})$ be the number of occurrences of spheres of type σ in $\mathcal{A}$. We shall see that any first-order formula is equivalent (over graphs of degree $\leq d$) to a statement on these occurrence numbers for finitely many types σ. Moreover, for any given formula the values $occ(\sigma, \mathcal{A})$ are relevant only up to a certain threshold $t \in \mathbb{N}$.

Formally, define $\mathcal{A} \sim_{r,t} \mathcal{B}$ if for any isomorphism type σ of spheres of radius r the numbers $occ(\sigma, \mathcal{A})$ and $occ(\sigma, \mathcal{B})$ are either both $> t$ or else coincide. The following "sphere lemma" states that $\sim_{r,t}$-equivalence (for suitable r, t) is fine enough to capture m-equivalence over graphs of degree $\leq d$.

Sphere Lemma. *For any $m \geq 0$ there are $r, t \geq 0$ such that for any two graphs $\mathcal{A}, \mathcal{B}$ (of degree $\leq d$) we have: If $\mathcal{A} \sim_{r,t} \mathcal{B}$ then $\mathcal{A} \equiv_m \mathcal{B}$.*

The proof, due to Hanf [Hf65], uses Fraïssé's Theorem: It suffices to ensure $\mathcal{A} \cong_m \mathcal{B}$ for suitable r, t. Set $r = 3^{m+1}$ and $t = m \cdot d^{3^{m+1}}$. The required sequence of sets $I_0, \ldots, I_m$ of partial isomorphisms is defined as follows: Let $p : (a_1, \ldots, a_{m-k}) \mapsto (b_1, \ldots, b_{m-k})$ belong to I_k iff

$$\bigcup_{i=1}^{m-k} 3^k - \mathrm{sph}(\mathcal{A}, a_i) \quad \cong \quad \bigcup_{i=1}^{m-k} 3^k - \mathrm{sph}(\mathcal{B}, b_i)$$

i.e., the two induced subgraphs formed from the 3^k-spheres around the a_i, resp. the b_i, are isomorphic. To verify e.g. the forth property, assume this condition holds for p and let $a (= a_{m-(k-1)}) \in A$. We have to find $b (= b_{m-(k-1)}) \in B$ such that

$$\bigcup_{i=1}^{m-(k-1)} 3^{k-1} - \mathrm{sph}(\mathcal{A}, a_i) \quad \cong \quad \bigcup_{i=1}^{m-(k-1)} 3^{k-1} - \mathrm{sph}(\mathcal{B}, b_i).$$

If $a \in \frac{2}{3} \cdot 3^k$–sph$(\mathcal{A}, a_i)$ for some a_i, we may choose b from $\frac{2}{3} \cdot 3^k$–sph$(\mathcal{A}, b_i)$ correspondingly; note that 3^{k-1}–sph$(\mathcal{A}, a)$ is contained in 3^k–sph$(\mathcal{A}, a_i)$ and thus 3^{k-1}–sph$(\mathcal{A}, b)$ in 3^k–sph$(\mathcal{A}, b_i)$. So 3^{k-1}–sph$(\mathcal{A}, a) \cong 3^{k-1}$–sph$(\mathcal{B}, b)$ holds. Otherwise, 3^{k-1}–sph$(\mathcal{A}, a)$, say of type σ, is disjoint from all 3^{k-1}–sph$(\mathcal{A}, a_i)$, and it suffices to find a sphere of type σ in $\mathcal{B}$ which is disjoint from all spheres 3^{k-1}–sph$(\mathcal{B}, b_i)$. This will be possible if the number of occurrences of spheres of type σ in $\mathcal{B}$ is large enough. But this is guaranteed by the assumption $\mathcal{A} \sim_{r,t} \mathcal{B}$.

By the Sphere Lemma and the Distributive Normal Form, any first-order formula is equivalent (over graphs of degree $\leq d$) to a boolean combination of statements "there are $\geq k$ occurrences of spheres of type σ". So first-order logic can express only "local" graph properties and hence is too weak for many applications. (In the terminology of formal language theory, a first-order definable set of graphs of bounded degree is "locally threshold testable"; see [Th91].) This fact, which has also been shown by a different method (quantifier elimination) in [Ga82], has motivated the consideration of several extensions of first-order logic over graphs. Suitable extensions of the Ehrenfeucht-Fraïssé game serve to analyze their expressive power. Such an analysis has been carried out, for example, for existential monadic second-order logic ([Fag75], [FSV92]), for transitive closure logic (e.g. [Gr92]), and for different kinds of fixed point logic ([Bo92]).

3.2 Labelled Linear Orders and Congruence Lemmas

A word w over an alphabet $\Sigma = \{s_1, \ldots, s_k\}$ can be represented by the structure $\underline{w} = (\{1, \ldots, |w|\}, <, P_1^w, \ldots, P_k^w)$ with unary relations P_i^w, where $j \in P_i^w$ iff the j-th letter of w is s_i. A formal language $L \subseteq \Sigma^+$ is called first-order definable if there is a first-order sentence φ (in the signature $\{<, P_1, \ldots, P_k\}$) such that $L = \{w \in \Sigma^+ \mid \underline{w} \models \varphi\}$. Ehrenfeucht-Fraïssé games have been useful in clarifying the relation between first-order logic and definability notions from formal language theory, in particular concerning *star-free regular languages*. A language $L \subseteq \Sigma^+$ is called *star-free* if it can be constructed from finite languages by applications of boolean operations and concatenation. By a well-known result of McNaughton, a language is first-order definable iff it is star-free. The difficult direction is from left to right, and usually proved by induction on quantifier-depth (e.g. in [Lad77]). It turns out that the essential point of the induction step (concerning the existential quantifier) is the following claim:

Congruence Lemma. If $\underline{u} \equiv_m \underline{u'}$ and $\underline{v} \equiv_m \underline{v'}$, then $\underline{u \cdot v} \equiv_m \underline{u' \cdot v'}$.

The proof is straightforward if we refer to $\cong_m$ instead of $\equiv_m$ and think in terms of the Ehrenfeucht-Fraïssé game: The assumption tells us that player II has winning strategies for the games $G_m(\underline{u}, \underline{u'})$ and $G_m(\underline{v}, \underline{v'})$. An obvious composition of these two strategies ("on the segments u and u' use the first strategy, on the segments v and v' use the second strategy") guarantees her or him to win also the game $G_m(\underline{u \cdot v}, \underline{u' \cdot v'})$.

Congruence lemmas are a typical application of Ehrenfeucht-Fraïssé games; they have been proved also for other logics and over more complex structures than words. A congruence lemma states that properties of a structure as a whole are determined by (and hence can be composed from) properties of its parts. In [Th84], [Th87a], Ehrenfeucht-Fraïssé games are applied to obtain congruence lemmas for two modified versions of first-order logic: first-order formulas in prenex normal form with a fixed quantifier prefix type, and star-free regular expressions. A small but interesting difference between first-order logic and star-free expressions is worked out in [LT88], also via the game method.

Highly intricate examples of congruence lemmas were given by Shelah [Sh75], concerning the the monadic second-order theory of (arbitrary) linear orderings, and obtained by an extension of the Ehrenfeucht-Fraïssé technique. Later, congruence lemmas were proved also for tree structures, e.g. in [GS83] for monadic second-order logic or in [Th87b] for path-oriented logics. The extension of first-order logic over trees by "modulo counting quantifiers" was analyzed in [Pot92], again using appropriate Ehrenfeucht-Fraïssé games.

3.3 Labelled Transition Systems and Modal Logic

The notion of bisimulation was introduced by Park [Pa81] and related to modal logic by Hennessy and Milner ([HM85], [Mil90]). There is a close connection between bisimulations and Ehrenfeucht-Fraïssé games, although they were developed quite independently. Here we describe some aspects of this connection, but to avoid technical overhead we consider only a very restricted form of bisimulation and observational equivalence, in which the special role of the so-called "silent transition" is suppressed. (For a different approach to treat behavioral equivalences in first-order logic see [Og92].)

We refer to structures $\mathcal{A} = (A, R_1^A, \ldots, R_l^A)$, which serve as the model theoretic representation of "labelled transition systems" ([Mil90]): The elements of A are "states", and $R_1^A, \ldots, R_l^A$ are "transition relations". Hennessy-Milner Logic is a system of modal logic to be interpreted over labelled transition systems. We introduce this logic here directly as a fragment of first-order logic, given by special "admissible" formulas. An admissible formula has exactly one free variable. As basic atomic formulas we allow only $tt(x_i)$ (always true). Binary boolean connectives are applicable only to formulas with the same free variable, negation is always applicable, and quantifiers are allowed only in "R_j-relativized form", observing certain conditions on the indexing of variables: Given $\varphi(x_{i+1})$ one may proceed to $\psi(x_i)$ of the form $\exists x_{i+1}(R_j x_i x_{i+1} \wedge \varphi(x_{i+1}))$ or $\forall x_{i+1}(R_j x_i x_{i+1} \rightarrow \varphi(x_{i+1}))$. To normalize the indexing of variables, we finish the construction of a formula always with x_0 as free variable (to be interpreted by a designated element of a labelled transition system). In this framework, a formula such as

$$\Diamond_i \Box_i (\Diamond_j tt \wedge \Diamond_k tt)$$

of Hennessy-Milner Logic is written as the following admissible formula $\varphi(x_0)$:

$$\exists x_1(R_i x_0 x_1 \wedge \forall x_2(R_i x_1 x_2 \rightarrow (\exists x_3(R_j x_2 x_3 \wedge tt(x_3)) \wedge \exists x_3(R_k x_2 x_3 \wedge tt(x_3)))))$$

Define $(\mathcal{A}, a_0) =_m (\mathcal{B}, b_0)$ in the same way as $(\mathcal{A}, a_0) \equiv_m (\mathcal{B}, b_0)$, however referring to admissible formulas of quantifier rank $\leq m$. By the lack of equality and because of the restricted use of the symbols R_j, the appropriate notion of "partial isomorphism" is weaker than before; in particular, we do no more require that it represents an injective function: Let us call a relation $r : (a_0, \ldots, a_n) \mapsto (b_0, \ldots, b_n) \subseteq A \times B$ a *correspondence* if $R_j^A a_i a_{i+1} \Leftrightarrow R_j^B b_i b_{i+1}$ for $i = 0, \ldots, n-1$ and $j = 1, \ldots, l$. Finally, define $(\mathcal{A}, a_0) \simeq_m (\mathcal{B}, b_0)$ in the same way as $(\mathcal{A}, a_0) \cong_m (\mathcal{B}, b_0)$, with correspondences replacing partial isomorphisms. Now the proof of Fraïssé's Theorem, adjusted to this context, shows

$$(\mathcal{A}, a_0) =_m (\mathcal{B}, b_0) \quad \text{iff} \quad (\mathcal{A}, a_0) \simeq_m (\mathcal{B}, b_0).$$

This equivalence may be regarded as a restricted form of the Hennessy-Milner characterization of bisimilarity (as formulated, for example, in [Mil90, Theorem 5.2.5 (1)]). Two points should be mentioned: Usually, in semantics of concurrency one deals with one transition system only, i.e. one considers equivalences between structures $(\mathcal{A}, a_0)$ and $(\mathcal{A}, b_0)$. More important, in the theory of bisimulations one does not refer to the existence of some sequence $(I_0, \ldots, I_m)$ of sets of correspondences (as one does in the definition of "m-isomorphism"), but works with a fixed canonical sequence $(J_0, \ldots, J_m)$ of correspondence sets: J_0 contains just the universal relation, and J_{k+1} contains *all* relations which allow back and forth extensions in J_k. This does not change the equivalence result above, but it prevents the definition of specific winning strategies by the correspondence sets.

In the general framework of bisimulations and observational equivalences (see [HM85], [Mil90]), the considered "actions" in transition systems are more complex, consisting of sequences of R_j-transitions and depending in different ways on occurrences of a designated "silent" transition. This general situation suggests to include infinite signatures. A corresponding extension of first-order logic is the system $L^{\omega}_{\infty\omega}$, allowing infinite disjunctions and conjunctions (however only finitely many variables in each formula, which are reusable within a formula). The appropriate extension of the Ehrenfeucht-Fraïssé technique has been developed and applied in other fields of computer science, e.g. in complexity theory (Immerman [Im82]) and data base theory (Kolaitis and Vardi [KV90]). The topic of reusable variables is treated, using special Ehrenfeucht-Fraïssé games, in [IK89] and [Fl92].

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