Term Rewriting in CT_Σ

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Abstract. We extend the classical theory of term rewriting systems to infinite and partial terms (i.e., to the elements of algebra CT_Σ), fully exploiting the complete partially ordered structure of CT_Σ. We show that redexes and rules, as well as other operations on terms, can be regarded as total functions on CT_Σ. As a consequence, we can study their properties of monotonicity and continuity. For rules, we show that non-left-linear rules are in general not monotonic, and that left-infinite rules are in general not continuous. Moreover, we show that the well-known Church-Rosser property of non-overlapping redexes holds for monotonic redexes. This property allows us to define a notion of parallel application of a finite set of monotonic redexes, and, using standard algebraic techniques, we extend the definition to the infinite case. We also suggest that infinite parallel term rewriting has interesting potential applications in the semantics of *cyclic term graph rewriting.*

1 Introduction

Term Rewriting is a model of computation that is employed in various areas of computer science, including symbolic algebraic computation, functional and logic programming, automated theorem proving, and execution of algebraic specifications. The 'computations' of a term rewriting system consist of repeatedly replacing subterms of a given expression with equal terms, until the simplest form possible is obtained. The theory of rewriting systems is nowadays well established within Theoretical Computer Science, at least for what concerns the rewriting of *finite* terms (see [DJ90, Kl91] for two recent surveys).

For a long time (at least to our knowledge) the extension of term rewriting to infinite terms has not been considered in the literature, probably because of the lack of motivations and/or interesting applications. Only recently this topics has become the target of an intense research activity, carried on by many research groups ([FRW88, FW89, DKP89, DK89, KKSV90]). The main motivation for the recent interest in infinite term rewriting is undoubtedly the need of extending the theory of term rewriting in order to provide a satisfactory interpretation for *cyclic term graph rewriting.*

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Acyclic term graph rewriting (i.e., the issue of representing finite terms with directed, acyclic graphs, and of modelling term rewriting via graph rewriting) has been addressed in a number of places [Ra84, BvEGKPS87, Ke87, HP91, CR93], and is now well understood. The main advantage of this approach (with respect to classical term rewriting) is that the sharing of common subterms can be represented explicitly in the graph. Therefore the rewriting process is speeded up, because the rewriting steps do not have to be repeated for each copy of an identical subterm. For example, the rewrite rule $R : f(x) \to g(x)$ can be applied twice to term $t \equiv k(f(a), r(f(a)))$, yielding in two steps term $t' \equiv k(g(a), r(g(a)))$. If instead t is represented as a graph, and the two identical subterms are shared (as in graph G of Fig. 1), then a single application of the rule is sufficient to reduce it to graph G' of Fig. 1, which clearly represents term t'. Thus a single graph rewriting step may correspond to n term rewriting steps, where n is the 'degree of sharing' of the rewritten subterm.

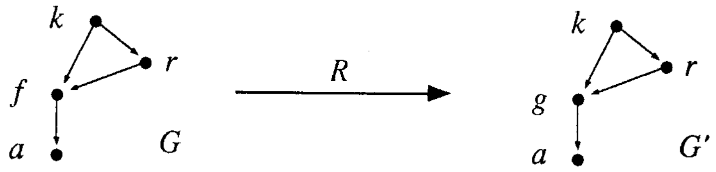

Fig. 1. An example of term graph rewriting

During the last years, many authors considered the extension of term graph rewriting to the cyclic case, allowing (finite, directed) cyclic graphs as well. The first consequence of this extension is that infinite terms (or, more precisely, *rational* terms, i.e., infinite terms with a finite number of distinct subterms) can be represented as well, exploiting cycles. The second effect is that a single graph rewriting step may now correspond to some infinite term rewriting. Consider for example rule R above: by applying it to graph H of Fig. 2 one obtain graph H' (in any reasonable definition of graph rewriting). Clearly, H represents the infinite term $f^\omega \equiv f(f(f(\ldots)))$, while H' represents term g^ω. There are (at least) two possible ways of interpreting the rewriting of term f^ω to term g^ω via some number of applications of rule R:

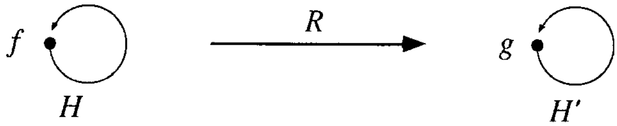

Fig. 2. An example of cyclic term graph rewriting

1. g^ω is the limit of an infinite sequence of applications of R, i.e., $f^\omega \to_R g(f^\omega) \to_R g(g(f^\omega)) \to_R \cdots \rightsquigarrow_\omega g^\omega$.

2. g^ω is the result of the simultaneous application of R to an infinite number of redexes in f^ω: in a single step all the occurrences of f in f^ω are replaced by g.

The relevant fact is that, unlike the example of Fig. 2, there are cases where these two interpretations lead to different results. This happens, for example, when *collapsing* rules are considered, i.e., rules having a variable as right-hand side. The most famous collapsing rule is the rule for identity, $R_I : I(x) \to x$, and the pathological case (considered already by many authors) is the application of R_I to I^ω. Using the first interpretation above, we have that $I^\omega \to_{R_I} I^\omega \to_{R_I} \ldots$, and clearly the limit of this sequence is I^ω itself. On the other hand, if we follow the second interpretation, all the occurrences of I in I^ω are deleted in a single step, and thus we should obtain as result a term not containing function symbols, i.e., some sort of 'undefined' term. It is worth stressing that *both* interpretations are meaningful from the point of view of cyclic term graph rewriting, and they correspond to two different choices of the graph rewriting algorithm.

In fact, if one uses the term graph rewriting model defined in [BvEGKPS87], the 'circular-I' (i.e., graph G_I in Fig. 3, representing I^ω) rewrites via R_I to itself: therefore the first interpretation must be used. This is the approach followed in [FRW88, FW89, DKP89, DK89, KKSV90], where they elaborated a theory of *transfinite term rewriting*, showing its adequacy for modelling finite, cyclic graph rewriting. In essence, a finite graph derivation has the 'same effect' of a converging transfinite term rewriting sequence: for the notion of convergence they used the well-known topological structure of (possibly infinite) terms, which, equipped with a suitable notion of distance, form a complete ultra-metric space [AN80].

If instead one uses as term graph rewriting model the so-called 'algebraic' or 'double-pushout' approach [EPS73], as done for the acyclic case in [HP91, CR93], the circular-I rewrites via R_I to a graph consisting of a single node. The situation is summarized in the lower part of Fig. 3. This is the approach taken by the author in a forthcoming paper with Frank Drewes. In order to explain this result from the perspective of term rewriting, the second of the above interpretations must be used. In this case the theory of transfinite term rewriting is no more helpful, because it cannot justify this result. We need instead some notion of 'infinite parallel rewriting', that could explain the fact that all the occurrences of the operator I in I^ω are deleted in a single step.

The notion of infinite parallel rewriting can be defined in a satisfactory way (as shown in this paper) by exploiting the well-known algebraic structure of (possibly infinite, possibly partial) terms over a signature Σ, which form a complete partial ordering (CPO) denoted CT_Σ. CT_Σ has a least element denoted $\perp$ (the undefined term), and the order relation is defined as $t < t'$ if t is a partial term that is 'less defined' than t'. CT_Σ enjoys several nice algebraic properties, which are studied in depth in the seminal work by the ADJ group, [ADJ77].

We show informally how the algebraic structure of CT_Σ can be exploited to show that I^ω rewrites via R_I to $\perp$ (by the way, this also gives a precise interpretation of the unlabelled node of Fig. 3: it denotes the undefined term

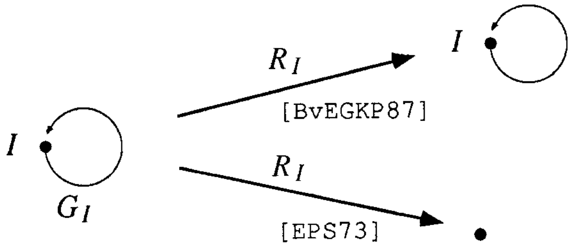

Fig. 3. The two possible results of applying R_I to the 'circular-I'

$\perp$). The infinite term I^ω is the least upper bound of an infinite chain of finite, partial terms of CT_Σ: in fact, $I(\perp) \leq I(I(\perp)) \leq \ldots \leq I^n(\perp) \leq \ldots \quad \rightsquigarrow_\omega I^\omega$. Now, if we apply rule R_I to each term of the chain as many times as possible, all those terms reduce to $\perp$ (in particular, $I^n(\perp)$ reduces to $\perp$ in n steps, or, and we prefer this interpretation, in a single finite parallel step, where the rule is applied simultaneously to the n occurrences of I). Therefore, after applying rule R_I as many times as possible, the above chain is reduced to $\perp \leq \perp \leq \ldots$: we define the least upper bound of this chain (i.e., $\perp$ itself) as the result of the infinite parallel rewriting of I^ω via R_I.

The last example shows that the CPO structure of (infinite) terms can be exploited fruitfully in the framework of term rewriting systems. But, as far as we know, the algebraic structure of CT_Σ has never been taken into account in the term rewriting literature. Therefore one of the goals of this paper is to revisit some definitions and some results of the classical theory of term rewriting, extending them to the case of infinite or partial terms, and taking care of the CPO structure of CT_Σ.

As expected, except for the original definition of infinite parallel rewriting in Section 5, no really new results come out from this reworking of well-known notions. Nevertheless, we think that our presentation sheds new light on some concepts which are a bit obscure in the term rewriting literature, like, for example, the real nature of non-left-linear rules, and why they behave so badly with respect to Church-Rosser properties. Moreover, the presence of partial terms and the CPO structure of CT_Σ allow us to define rewrite rules, redexes, and also basic operations like subterm selection and subterm replacement in an original way as *total* functions operating on terms. Like for every function on a CPO, we can ask ourselves if those functions are monotonic or continuous: in this way we can classify the rules of a term rewriting system w.r.t. to their algebraic properties. An interesting result shows that the classical Church-Rosser property of independent redexes always holds for *monotonic* rewrite rules; *continuous* rules are instead required for a correct definition of infinite parallel rewriting. These results are reconciled with the traditional Church-Rosser properties of orthogonal (i.e., left-linear, left-finite, non-overlapping) term rewriting systems by a proposition that shows that all left-linear rules are monotonic, while all left-linear and left-finite rules are continuous.

The paper is organized as follows. In Section 2 we introduce the CPO structure of terms and some basic operations on terms, regarding them as (continuous) total functions. Next in Section 3 we introduce term rewriting of (possibly partial) terms: the main contribution here is an extension of the definition of redex and of redex application which takes into account the possibility that the left-hand side of a rule matches just partially the term to be rewritten. In Section 4 we study the properties of monotonicity and continuity of rewrite rules, showing, among other things, that the customary discrimination against not left-linear rules is fully justified in this clean algebraic setting, because they are even not monotonic (at least in general). Furthermore, we present the natural extension of the Church-Rosser theorem for orthogonal TRS's to the case of partial, infinite terms, showing that it holds for all monotonic rules. The Church-Rosser theorem is then exploited in Section 5 in order to define the notion of *finite* parallel rewriting via the application of two monotonic rules to two independent redexes in a term. In the same section we introduce our original definition of *infinite* parallel rewriting for continuous rules, which is defined via a suitable limit construction: the well-definedness of the definition and its consistency with the finite case are the main results of the section. Finally, Section 6 summarizes the main results of the paper and suggests some topics for future research. Because of space limitations, most of the proofs are not included in the paper.

2 The complete partial ordering of partial, infinite terms

We introduce here the notion of possibly partial, possibly infinite terms, borrowing their definition from [ADJ77], where they are called Σ-trees.

Definition 1 (occurrences). Let ω^* be the set of all finite strings of natural numbers. Elements of ω^* are called **occurrences**. The empty string is denoted by λ. The set ω^* is equipped with a binary relation (which is obviously a partial ordering), defined as $u \leq w$ iff u is a prefix of w. Two occurrences u, w are called *disjoint* (written $u|w$) if they are incomparable w.r.t. $\leq$. The *length* of an occurrence w, denoted $|w|$, is defined as $|\lambda| = 0$ and $|wi| = |w| + 1$ for $w \in \omega^*$ and $i \in \omega$.

Definition 2 (terms). Let Σ be a (one-sorted) signature, i.e., a ranked alphabet of operator symbols $\Sigma = \bigcup_n \Sigma_n$, and let X be a set of variables. A **term over** (Σ, X) is a partial function $t : \omega^* \to \Sigma \cup X$, such that for all $w \in \omega^*$ and all $i \in \omega$, the domain of definition of t, $\mathcal{O}(t)$, satisfies the following:

- $wi \in \mathcal{O}(t) \Rightarrow w \in \mathcal{O}(t)$
- $wi \in \mathcal{O}(t) \Rightarrow t(w) \in \Sigma_n$ and $i \leq n$ for some $n > 0$.

Set $\mathcal{O}(t)$ is also called the **set of occurrences** of t. We will denote by $\mathcal{O}_X(t)$ the *set of occurrences of variables* of t, i.e., $\mathcal{O}_X(t) = \{v \in \mathcal{O}(t) \mid t(v) \in X\}$, and by $\mathcal{O}_\Sigma(t)$ the *set of occurrences of operators* of t, i.e., $\mathcal{O}_\Sigma(t) = \{v \in \mathcal{O}(t) \mid t(v) \in \Sigma\}$. The set of variables of a term t, $var(t)$, is defined as $var(t) = \{x \in X \mid \exists v \in \mathcal{O}(t) \,.\, t(v) = x\}$.

A term t is **finite** if $\mathcal{O}(t)$ is finite. The **depth** of a term t is defined only if t is finite; in this case, $depth(t) \equiv max\{|w| \mid w \in \mathcal{O}(t)\}$. A term t is **total** if $t(w) \in \Sigma_n \Rightarrow wi \in \mathcal{O}(t)$ for all $0 < i \leq n$. The set of terms over (Σ, X) is denoted by $CT_\Sigma(X)$ (with the convention that CT_Σ stays for $CT_\Sigma(\emptyset)$).

Throughout the paper we will often use (for finite terms) the equivalent and more usual representation of terms as operators applied to other terms. Partial terms are made total in this representation by introducing the undefined term $\perp$ (called *bottom*), which represents the empty function $\perp : \emptyset \to \Sigma \cup X$. Thus, for example, if $x \in X$, $t = f(\perp, g(x))$ is the term such that $\mathcal{O}(t) = \{\lambda, 2, 21\}$, $t(\lambda) = f \in \Sigma_2$, $t(2) = g \in \Sigma_1$, and $t(21) = x \in X$.

We introduce now the relevant algebraic structure of terms ([ADJ77]).

Definition 3 ($CT_\Sigma(X)$ as ω-complete lower semi-lattice). Given two terms $t, t' \in CT_\Sigma(X)$, t **approximates** t' (written $t \leq t'$) iff t is less defined than t' as partial function. In the proofs throughout the paper, we will use the following characterization of term approximation:

$$t \leq t' \quad \Leftrightarrow \quad \forall w \in \mathcal{O}(t)\,.\, t(w) = t'(w)$$

Equivalently, relation '$\leq$' can be defined as the minimal relation such that $\perp \leq t$ for all t; $x \leq x$ for all $x \in X$; and $f(t_1, ..., t_n) \leq f(t'_1, ..., t'_n)$ if $t_1 \leq t'_1, \ldots, t_n \leq t'_n$, for all $f \in \Sigma_n$.

An ω-**chain** $\{t_i\}_{i<\omega}$ is an infinite sequence of terms $t_0 \leq \ldots \leq t_n \leq \ldots$. Every ω-chain $\{t_i\}_{i<\omega}$ in $CT_\Sigma(X)$ has a **least upper bound** (*lub*) $\sqcup_{i<\omega}\{t_i\}$ characterized as follows:

$$t = \sqcup_{i<\omega}\{t_i\} \quad \Leftrightarrow \quad \forall w \in \omega^*\,.\,\exists i < \omega\,.\,\forall j \geq i\,.\,t_j(w) = t(w)$$

Formally this means that $CT_\Sigma(X)$ is ω-**complete**.

Given two terms t and t', their **greatest lower bound** $t \sqcap t'$ is uniquely characterized by the property $(t \sqcap t' \leq t) \wedge (t \sqcap t' \leq t) \wedge (\forall t''\,.\,(t'' \leq t) \wedge (t'' \leq t') \Rightarrow t'' \leq t \sqcap t')$. It can be proved that $t \sqcap t'$ exists for all $t, t' \in CT_\Sigma(X)$, and that it is defined as follows. Let $D = \{w \mid w \in \mathcal{O}(t) \cap \mathcal{O}(t') \wedge t(w) = t'(w)\}$ be the subset of the intersection of the domains of t and t' where their values agree, and let $D' \subseteq D$ be the largest prefix-closed subset of D, i.e., such that $wi \in D' \Rightarrow w \in D'$. Then $t \sqcap t'$ is defined as

$$t \sqcap t'(u) = \begin{cases} t(u) \text{ if } u \in D' \\ \perp \quad \text{otherwise.} \end{cases}$$

Finally, $CT_\Sigma(X)$ has a least element w.r.t. $\leq$, which is $\perp$ (bottom). All this amounts to say that $CT_\Sigma(X)$ is an ω-**complete lower semilattice**.

Definition 4 (monotonic and continuous functions). A function $f : D \to D'$ between ω-complete partial orderings D and D' is said **monotonic** if $d \leq d' \Rightarrow f(d) \leq f(d')$. It is ω-**continuous** if for all ω-chain $\{d_i\}_{i<\omega} \subseteq D$, $\sqcup_{i<\omega}\{f(d_i)\} = f(\sqcup_{i<\omega}\{d_i\})$, i.e., the lub of ω-chains are preserved.

In the rest of the paper we will omit the 'ω-' qualification of chains, completeness, and continuity. We introduce now two well-known operations on terms, namely *subterm selection* and *subterm replacement*. We define them in an original way: by exploiting the existence of partial terms, we can turn them into *total functions* on terms. We also state that, as functions on CT_Σ, both operations are continuous.

Definition 5 (subterm selection). Given an occurrence $w \in \omega^*$ and a term $t \in CT_\Sigma(X)$, the **subterm of** t **at** *(occurrence)* w is the term t/w defined as $t/w(u) = t(wu)$ for all $u \in \omega^*$. Using the alternative representation of terms, t/w is equivalently defined by the following clauses:

- $\perp/w = \perp$
- $t/\lambda = t$
- $x/iw = \perp$ if $x \in X$
- $f(t_1, ..., t_n)/iw = t_i/w$ if $f \in \Sigma_n$ and $i \leq n$
- $f(t_1, ..., t_n)/iw = \perp$ if $f \in \Sigma_n$ and $i > n$.

It is easy to check that $t/w = \perp$ iff $w \notin \mathcal{O}(t)$.

Proposition 6 (subterm selection is continuous). *For all $w \in \omega^*$, the function $\_/w : CT_\Sigma(X) \to CT_\Sigma(X)$ mapping t to t/w is continuous.*

Definition 7 (subterm replacement). Given terms $t, s \in CT_\Sigma(X)$ and an occurrence $w \in \omega^*$, the **replacement of** s **in** t **at** *(occurrence)* w, denoted $t[w \leftarrow s]$, is the term defined as $t[w \leftarrow s](u) = t(u)$ if $w \not\leq u$ or $t/w = \perp$, and $t[w \leftarrow s](wu) = s(u)$ otherwise. Equivalently, it can be defined as follows:

- $t[w \leftarrow s] = t$ if $t/w = \perp$ (i.e., if $w \notin \mathcal{O}(t)$)
- $t[\lambda \leftarrow s] = s$ if $t \neq \perp$
- $f(t_1, ..., t_n)[iw \leftarrow s] = f(t_1, \ldots, t_i[w \leftarrow s], \ldots, t_n)$ if $i \leq n$.

The first clause also implies that $\perp[w \leftarrow s] = \perp$ for all w, s (even if $w = \lambda$).

Proposition 8 (subterm replacement is continuous). *For all $w \in \omega^*$, the function $\_[w \leftarrow \_] : CT_\Sigma(X) \times CT_\Sigma(X) \to CT_\Sigma(X)$ mapping (t, s) to $t[w \leftarrow s]$ is continuous, that is, it is continuous in the two arguments separately.*

The next statement collects some equalities relating subterm selection and subterm replacement. They follow directly from the definitions. The names are taken from [Hu80].

Proposition 9 (properties of subterm replacement and selection). *The following equalities hold for all occurrences w, v, and for all terms t, s, and s':*

[commutativity] $t[w \leftarrow s][v \leftarrow s'] = t[v \leftarrow s'][w \leftarrow s]$ *if* $w|v$;
[dominance] $t[wv \leftarrow s][w \leftarrow s'] = t[w \leftarrow s']$;
[distributivity] $t[wv \leftarrow s]/w = t/w[v \leftarrow s]$.

Definition 10 (substitutions). Let X and Y be two sets of variables. A **substitution** *(from X to Y)* is a function $\sigma : X \to CT_\Sigma(Y)$ (used in postfix notation). The collection of all substitutions from X to Y is denoted by $Subs_\Sigma(X,Y)$. A substitution $\sigma \in Subs_\Sigma(X,Y)$ uniquely determines a function (also denoted by σ) from $CT_\Sigma(X)$ to $CT_\Sigma(Y)$, which extends σ as follows

- $\perp\sigma = \perp$,
- $f(t_1,...,t_n)\sigma = f(t_1\sigma,...,t_n\sigma)$.

The partial ordering on terms can be extended to elements of $Subs_\Sigma(X,Y)$ as follows: $\sigma \leq \sigma'$ iff for all $x \in X$, $x\sigma \leq x\sigma'$. $Subs_\Sigma(X,Y)$ is an ω-complete lower semilattice under this ordering. If $\sigma \in Subs_\Sigma(X,Y)$ and $\sigma' \in Subs_\Sigma(Y,Z)$, then their composition $\sigma' \circ \sigma$ is a substitution from X to Z defined as $x(\sigma' \circ \sigma) = (x\sigma)\sigma'$ for all $x \in X$.

If X is finite, a substitution $\sigma \in Subs_\Sigma(X,Y)$ will be represented sometimes as a finite set of the form $\{x_1/t_1,...,x_n/t_n\}$ with $t_i = x_i\sigma$ for all $1 \leq i \leq n$.

Proposition 11 (substitution is continuous). *Every substitution σ from X to Y, regarded as a function from $CT_\Sigma(X)$ to $CT_\Sigma(Y)$, is continuous. Moreover, for all sets of variables X, Y, and Z, the composition of substitutions $\_ \circ \_$: $Subs_\Sigma(X,Y) \times Subs_\Sigma(Y,Z) \to Subs_\Sigma(X,Z)$ is continuous, i.e., it is continuous separately on both arguments.*

Proof. See [ADJ77].

3 Term rewriting systems: basic definitions

In this section we introduce some basic definitions about term rewriting systems, like rules, redexes, and the application of a redex to a term, taking into account the rich algebraic structure of CT_Σ.

Definition 12 (rewrite rule, term rewriting system). A **rewrite rule** $R = (l,r)$ is a pair of terms of $CT_\Sigma(X)$, where $var(r) \subseteq var(l)$, and l is not a variable. Terms l and r are called the left- and the right-hand side of R, respectively. A rule is called *left-linear* if no variable occurs more than once in l. A rule is *left-finite* if l is finite, and it is *total* if l and r are total terms (see Definition 2). In the paper we will consider total rules only. A **term rewriting system** $\mathbf{R}$ is a finite set of rewrite rules, $\mathbf{R} = \{R_i\}_{i<n}$.

A *redex* (for *REDucible EXpression*) is usually defined in the literature as an occurrence of the left-hand side of a rule in a given term at a certain occurrence. We will use a slightly different definition, which does not involve any term: a redex is just a pair $\Delta = (w,R)$ where w is an occurrence and R is a rewrite rule. Then given any term t, a redex $\Delta = (w,R)$ can be *total* in t (if the subterm of t at w matches the left-hand side of R), or *partial* in t (if the matching is only partial), or *null* in t (if there is no matching at all). This definition allows us to regard a redex as a total function from terms to terms, whose behaviour on a term is determined by its type w.r.t. that term. More precisely, when a redex $\Delta = (w, R)$ is applied to a term t, there are three possible effects: if Δ is total in t, then the result is the usual one, i.e., the application of rule R to t at occurrence w; if Δ is null in t, then the result is t itself; and finally if Δ is partial in t, then the result of the application of Δ to t is the best approximation we can determine of the actual result. For the sake of simplicity (and without loss of generality) we consider just the rewriting of ground terms (i.e., elements of CT_Σ): the same definitions can be applied to the rewriting of terms of $CT_\Sigma(X)$ as well, because the variables in the term to be rewritten play no role during rewriting.

Definition 13 (redex). Given a term rewriting system $\mathbf{R}$, a **redex** Δ (w.r.t. $\mathbf{R}$) is a pair $\Delta = (w, R)$ where $w \in \omega^*$ is an occurrence, and $R : l \to r \in \mathbf{R}$ is a rule. Given a term $t \in CT_\Sigma$, a redex $\Delta = (w, R)$ can be of three kind w.r.t. t:

- *Δ is* **total** *in t* if there exists a substitution $\sigma : var(l) \to CT_\Sigma$ such that $t/w = l\sigma$. In this case we say that *substitution σ makes Δ total in t.*
- *Δ is* **partial** *in t* if there is no σ such that $t/w = l\sigma$, but there exists a term $l' < l$ and a substitution $\sigma' : var(l') \to CT_\Sigma$ such that $t/w = l'\sigma'$. In this case we say that *the pair (l', σ') makes Δ partial in t.*
- *Δ is* **null** *in t* if it is neither total nor partial in t.

For a given rule R, we will often denote the redex (λ, R) improperly by R itself.

Example 1 (redexes). Given the rule $R : f(g(x), y) \to h(x)$, the redex $\Delta = (1, R)$ is made total in $k(f(g(\bot), b))$ by substitution $\{x/\bot, y/b\}$ and in $f(g(\bot), \bot)$ by substitution $\{x/\bot, y/\bot\}$; it is made partial in $k(f(\bot, \bot))$ by pairs $(f(\bot, \bot), \{\})$ and $(f(\bot, y), \{y/\bot\})$; and it is null in $k(f(k(a), b))$. Notice that, by the definition, Δ is also made partial in term c by the pair $(\bot, \{\})$.

Consider now the redex $\Delta' = (\lambda, R')$, where $R' : f(g(x), x) \to h(x)$ is a non-left-linear rule. Then Δ' is made total in $f(g(k(\bot)), k(\bot))$ by substitution $\{x/k(\bot)\}$; it is made partial in $f(\bot, k(\bot))$ by pair $(f(\bot, x), \{x/k(\bot)\})$; and it is null in $f(g(k(\bot)), k(a))$.

Proposition 14 (characterization of the kind of rexedes). *Let $\Delta = (w, R : l \to r)$ be a redex and $t \in CT_\Sigma$ be a term.*

1. *The kind of Δ in t (i.e., total, partial, or null) is uniquely determined.*
2. *Δ is total in t iff the following two conditions are satisfied:*
   (a) for each occurrence $v \in \mathcal{O}_\Sigma(l)$, $t(wv) = l(v)$.
   (b) for each pair of distinct occurrences of variables $v, v' \in \mathcal{O}_X(l)$ such that $l(v) = l(v')$, $t/wv = t/wv'$.
   Moreover, the unique substitution $\sigma : var(l) \to CT_\Sigma$ making Δ total in t (i.e., such that $l\sigma = t/w$) is determined as $x\sigma = t/wv$, if $x \in var(l)$ and v is any occurrence in $\mathcal{O}_X(l)$ such that $l(v) = x$.

3. *Δ is partial in t iff the following three conditions are satisfied:*
   (a) *for each occurrence $v \in \mathcal{O}_\Sigma(l)$, either $t(wv) = l(v)$ or $t(wv) = \bot$;*
   (b) *for each pair of distinct occurrences of variables $v, v' \in \mathcal{O}_X(l)$ such that $l(v) = l(v')$, either $t/wv = t/wv'$, or $t/wv = \bot$, or $t/wv' = \bot$;*
   (c) *there exists an occurrence $v \in \mathcal{O}_\Sigma(l)$ such that $t(wv) = \bot$, or there exist two occurrences $v, v' \in \mathcal{O}_X(l)$ such that $l(v) = l(v')$, and $t/wv \neq t/wv'$.*

   As a consequence, for every occurrence w and rule R, if $w \notin \mathcal{O}(t)$ then the redex (w, R) is partial in t.
4. *Δ is null in t iff one of the following conditions hold:*
   (a) *there exists an occurrence $v \in \mathcal{O}_\Sigma(l)$ such that $t(wv) \neq \bot$ and $t(wv) \neq l(v)$,*
   (b) *there exist two distinct occurrences $v, v' \in \mathcal{O}_X(t)$, with $l(v) = l(v')$, $t/wv \neq \bot$, $t/wv' \neq \bot$, and $t/wv \neq t/wv'$.*

It is worth stressing that the conditions characterizing total, partial, and null redexes can be simplified in the case of left-linear rules. In fact, if R is left-linear then the second condition of the characterization of total and partial redexes always holds (it is vacuous), while condition 4.b and the second part of condition 3.c cannot be satisfied. Exploiting this fact it is possible to prove the following lemma.

Lemma 15 (properties of left-linear rules). *Let R be a left-linear rule. If $t \leq t'$, then*

1. *if (w, R) is total in t, then it is total in t';*
2. *if (w, R) is null in t, then it is null in t';*
3. *if (w, R) is partial in t', then it is partial in t.*

We define now what does it mean to apply a redex to a term t. As expected, the result of this operation depends on the kind of the redex in t.

Definition 16 (redex application). Given a redex $\Delta = (w, R : l \to r)$, the result of its **application** to a term t, denoted $\Delta(t)$, is defined by the following clauses:

- $\Delta(t) = t[w \leftarrow r\sigma]$ if σ makes Δ total in t (i.e., $l\sigma = t/w$);
- $\Delta(t) = t[w \leftarrow (l' \sqcap r)\sigma]$ if (l', σ) makes Δ partial in t (i.e., $l' < l$ and $l'\sigma = t/w$);
- $\Delta(t) = t$ if Δ is null in t.

We also write $t \to_\Delta s$ to mean $\Delta(t) = s$, and we say that t **rewrites to s via** Δ. Recalling that R also denotes the redex (λ, R) (see Definition 13), it follows that $R(t)$ denotes the result of the application of R to the topmost operator of t.

The second clause of the last definition may be not obvious. The idea is that as far as we have just a partial matching of the lhs of a rule with a term, we cannot specify completely the term resulting from the application of the redex: we should wait for additional information about the term, which can either complete the matching successfully (and then the first clause is applied) or can cause a clash (in this case the third clause is applied). Thus if the matching is just partial, we can specify the result just as far as the first and the third clauses agree: this is expressed by the term $(l' \sqcap r)\sigma$ obtained by applying the matching substitution to the largest term included both in the approximation l' of the left-hand side, and in its right-hand side r (see Definition 3 for the definition of $t \sqcap t'$).

In order to prove the well-definedness of the last definition, we need the following technical lemma.

Lemma 17. *Let l', l'' be two terms and let $\sigma' : var(l') \to CT_\Sigma$, $\sigma'' : var(l'') \to CT_\Sigma$ be two substitutions. If $l'\sigma' \leq l''\sigma''$ and l' and l'' have a common upper bound (i.e., $l' \leq l$ and $l'' \leq l$ for some l), then for each term r it holds $(l' \sqcap r)\sigma' \leq (l'' \sqcap r)\sigma''$.*

Proposition 18 (redex application is well-defined). *The application of a redex Δ to a term t is well-defined, that is, $\Delta(t)$ is uniquely determined. Thus Δ is a total function $\Delta : CT_\Sigma \to CT_\Sigma$.*

Proof. By Proposition 14.1 the three clauses of Definition 16 are applicable in mutually disjoint cases. The fact that $\Delta(t)$ is uniquely determined is obvious if Δ is total or null in t (if it is total, the substitution σ making it total is unique, as stressed in Proposition 14.2)

If the second clause is applied, we have to show that if (l', σ') and (l'', σ'') make Δ partial in t, then $(l' \sqcap r)\sigma' = (l'' \sqcap r)\sigma''$. But the fact that (l', σ') and (l'', σ'') make Δ partial in t means that $l', l'' < l$ and that the two substitutions $\sigma' : var(l') \to CT_\Sigma$ and $\sigma'' : var(l'') \to CT_\Sigma$ are such that $l'\sigma' = t/w = l''\sigma''$. Then $(l' \sqcap r)\sigma' = (l'' \sqcap r)\sigma''$ follows by two applications of Lemma 17. Therefore $\Delta(t)$ is well-defined also when Δ is partial in t.

The following statement stresses some properties of rule application which will be helpful later on. The listed properties can easily be checked by a careful inspection of the corresponding definitions.

Proposition 19 (properties of rule application). *The following properties hold for every rule R, for all t, $s \in CT_\Sigma$, and v, $w \in \omega^*$:*

1. $(vw, R)(t) = t[v \leftarrow (w, R)(t/v)]$
   If $w = \lambda$, this simplifies to $(v, R)(t) = t[v \leftarrow R(t/v)]$.
2. $(wv, R)(t[w \leftarrow s]) = t[w \leftarrow (v, R)(s)]$.
3. *If $\Delta = (w, R)$ and $v|w$ then $\Delta(t)/v = t/v$.*

4 On monotonicity and continuity of rules

Since redexes are just functions from CT_Σ to itself (as shown in Proposition 18), it makes sense to talk about *monotonic* and *continuous* redexes. These notions may be extended to rules in an obvious way.

Definition 20 (monotonic and continuous rules). A redex Δ is *monotonic* (*continuous*) if so is the corresponding function $\Delta : CT_\Sigma \to CT_\Sigma$. A rewrite rule R is **monotonic** (**continuous**) if for every occurrence $w \in \omega^*$, (w, R) is monotonic (continuous).

The following proposition shows that in order to check the continuity (or monotonicity) of a rewrite rule, it is sufficient to examine its effect when applied at the topmost occurrence of a term.

Proposition 21 (rules and topmost redexes). *A rewrite rule R is monotonic (continuous) iff redex (λ, R) is monotonic (continuous).*

Proof. The *only if* part is obvious. For the *if* part, by Proposition 19.1 we have that for every occurrence $w \in \omega^*$, $(w, R)(t) = t[w \leftarrow (\lambda, R)(t/w)]$. Therefore (w, R) is a suitable composition of subterm selection $(\_/w)$, of subterm replacement $(\_[w \leftarrow \_])$, and of the redex (λ, R). Then the thesis follows by observing that for all $w \in \omega^*$, functions $\_[w \leftarrow \_]$ and $\_/w$ are continuous (see Propositions 6 and 8).

The next interesting result shows that the classification of rewrite rules (regarded as functions) with respect to their algebraic properties is consistent with the usual classification based on the properties of the left-hand side. In fact, left-linearity and left-finiteness of rules are proved to be strictly related to monotonicity and continuity.

Theorem 22 (characterization of monotonic and continuous rules). *Let $R : l \to r$ be a rewrite rule. Then*

1. *If R is left-linear then it is monotonic.*
2. *If R is left-linear and left-finite then it is continuous.*
3. *If R is not left-linear, then it is monotonic iff $l = r$.*
4. *If R is not left-finite, then it is continuous iff $l = r$.*

Example 2 (non-monotonic and non-continuous rules). Let $R = f(x, x) \to k$ be a non-left-linear rule. Consider the terms $t_1 = f(g(\bot), g(\bot))$ and $t_2 = f(g(b), g(c))$. Clearly, $t_1 < t_2$, $t_1 \to_R k$ and $t_2 \to_R f(g(b), g(c))$, but $k \not\leq f(g(b), g(c))$. Thus R is not monotonic.

Let $R = f(f(f(\ldots))) \to k$ be a left-infinite rule, let $f^n(\bot) = f(f^{n-1}(\bot))$ for each $1 < n < \omega$, and let $f^1(\bot) = f(\bot)$. Then $f^n(\bot) \to_R \bot$ for all n, but $f^\omega \to_R k$. Thus R is not continuous.

We restate now the well-known Church-Rosser theorem for independent, finite set of redexes (see for example [Ros73, Kl91]). We show that the Church-Rosser property holds for *monotonic* rules. It is worth stressing that, as for redexes, also the definition of *independence of redexes* presented below is given in a way which is independent of the actual term to be rewritten, unlike the related literature. The Church-Rosser theorem will allow us to define in the next section the parallel rewriting of a term via a finite set of independent redexes.

Definition 23 (independent redexes). Two redexes $(w, R : l \rightarrow r)$ and $(w', R' : l' \rightarrow r')$ are **independent** if their left hand sides do not overlap on occurrences of operators, that is, if $w \cdot \mathcal{O}_\Sigma(l) \cap w' \cdot \mathcal{O}_\Sigma(l') = \emptyset$ (if V is a set of occurrences and w is an occurrence, by $w \cdot V$ we denote the set $\{wv \mid v \in V\}$).

Theorem 24 (Church-Rosser property for monotonic redexes). *Let $R : l \rightarrow r$ and $R' : l' \rightarrow r'$ be two* monotonic *rules, and let $\Delta = (w, R)$ and $\Delta' = (w', R')$ be two independent redexes. For every term $t \in CT_\Sigma$ there exist two natural numbers $1 \leq n, n' < \omega$ and occurrences $v_1, ..., v_n, v'_1, ..., v'_{n'}$ such that $(v_1, R) \circ \ldots \circ (v_n, R) \circ \Delta'(t) = (v'_1, R') \circ \ldots \circ (v'_{n'}, R') \circ \Delta(t)$. Moreover, if $w \not> w'$ (i.e., $w|w'$ or $w < w'$, because $w \neq w'$ by independence of Δ and Δ'), then the last statement holds for $n = 1$ and $v_1 = w$ (that is, there exists a number $1 \leq n' < \omega$ and occurrences $v'_1, ..., v'_{n'}$ such that $\Delta \circ \Delta'(t) = (v'_1, R') \circ \ldots \circ (v'_{n'}, R') \circ \Delta(t)$).*

5 Infinite Parallel Rewriting

Exploiting the Church-Rosser theorem presented in the last section, it is easy to define a notion of parallel term rewriting. To this aim, we need to stress that the theorem not only shows that one can build the classical 'diamond' when two different (monotonic, independent) redexes are applied to a term, but it also shows that the term 'closing the diamond' can be characterized easily as $\Delta \circ \Delta'(t)$, provided that $w|w'$ or $w < w'$ (symmetrically, if instead $w > w'$, then the term closing the diamond is $\Delta' \circ \Delta(t)$). We will use the Church-Rosser diamond to define in an obvious way the parallel application of two redexes to a term. Actually, we consider the parallel application of any *finite* set of independent, monotonic redexes.

Definition 25 (parallel redexes). A **parallel redex** Φ is a (possibly infinite, necessarily countable) set of *monotonic*, mutually independent redexes. The set of *root occurrences* of a parallel redex Φ is defined as $\mathcal{O}_{rt}(\Phi) = \{w \in \omega^* \mid \exists R . (w, R) \in \Phi\}$. A parallel redex Φ is *continuous* if all the redexes in Φ are continuous. If $t \in CT_\Sigma$ and Φ is a parallel redex, then the parallel redex $\Phi \cap t$ is defined as the subset of Φ including all redexes whose occurrence is an occurrence of t, i.e., $\Phi \cap t = \{\Delta \in \Phi \mid \Delta = (w, R) \wedge w \in \mathcal{O}(t)\}$.

Definition 26 (application and composition of finite parallel redexes). Let $\Phi = \{\Delta_1, \ldots, \Delta_n\}$ be a *finite* parallel redex with $\Delta_i = (w_i, R_i)$ for all $1 \leq i \leq n$. Then Φ is also a function (called *finite parallel redex application*)

$\Phi : CT_\Sigma \to CT_\Sigma$, defined as $\Phi = \Delta_{i_1} \circ \ldots \circ \Delta_{i_n}$, where $(\Delta_{i_1}, \ldots, \Delta_{i_n})$ is any permutation of Φ such that for all $1 \le j \le k \le n$, $w_{i_j} \not> w_{i_k}$ (i.e., either $w_{i_j} | w_{i_k}$ or $w_{i_j} \le w_{i_k}$). If Φ and Φ' are two parallel redexes, their *parallel composition* is the parallel redex $\Phi || \Phi' = \{\Delta \mid \Delta \in \Phi \text{ or } \Delta \in \Phi'\}$, and it is defined only if all redexes in $\Phi \cup \Phi'$ are mutually independent.

The well-definedness of finite parallel redex application is ensured by the next result.

Proposition 27 (finite parallel redex application is well-defined). *Let $\Phi = \{\Delta_1, \ldots, \Delta_n\}$ be a finite parallel redex. Then the finite parallel redex application $\Phi : CT_\Sigma \to CT_\Sigma$ is well defined. That is, if $(\Delta_{i_1}, \ldots, \Delta_{i_n})$ and $(\Delta_{l_1}, \ldots, \Delta_{l_n})$ are two permutations of Φ such that for all $1 \le j \le k \le n$ both $w_{i_j} \not> w_{i_k}$ and $w_{l_j} \not> w_{l_k}$, then $\Delta_{i_1} \circ \ldots \circ \Delta_{i_n} = \Delta_{l_1} \circ \ldots \circ \Delta_{l_n}$.*

Fact 28 (monotonicity and continuity of parallel redexes). *If Φ is a finite parallel redex, then $\Phi : CT_\Sigma \to CT_\Sigma$ is monotonic. Moreover, if all the redexes in Φ are continuous, then function Φ is continuous as well. This follows directly from the definitions, because Φ, regarded as a function, is a suitable compositions of all the redexes it contains.*

We are now ready to extend the definition of application of parallel redexes to the infinite case. Since in the finite case the application of a parallel redex is defined as the sequential application of all contained redexes (in a suitable order), a naive extension to infinity wouldn't work, because it would correspond to an infinite composition of functions. We propose therefore a definition which makes use of a suitable limit construction. The main result of this section (Theorem 32 below) proves that the definition is well-given and consistent with the definition of finite parallel redex application, provided that all the involved redexes are continuous. Thus we restrict the definition to *continuous* parallel redexes.

Definition 29 (parallel redex application). Let Φ be a continuous, possibly infinite parallel redex, and let $t \in CT_\Sigma$. Let $\{t_i\}_{i<\omega}$ be any chain of terms such that $t = \sqcup_{i<\omega}\{t_i\}$, and such that for all $i < \omega$ the parallel redex $\Phi_i \equiv \Phi \cap t_i$ is finite. Then the *application* of Φ to t is defined as $\Phi(t) = \sqcup_{i<\omega}\{\Phi_i(t_i)\}$.

In order to prove the well-definedness of the last definition, we need two technical lemmas that state some important properties of parallel redexes.

Lemma 30 (some properties of parallel redexes).

1. *Let Φ and Φ' be two finite, parallel redexes, such that all redexes in $\Phi \cup \Phi'$ are mutually independent. If for all $w \in \mathcal{O}_{rt}(\Phi)$ and for all $w' \in \mathcal{O}_{rt}(\Phi')$ $w \not> w'$, then $\Phi || \Phi' = \Phi \circ \Phi'$.*
2. *If Φ is a finite parallel redex, then for all $t \in CT_\Sigma$, $\Phi(t) = (\Phi \cap t)(t)$.*
3. *If Φ is a parallel redex, $t \in CT_\Sigma$, and $\Phi \cap t$ is finite, then for all t' such that $t \le t'$ and for all finite Φ' such that $\Phi \cap t \subseteq \Phi' \subseteq \Phi$, it holds $(\Phi \cap t)(t) \le \Phi'(t')$.*

Lemma 31 (a property of continuous parallel redexes). *Let $t \in CT_\Sigma$ and let Φ be a* continuous *parallel redex such that $\Phi \cap t$ is finite. Then for all $v \in \mathcal{O}((\Phi \cap t)(t))$ there exists a* finite *term $t_v \leq t$ such that $(\Phi \cap t)(t)(v) = (\Phi \cap t_v)(t_v)(v)$.*

Theorem 32 (parallel redex application is well-defined). *Let Φ be a continuous, possibly infinite parallel redex.*

1. *If $\{t_i\}_{i<\omega}$ is any chain of terms such that $t = \sqcup_{i<\omega}\{t_i\}$, and such that for all $i < \omega$ the parallel redex $\Phi_i = \Phi \cap t_i$ is finite, then $\{\Phi_i(t_i)\}_{i<\omega}$ is a chain.*
2. *Definition 29 is well given, i.e., $\Phi(t)$ does not depend on the choice of the chain approximating t.*
3. *Definition 29 is consistent with the definition of* finite *parallel redex application (Definition 26).*

Proof. 1. We have to show that if $t \leq t'$ and both $\Phi \cap t$ and $\Phi \cap t'$ are finite, then $(\Phi \cap t)(t) \leq (\Phi \cap t')(t')$. This follows directly from Lemma 30.3, observing that $(\Phi \cap t) \subseteq (\Phi \cap t') \subseteq \Phi$.

2. Let $t \in CT_\Sigma$, and let $\{s_i\}_{i<\omega}$ and $\{t_i\}_{i<\omega}$ be two chains approximating t, such that for all $i < \omega$ both $\Phi \cap s_i$ and $\Phi \cap t_i$ are finite. Moreover, let $s' = \sqcup_{i<\omega}\{(\Phi \cap s_i)(s_i)\}$ and $t' = \sqcup_{i<\omega}\{(\Phi \cap t_i)(t_i)\}$ Then we have to show that $s' = t'$: we show just that $s' \leq t'$ (i.e., that $s'(v) = t'(v)$ for all $v \in \mathcal{O}(s')$), the converse being symmetrical.
   Let $v \in \mathcal{O}(s')$. Since $s' = \sqcup_{i<\omega}\{(\Phi \cap s_i)(s_i)\}$, there exists a $k < \omega$ such that $v \in \mathcal{O}((\Phi \cap s_k)(s_k))$ and $(\Phi \cap s_k)(s_k)(v) = s'(v)$. By Lemma 31, there exists a finite term $\hat{s}_k \leq s_k \leq t$ such that $v \in \mathcal{O}((\Phi \cap \hat{s}_k)(\hat{s}_k))$ and $(\Phi \cap \hat{s}_k)(\hat{s}_k)(v) = (\Phi \cap s_k)(s_k)(v) = s'(v)$. Since $\hat{s}_k \leq t$ is finite, there exists an n such that $\hat{s}_k \leq t_n$, and therefore $(\Phi \cap \hat{s}_k)(\hat{s}_k) \leq (\Phi \cap t_n)(t_n)$ (by Lemma 30.3). As a consequence, $s'(v) = (\Phi \cap \hat{s}_k)(\hat{s}_k)(v) = (\Phi \cap t_n)(t_n)(v) = t'(v)$.
3. Let Φ be a finite, continuous parallel redex and let $t \in CT_\Sigma$. Let $t = \sqcup_{1 \leq i<\omega}\{t_i\}$, and $t' = \sqcup_{i<\omega}\{(\Phi \cap t_i)(t_i)\}$ ($\Phi \cap t_i$ is clearly finite for all i). Then we have to show that $t' = \Phi(t)$. In fact, since Φ is finite there must exist a $k < \omega$ such that for all $k \leq j < \omega$, $\Phi \cap t_j = \Phi \cap t$. Thus $\Phi(t) = (\Phi \cap t)(t) = (\Phi \cap t_k)(\sqcup_{i<\omega}\{t_i\}) = (\Phi \cap t_k)(\sqcup_{k \leq i<\omega}\{t_i\}) = \sqcup_{k \leq i<\omega}\{(\Phi \cap t_k)(t_i)\} = \sqcup_{k \leq i<\omega}\{(\Phi \cap t_i)(t_i)\} = \sqcup_{i<\omega}\{(\Phi \cap t_i)(t_i)\} = t'$. □

The next example shows that the continuity of Φ, required in the last theorem, is a necessary condition, at least for points 2 and 3.

Example 3. Let $\Phi = \{(\lambda, R)\}$ be a finite parallel redex, where $R = f(f(f(\ldots))) \to k$ is a left-infinite rule that, as shown in Example 2, is not continuous. Clearly, $\Phi(f^\omega) = R(f^\omega) = k$ using Definition 26. Now, let $t_i = f^\omega$ and $s_i = f^i(\bot)$ for all $1 \leq i < \omega$. Then, obviously, $\sqcup_{i \leq \omega}\{t_i\} = f^\omega = \sqcup_{i \leq \omega}\{s_i\}$, and $\Phi \cap t_i$ and $\Phi \cap s_i$ are finite for all i. Moreover, for all $i < \omega$, $\Phi(t_i) = k$ (because R is total in t_i) and $\Phi(s_i) = s_i \cap k = \bot$ (because R is partial in s_i). As a consequence, $\sqcup_{i \leq \omega}\{\Phi(t_i)\} = k$, but $\sqcup_{i \leq \omega}\{\Phi(s_i)\} = \bot$, showing that infinite parallel redex application is not well-defined if one wants to extend it to non-continuous redexes.

6 Conclusion and future work

In this paper we extended the theory of term rewriting systems to infinite and partial terms (i.e., the elements of algebra CT_Σ), fully exploiting the complete partially ordered structure of CT_Σ. By regarding redexes and rules, as well as the main operations on terms, as total functions, we studied their properties of monotonicity and continuity. For rules, we showed that non-left-linear rules are in general not monotonic, and that left-infinite rules are in general not continuous, unless the left- and right-hand sides are identical. Then we showed that the well-known Church-Rosser property of non-overlapping redexes always holds for monotonic rules. We exploited this property in order to give meaning to the parallel application of a finite set of monotonic redexes, and using standard algebraic techniques we extended the definition to the infinite case. Infinite parallel term rewriting, which is the main contribution of this paper, is well-defined if all the involved rewrite rules are continuous.

This paper is just a first step towards a complete theory of term rewriting in CT_Σ. A lot of work remains to be done in many directions. For example one should consider the extension of other classical notions of the term rewriting literature (like confluence and termination, just to mention two of them) to the setting described in this paper.

Coming back to the motivating example presented in the introduction, the author is currently working with Frank Drewes, Berthold Hoffmann and Detlef Plump of Bremen in order to show that infinite parallel term rewriting as defined here is adequate with respect to cyclic term graph rewriting. A single graph rewriting step would correspond to a single infinite parallel term rewriting step, where the same rule can be applied to an infinite ('rational') number of independent redexes. In our view this solution is more satisfactory than the one proposed for example in [KKSV90], for at least two reasons. First, finite graph derivations can be modelled by *finite* term reductions, while in [KKSV90] a finite graph derivation may correspond to an *infinite* term derivation. Second, the collapsing rules (like R_I in the Introduction) are handled in a completely uniform way in our approach, while in the mentioned paper they are treated in an ad hoc manner in many definitions and proofs.

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