The Möbius Strip Parameterization for Line Extraction *

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Abstract. A parameter mapping well suited for line segmentation is described. We start with discussing some intuitively natural mappings for line segmentation, including the popular Hough transform. Then, we proceed with describing the novel parameter mapping and argue for its properties. The topology of the mapping introduces its name, the "Möbius strip" parameterization. This mapping has topological advantages over previously proposed mappings.

1 Introduction

The reason for using a parameter mapping is often to convert a difficult global detection problem in image space into a local one. Spatially extended patterns are transformed so that they produce spatially compact features in a space of parameter values. In the case of line segmentation the idea is to transform the original image into a new domain so that colinear subsets, i.e. global lines, fall into clusters. The topology of the mapping must reflect closeness between wanted features, in this case features describing properties of a line. The metric describing closeness should also be uniform throughout the space with respect to the features. If the metric and topology do not meet these requirements, significant bias and ambiguities will be introduced into any subsequent classification process.

2 Parameter Mappings

In this section some problems with standard mappings for line segmentation will be illuminated.
The Hough transform, HT, was introduced by P. V. C. Hough in 1962 as a method for detecting complex patterns [Hough, 1962]. It has found considerable application due to its robustness when using noisy or incomplete data. A comprehensive review of the Hough transform covering the years 1962-1988 can be found in [Illingworth and Kittler, 1988].
Severe problems with standard Hough parameterization are that the space is unbounded and will contain singularities for large slopes. The difficulties of unlimited ranges of the values can be solved by using two plots, the second corresponding to interchanging the axis. This is of course not a satisfactory solution. Duda and Hart [Duda and Hart, 1972] suggested that straight lines might be most usefully parameterized by the length, ρ, and orientation φ, of the normal

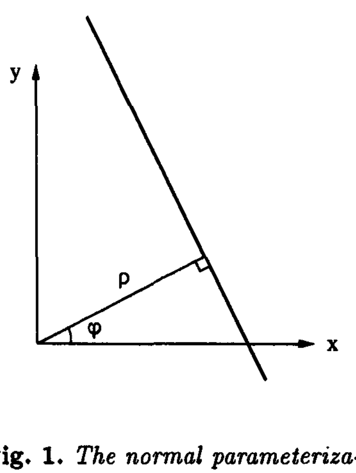

Fig. 1. *The normal parameterization, (ρ, φ), of a line. ρ is the magnitude of displacement vector from the origin and φ is its argument.*

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vector to the line from the origin, the **normal parameterization**, see figure 1. This is a mapping has the advantage of having no singularities.

Measuring local orientation provides additional information about the slope of the line, or the angle φ when using the normal parameterization. This reduces the standard HT to a one-to-one mapping. With one-to-one we do not mean that the mapping is invertible, but that there is only one point in the parameter space that defines the parameters that could have produced it.

Duda and Hart discussed this briefly in [Duda and Hart, 1973]. They suggested that this mapping could be useful when fitting lines to a collection of short line segments. Dudani and Luk [Dudani and Luk, 1978] use this technique for grouping measured edge elements. Princen, Illingworth and Kittler do line extraction using a pyramid structure [Princen et al., 1990]. At the lowest level they use the ordinary $\rho\varphi$-HT on subimages for estimating small line segments. In the preceding levels they use the additional local orientation information for grouping the segments.

Unfortunately, however, the normal parameterization has problems when ρ is small. The topology here is very strange. Clusters can be divided into parts very far away from each other. Consider for example a line going through the origin in a xy-coordinate system. When mapping the coordinates according to the normal parameterization, *two* clusters will be produced separated in the φ-dimension by π, see figure 2. Note that this will happen even if the orientation estimates are perfectly correct. A line will always have at least an infinitesimal thickness and will therefore be projected on *both* sides of the origin. A final point to note is that a translation of the origin outside the image plane will not remove this topological problem. It will only be transferred to other lines.

Granlund introduced a double angle notation [Granlund, 1978] in order to achieve a suitable continuous representation for local orientation. However, using this double angle notation for *global* lines, removes the ability of distinguishing between lines with the same orientation and distance, ρ, at opposite side of the origin. The problem near $\rho = 0$ is removed, but unfortunately we have introduced another one. The two horizontal lines (marked a and c), located at the same distance ρ from the origin, are in the double angle normal parameterization mixed into one cluster, see figure 2.

It seems that we need a "double angle" representation around the origin and a "single angle" representation elsewhere. This raises a fundamental dilemma: is it possible to achieve a mapping that fulfills both the single angle and the double angle requirements simultaneously?

We have been concerned with the problem of the normal parameterization spreading the coordinates around the origin unsatisfactorily although they are located very close in the cartesian representation. Why do we not express the displacement vector, i.e. the normal vector to the line from the origin, in *cartesian* coordinates, (X, Y), since the topology is satisfactory? This parameterization is defined by

$$\begin{cases} X = x\cos^2(\varphi) + y\cos(\varphi)\sin(\varphi) \\ Y = y\sin^2(\varphi) + x\cos(\varphi)\sin(\varphi) \end{cases}$$

where φ is as before the argument of the normal vector (same as the displacement vector of the line).

Davis uses the $\rho\varphi$-parameterization in this way by storing the information in a cartesian array [Davis, 1986]. This gives the (X, Y) parameterization. There are two reasons for not using this parameterization. First, the spatial resolution is very poor near the origin. Secondly, and worse, all lines having ρ equal to 0 will be mapped to the same cluster.

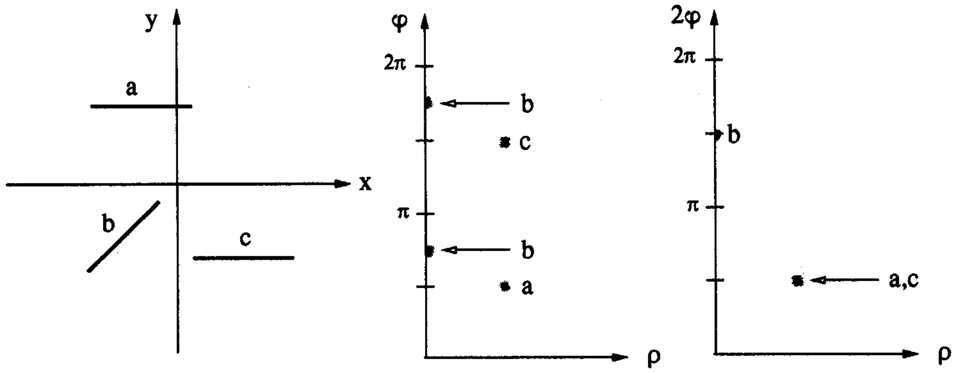

Fig. 2. *A test image containing three lines and its transformation to the $\rho\varphi$-domain, the normal parameterization of a line. The cluster from the line at 45° is divided into two parts. This mapping has topological problems near $\rho = 0$. The $\rho 2\varphi$-domain, however, folds the space so the topology is good near $\rho = 0$, but unfortunately it is now bad elsewhere. The two horizontal lines, marked* a *and* c*, have in this parameter space been mixed in the same cluster.*

The first problem, the poor resolution near the origin, can at least be solved by mapping the XY-plane onto a logarithmic cone. That would stretch the XY-plane so the points close to the origin get more space. However, the second problem still remains.

3 The Möbius Strip Parameterization

In this section we shall present a new parameter space and discuss its advantages with respect to the arguments of the previous section. The Möbius Strip mapping is based on a transformation to a 4D space by taking the **normal parameterization** in figure 1, expressed in cartesian coordinates (X, Y) and adding a "double angle" dimension, (consider the Z-axis in a XYZ-coordinate system). The problem with the cartesian normal parameterization is as mentioned that all clusters from lines going through the origin mix into one cluster. The additional dimension, $\phi = 2\varphi$, separates the clusters on the origin and close to the origin if the clusters originate from lines with different orientation. Moreover, the wrap-around requirement for ϕ is ensured by introducing a fourth dimension, R.

The 4D-mapping

$$\begin{cases} X = x\cos^2(\varphi) + y\cos(\varphi)\sin(\varphi) \\ Y = y\sin^2(\varphi) + x\cos(\varphi)\sin(\varphi) \\ \phi = 2\varphi \\ R = R_0 \in \mathbb{R}^+ \end{cases}$$

The two first parameters, X and Y, is the normal vector in fig 1, expressed in cartesian coordinates. The two following parameters, ϕ and R, define a circle with radius R_0 in the $R\phi$-subspace. Any $R_0 > 0$ is suitable. This gives a $XY\phi$-system with wrap-around in the ϕ-dimension.

In the mapping above, the parameters are dependent. As the argument of the vector in the XY-plane is φ and the fourth dimension is constant, it follows that for a specific

(X, Y) all the parameters are given. Hence, the degree of freedom is limited to two, the dimension of the XY-plane. Thus, all the mapped image points lie in a 2D subspace of the 4D parameter space, see figure 3.

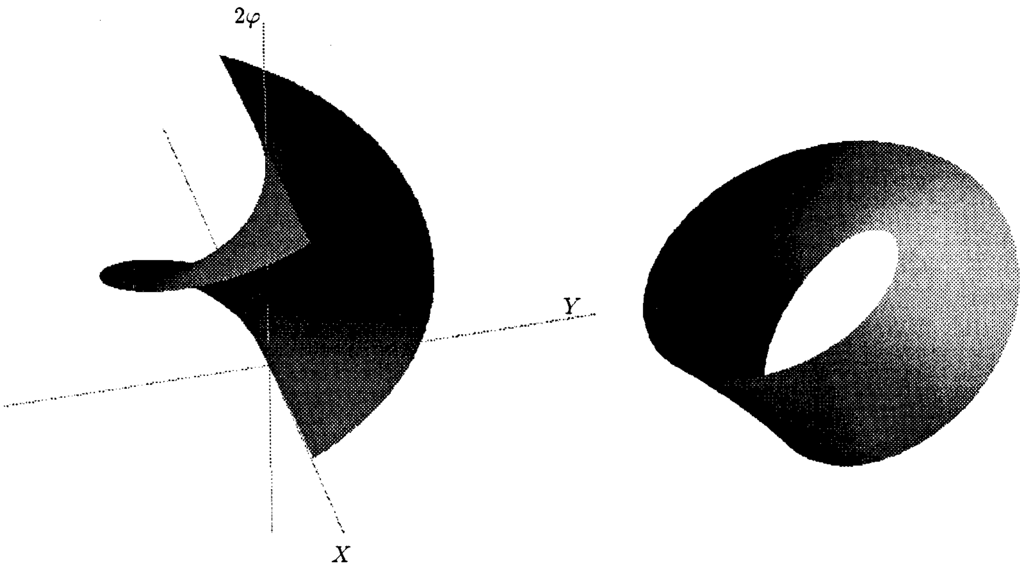

Fig. 3. *The $XY2\varphi$ parameter mapping. The wrap around in the double phi dimension gives the interpretation of a Möbius strip*

The 2D-surface

The regular form of the 2D-surface makes it possible to find a two-parameter form for the wanted mapping. Let us consider a $\eta\phi$-plane corresponding to the flattened surface in figure 3. Let

$$\rho^2 = X^2 + Y^2 = (x\cos^2(\varphi) + y\cos(\varphi)\sin(\varphi))^2 + (y\sin^2(\varphi) + x\cos(\varphi)\sin(\varphi))^2$$

$$\Leftrightarrow \qquad \rho = x\cos(\varphi) + y\sin(\varphi)$$

Then the (η, ϕ) mapping can be expressed as

$$\eta = \begin{cases} \rho & 0 \leq \varphi < \pi \\ -\rho & \pi \leq \varphi < 2\pi \end{cases}$$

$$\phi = 2\varphi$$

η is the variable "across" the strip with 0 value meaning the position in the middle of the strip, i.e *on* the 2φ axis. The wrap-around in the ϕ dimension make the interpretation that the surface is a Möbius strip easy, see figure 3.

Finally, using the same test image as before, we can see that we can distinguish between the two lines at opposite side of the origin at the same time as the cluster corresponding to the line going through the origin is not divided, see figure 4.

4 Conclusion

The main contribution of this paper is the novel Möbius strip parameter mapping. The name, as mentioned above, reflects the topology of the $\eta\phi$-parameter surface, its *twisted* wrap-around in the ϕ-dimension. The proposed mapping has the following properties:

- The parameter space is bounded and has no singularities such as the standard HT parameterization for large slopes.
- The metric reflects closeness between features. The mapping does not share the topological problem of the $\rho\varphi$-mapping near $\rho = 0$. It has been shown that any line passing through the origin produces *two* clusters, separated by π in the φ-dimension.

Note that these properties have been achieved *without* increasing the dimension of the parameter space compared to for example the *normal parameterization*, see figure 1.

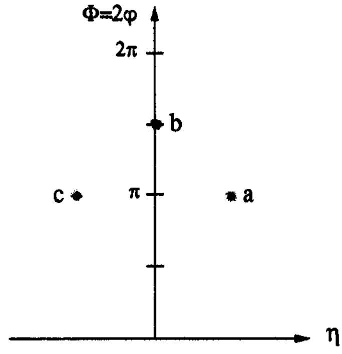

Fig. 4. *The (η, ϕ) parameter mapping (the "Möbius strip" mapping). We can see that we can distinguish between the two lines,* a *and* c*, at opposite side of the origin at the same time as the cluster corresponding to the line going through the origin is not divided.*

Line segmentation results from using the Möbius Strip Parameterization can be found in [Westin, 1991].

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